Chaotic Dynamical Systems
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Lecture – 16
Devaney's Chaos

Welcome to students. So, today we will be discussing a very popular definition of chaos that is Devaney's chaos. And we will also see that this definition of chaos almost implies almost all other definitions of chaos. So, what do we need here is as our hypothesis today we will just need that x d is a metric space.

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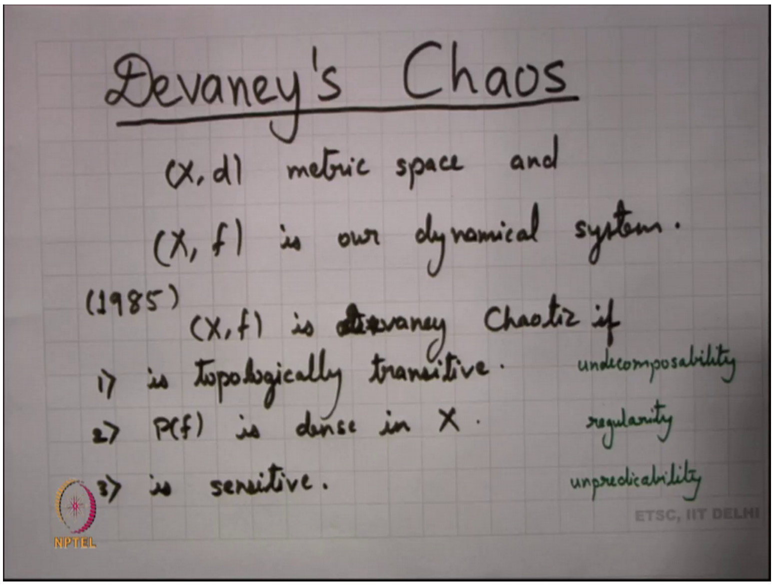

And we have our dynamical system what is chaos. So, basically, we try to term chaos as something which is like unpredictable something, which cannot be focused aid something which is sort of naturally occurring in various systems, and we are not able to specify or we are not able to identify lot of properties, right. Using because of the presence of this kind of behavior.

So, such kind of behavior is called chaos. And it was in almost 85 that devaney wrote a book on chaos, and he tried to put up 3 of course, there are were a lot of other definitions of chaos, but he tried to put up 3 ingredients. Basically, one ingredient was the in-decomposing ability, the second was regularity, the third was unpredictability. And he tried to put all these 3 ingredients to form a definition of chaos.

So, this was essentially in 1985 that this was defined. And we say that the system is devaney chaotic, if the first thing is the system is topologically transitive. The second property we want that the set of periodic points of f is dense in x. And the third property which would require is that this property is sensitive.

Now, we have already studied for the sensitivity, and we again try to recall this definition here, that you have a delta positive such that whenever x and y are 2 points very close by, right. At some instant their orbits move delta apart. So, that is what is our sensitivity. And if we try to see in fact, what devaney had in mind was that if you look into topological transitivity this gives some kind of un decomposability. If you try to look into a denseness of periodic points that gives because periodic points always try to come back to itself after finite times.

So, this gives of some kind of regularity. And of course, sensitivity as we have studied earlier also, this gives some kind of unpredictability. So, this was essentially the definition of chaos given by devaney. And now let us try to look into some examples.

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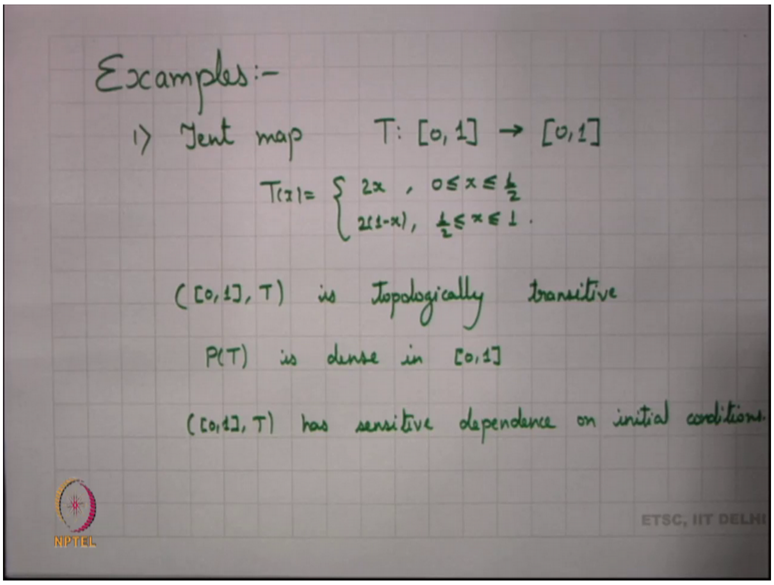

So, the first example we look into is where example of the tenth map. So, basically, we have this map T from 0 1 to 0 1 given by T x equal to twice x in twice 1 minus x.

Now we have already studied this map earlier also, and I think it was one of the previous lectures that, we saw that this map is topologically transitive right. So, this is transitive.

In one of the previous classes, we have also seen that this has a dense set of periodic points. In fact, it has periodic points of all periods. And it has a dense set of periodic points. So, what we have here is that p of T is dense 0 1. And also, one of the classes we had seen that this is sensitive. In fact, if you try to look into what is the derivative except at the point half, right. The derivative at each and every point is 2. So, that means, if you have 2 points closed by, right. Then their iterates will be moving it destroys the distance between them, right and that 2 2 keeps on increasing fine.

So, you will find that at some point, they would have at least reached the distance half between them. So, we can say that fine we could say half we could say one third we could say one 4th whatever it be, we have a delta such that, you start with any point there is a neighborhood point, right. Which moves delta apart, right at some instant it moves delta apart. So, this system is also sensitive.

So, this is a chaotic system. And we take this as a model chaotic system, because it is very will as we try to see we will try to formulate what kind of transitive points, we have here what kind of sensitive points we have here will try to look into that this will be our model system there is another system, which is closely related to the tent map, and which again we have already studied we are not introducing something new.

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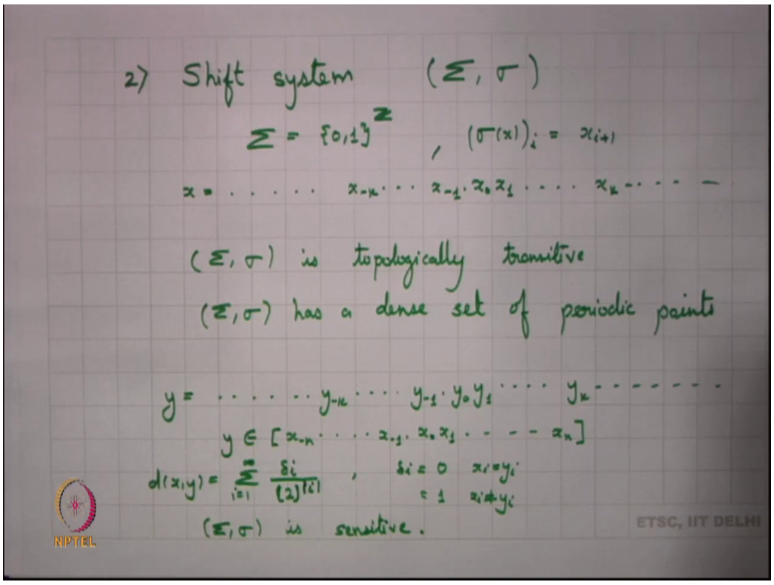

So, let us look into this second example. And the second example is that of the shift system. So, we are looking into the system, where my space is basically the set of all sequences 0 1 sequences by infinite sequences under the product topology. And my shift map is simply just the, right. Shift where my typical point x turns out to be of the form we just saw in one of the previous classes that this is a transitive system.

So, the system is transitive again this has dense set of periodic points, because each and every word here, right each and every finite sequence here gives rise to a periodic point. So, this and we know the construction the basic open sets are the cylinders sets here. So, this has a dense set of periodic points. And is it is sensitive look into the fact you have any x here, right. Then at some instant we know that there will be a point in the neighborhood.

So, will try to say that this is sensitive, let us say that this is my x, right. And if my y belongs to some neighborhood of x then; that means, y has to belong to. So, this point y which I am writing as y minus k y minus 1; if we try to look into the system this point, what we find is that for given epsilon right; that means, I will have a bigger block as epsilon is get smaller and smaller there is a bigger block on which y the central part of y, right. It agrees with the central part of x.

So, if I say that my y belongs to the cylinder set. So, I am talking of x minus n. So, if my y is here; that means, y agrees with x on this particularly instant what happens here once you move y, right. Because I my sequences can take anything right; that means, on the left and the, right. There can be any sequences. So, once you have fixed this point x, and you are now in the neighborhood of x. So, the neighborhood of x we can think of as a cylinder set around x. So, in this particular neighborhood of x, what you find here is that at some instant somewhere, right. Because all sequences are possible later on right.

At some instances x will have a 0, right. The sequence y will have a one there at the same place. And when x is 0 and y has a one there, right. What with the distance as we have seen what does a metric here. So, the metric here is dx y is let me just recall this I going from one to infinity, we have mode of we can think of because this is 0 1 0 2 matrix I can just think of it delta I. Write upon 2 to the power mod i. And where my delta i is 0 if my xi is equal to yi write, and this is equal to one when xi is not equal to yi.

So, this is what the metric that we had taken for this particular sequence. So, this matric gives us the product topology here. And we find that what happens here is that at some instant right. So, once you have taken sigma over some place, right. At the 0 th place you will find that this there is a y supposing there is a 0 in the x th placed there is a one in the yth place. So, after taking some maybe after moving, for some iterates or maybe some iterates of x and y, right will have at the central place 0 and central place one.

So, if we take central plus 0 and central place one, right. Because this is at the center place, right. The distance here is at least one, right. The distance there becomes at least one. And so, the system is sensitive also our shift system is also. A good model for a chaotic system and once I have the shift system, right. We have also seen that the shift system is topologically conjugated to the horseshoe attractor right, but in the horseshoe attractor again, we need to check out what happens to our sensitivity because we know that topological conjugates transitivity and periodic points are closed under topological conjugacy. What happens to sensitivity in that particular case?

So, it is very liable to say that the shift system since the shift system is sensitive our horseshoe attractor should also have been sensitive. We will see how that comes up, but before that will look into some examples, where which are not chaotic in the sense of devaney. Now devaney wants all these 3 properties to be satisfied for the system to be chaotic. Whereas, we know that all these 3 properties that we have taken up topological transitivity, dense periodic points and sensitivity they are independent properties on their own.

So, supposing now I take as my example. So, this is my example 3 that I put up.

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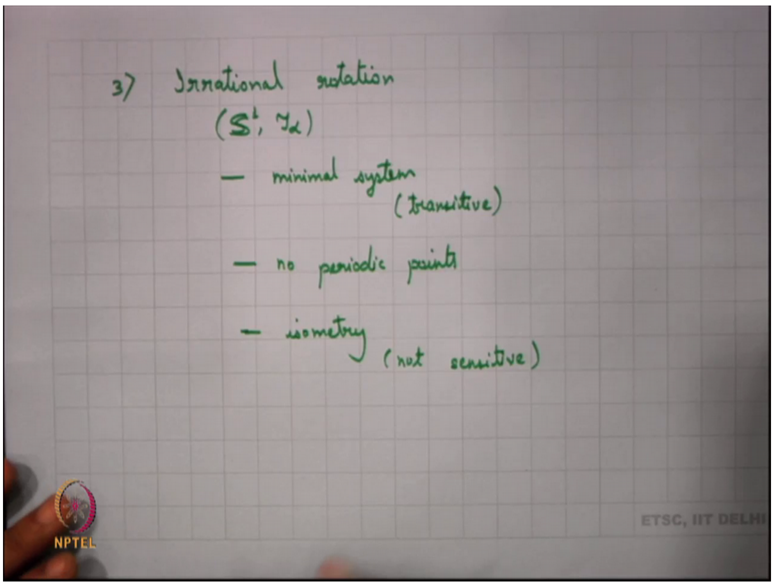

I take as my example the irrational rotation, then we all recall here that we are looking into the irrational rotation on the circle S 1. And my rotation tau alpha is rotation by an irrational multiple of 2 pi. And we have already seen that what happens in case of irrational rotation, this is basically a minimal system, since this is a minimal system, this is a compact metric space, right. We can say that the system is transitive. This is basically a transitive system. We have also seen for an international rotation, that this has no periodic points. Irrational rotation has no periodic points. And if we try to say what is this irrational rotation basically it is an example of an isometry. The distance between the images is same as the distance between the 2 points itself right.

So, because we are all moving with the same angle, right. With this all the points move with the same angle. So, that means, this is basically an isometry and when you have an isometry, right. The distance between x and y is always preserved right. So, there is no possibility of sensitivity here. So, this system is an isometry and hence not sensitive. So, this system is not sensitive. And so, what we have here is that for an irrational rotation, right. This is not devaney chaotic. So, this is a very nice interesting example from a dynamical system point of view, but this system is this example is basically not devaney chaotic.

now let us try to look into the rational rotation here.

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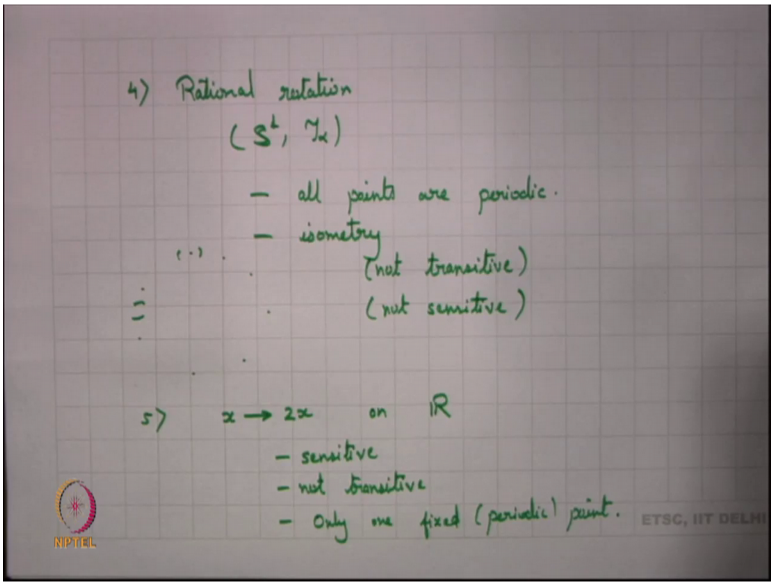

Now if we try to look into the rational, rotation again we have over circle unit circle and again we have our rational rotation; that means, we have a rational we are moving with rational multiple of 2 pi. And we have seen that in this case, right. Every point is a periodic point of a particular period n. So, all points periodic here, now since all points are periodic, right. We have this condition that the periodic points are dense here, right. Every point is periodic here, but this system is not transitive.

Since well where are the points moving, right. The points are just moving in some cycle right. So, I cannot say that is a point from say this neighborhood, right. We will never come up to this neighborhood. Because they are all moving in some kind of a symmetry here, right. This is this is again an isometric motion right. So, this is isometry here. So, this is not going to so, this is not. So, there is no orbit from some particular neighborhood which visits every other neighborhood, right. And hence this is not transitive and the isometry is the reason why this is not sensitive also.

So, we have an example of a system which has dense set of periodic points, but it is neither transitive nor sensitive. Now the previous example was an example of a transitive system, right. Which for which it is neither dense orbits, dense periodic or points nor it is sensitive. We can look for another example where we have sensitivity. So, I am just taking this map x going to 2 x, right on the real line. And this is definitely an example of a sensitive system right, but this is not transitive because nothing is going to just moves towards infinity, right moves towards infinity and minus infinity on the 2 sides. It never comes back again. So, this is not transitive. So, this is sensitive, this is not transitive. Again, what happens to the periodic points? This is only one fixed point 0 is the only fixed point right.

So, again here it has only one fixed point I should say periodic. So, this again the system is not chaotic although, if you try to look into this this is unpredictable because it is some kind of a sensitive system. So, the 3 properties that devaney brought together. They are basically independent of each other, but these are the ingredients of chaos as he believed it. But there was another problem with this definition. And the problem with this definition came up in looking into the concept of topological conjugacy.

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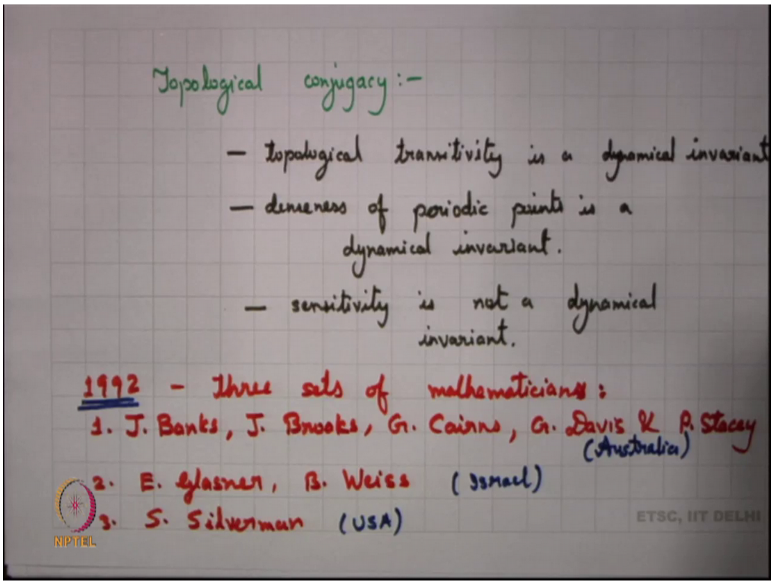

Now, as we recall, right topological conjugacy, we say that 2 systems are conjugate, if there is a homomorphism between them which preserves the dynamical property.

So, the dynamical properties preserved under topological conjugacy. And whatever properties are preserved we call them as dynamical invariants. So now, if we try to look into that part, right. Just as we had seen in the one of the previous classes; that topological transitivity is a dynamic invariant. So, topologically transitivity is a dynamically in variant. And we are not trying to look into what happens to dense set of periodic points. As we have seen that in topology in conjugacy, right periodic points are preserved. So, if the periodic points are dense in one system the periodic points will be dense in the other system also. So, denseness of periodic points, what happens to sensitivity?

Now, again when we had seen this chapter of sensitivity, we had seen that there is an equivalent metric, right. With for given any metric there is an equivalent metric under which the system does not become remain sensitive right. So, sensitivity is not a dynamical invariant. So, topologically this definition of chaos does not make sense, right because this definition of chaos is not a dynamically invariant. Of course, here I would like to mention here. That if instead of working with just a metric space if we try to look into compact metric spaces, then, for compact metric spaces, right. You take any equivalent metric, right. Sensitivity is preserved under all equivalent matrices matrix, right.

So, this is basically if we look into this part this in under in in the special condition of compactness definitely this becomes a dynamical invariant. But then this is not a dynamical invariant in general. So, in general if I am looking into this definition of chaos, this definition of chaos becomes almost unimportant. This definition of chaos was something which was widely accepted. Because as I said that will try to see the spat it was some kind of a logical terminology which defined chaos.

So, there was lot of research and going on, what do we really need sensitivity should we remove sensitivity, but sensitivity is giving you unpredictability there is nothing else that gives you unpredictability. So, what should we do for aiming this property of sensitivity. So, that it becomes some kind of a dynamical invariant. So, it was during all this study that it was in 1992. We had 3 sets of mathematicians. Now those 3 sets of mathematicians I am naming the first set here.

So, the first set here was J Banks, J Brooks, G Cain's G Davis and P Stacey. Now this way the set of mathematicians from Australia, then we had another set of mathematicians which is E Glassner and B Weiss. And this was the set of mathematicians from Israel. And then we had a third set of course, it is not a set this is just a single mathematician here, and we have studied something about him earlier also. So, this is s silver man from the united states.

So, this is the third set from the United States. So, it was like 92. So, it depends on when you conceive an idea when you are able to produce results on it and when that result gets published. So, there is some kind of an epsilon difference between when the 3 papers were published, but it was basically 1992 that they came up with this idea. And their idea was very simple they said that you look back to the devaneys definition. So, if we looked that to devaneys definition, right. Their idea was simple that 1 plus 2 implies 3. So, if the system is topologically transitive and if it has a dense set of periodic points, then the system will be sensitive. And we shall try to look into this theorem and this proof right now.

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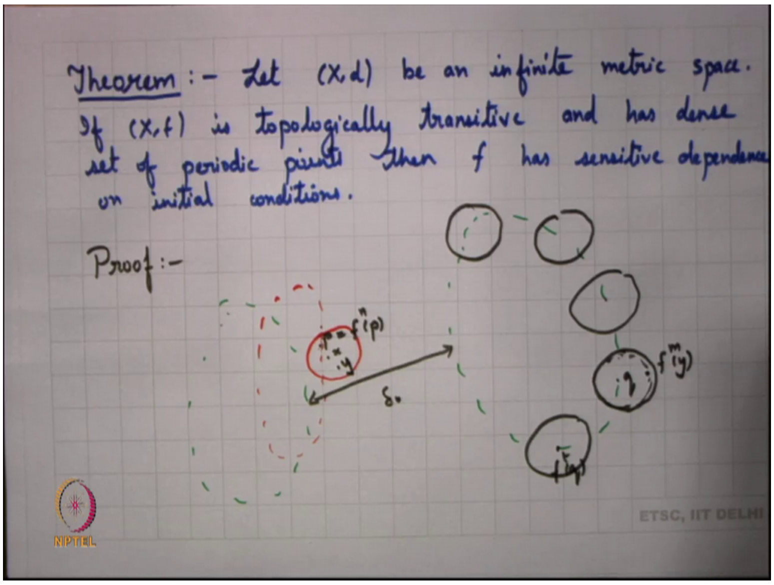

So, let us go back to stating the theorem. So, what we want is we want an infinite metric space. What happens in case of a finite metric space? We know that in the finite metric space if it is transitive it has to be a periodic orbit. So, you do have periodic orbit, but then again sensitivity is missing because of the finite space it is always discrete right. So, there is no sensitivity there. So, we would not our metric space to be infinite. So, let xd be an infinite metric space. If the system xf now, I am looking into the dynamical system the system xf is has is topologically transitive and has dense set of periodic points. Then our system has sensitive dependence on initial conditions. So, there was a sort of redundancy seen in devaneys definition of chaos, and after this whenever one talks of devaneys definition of chaos it is simply that the system is topologically transitive and it has dense set of periodic points, because sensitivity was found out to be redundant in that particular definition.

So, what is the idea behind it? So, we will try to look into the idea behind this proof first. So, let us look into this proof here, and we will try to look into some idea behind it now. The idea behind this proof is supposing I have a point x here, right. Since I have a dense set of periodic orbits, right periodic points are dense; that means, there are many many periodic orbits, because we have an infinite system. So, there are many many periodic orbits. So, you will find that there is some kind of a there is some periodic orbit here, right. And there is some periodic orbit here. So, there are 2 distinct periodic orbits. And when we have 2 distinct periodic orbits, I can always say that there will be some distance between these 2 periodic orbits. So, we just pick up 2 periodic orbits. We find the distance between these 2 periodic orbits. And we take say that fine we fix this distance. So, we have some kind of a delta naught we call delta naught here. So, we have some kind of a delta naught fixed here.

Now, think of this point x. You think any point x anywhere in the space, right. Then it will be either delta naught by 2 closer to this orbit or it will be delta by naught closer to this orbit. So, that means, there is one of the orbits from which is it is delta y not too far apart right. So, given any point there is one of the orbits where it is far apart.

Now the interesting fact comes here is that we start with this point x, and we try to take a neighborhood of x. Now if we try to take this neighborhood of x this neighborhood of x is also going to contain a periodic point. So, this periodic point let me call it say p here. So, this will also contain a periodic point p here. And this periodic point p will have it is own orbit. So, that means, after some it takes it is going to come back to itself. So, this has it is own orbit we do not worry about where it travels elsewhere we are only worried with what happens at this particular point p.

So, we know that this will have some period also. So, let us say the period is n. So, we know that this p is same as fnp. So, after every n instance it is coming back to itself. Now we have this point we have this particular periodic orbit, where which is say delta naught by 2 further from this particular point x. So, what we try to do is we try to see use topological transitivity, right. In order to for a point from here, we have another point here let me call that u, right. Let me call that y we have another point here y here such that, y is going to move to one of this somewhere closer to one of these periodic points right. So, let me call this periodic point as q right.

So, then it is going to move and let me say that this is some neighborhood of q. So, after some instance y is going to move here. So, let me say that here I have my fm of y. So, y moved to this particular orbit. Now if I look into q, right. Now I am looking into this neighborhood of q. Then I know that the inverse image because I am looking into inverse images now. So, what I will try to do is this is q, right. Then I will have f of q, right. Because this is any other periodic orbit, it moves into a periodic motion. So, I will take this neighborhoods of all fq f square q f cube q and so on. And then we can try to look into then once we know that q will also have certain orbit right.

So, once I take the inverse image of this, this inverse image overlaps with the inverse with the neighborhood here. This inverse image overlaps with a neighborhood here. This inverse image overlaps with the neighborhood here. This inverse image overlaps over here. So, if I try to take the inverse image of all this. So, the inverse image of this is taken say maybe this is my f T of q, then the inverse image of this is taken T times the inverse image of this is taken T minus 1 times, right.

If I keep on overlapping these images, what I get is; I get an open set of q. Now think of that part what happens over here is that this particular point has moved here right. So, once it comes here this is going to move in this direction, right close to each other. And now so, this particular iterate, right is just going to move around this particular sets right. So, it has now this gets a very nice motion that it is just basically moving around somewhere here, it is not moving around elsewhere.

So, finitely many steps it is just going to move around this place itself. But my p is something which is moving around here, right. And at every end stage it is coming back to. So, if I am super imposing my n over here, what I have here is that some iterate or some multiple of n, right. P comes over here, but if I look into my y, right. Some multiple of n, right; my y would be somewhere over here. So, the distance between these 2 would we say at least some delta apart, right which gives us sensitivity.

So, these are the basic idea of the proof and now let us try to write down the proof. Let us look into the proof.

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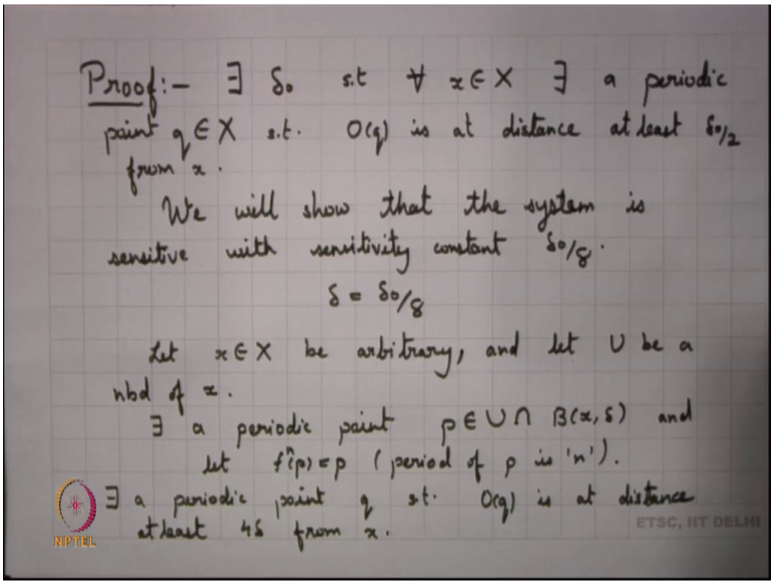

And as I said that this is a really very, very simple proof. As I have already mentioned, right. There exists a delta naught such that for every x in X a periodic point, the orbit of q is at distance at least delta naught by 2 from x. Now we will show that the system is sensitive. With sensitivity constant say delta not by 8. So, I am assuming my delta to be equal to delta not by 8, and we will show that this delta is the sensitivity constant. We have x in X this is an arbitrary point.

So, let x in X be arbitrary. So, with respect to this x we have this particular point. We have this particular orbit q which is at distance 4 delta apart, right which is at a distance 4 delta apart. Here now we take this point x to be arbitrary and let u be a neighborhood, neighborhood of x. So, we are just taken up some neighborhood of x. Now we any arbitrary neighborhood of x, what we have is there exist a point. So, I am now looking into a periodic point. So, we can say that there exists a periodic point p in u now I am taking a particular neighborhood of u of x. So, this is u intersection I am taking b x delta, right ball of radius delta intersection with u.

So, there exist a periodic point p, and let me test assume that n to be the period of p. So, fnp be equal to p. So, my period of p is n, right. Now we have assumed this and we have also seen as we have observed earlier, that there exist a periodic point q such that orbit of q is at distance at least 4 delta from x we make this observation. And now let us try to make another construction. So now, I am taking my set v and defining my set v to be the intersection of i going from 0 to n, what I have is f minus i have a ball centered at fiq, right of radius delta.

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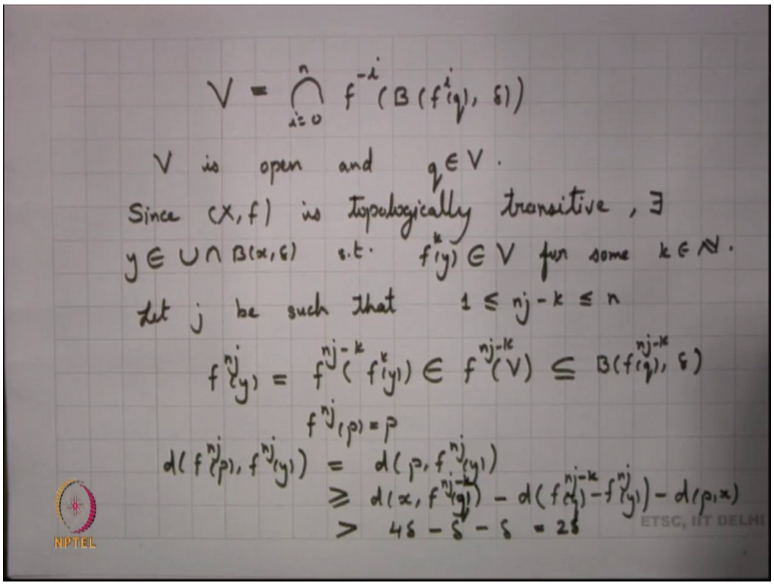

So, I am defining my v to be the intersection of these open sets. And my v is open now all we can guess here is all we can conclude is here that v is open. So, v is open and q belongs to v. So now, we have a non-empty set u, we have a non-empty set v. So, there exists a non-empty set u there exists a non-empty set v, right. So, you have you v you have v, right. And our system is topologically transitive right. So, since the system is topologically transitive, then purposely taking something else I am taking a y belonging to u intersection b x delta, right. Such that I would have fk of y belonging to v, right. For some k E N. My system is topologically transitive sorry, can think of my some y from here such that f k of y belongs to v.

Let my j be such I have 1, which is less than or equal to n times j minus k, right. And this is less than or equal n, right. Because we actually do not know whether there is any relation between k an n, right. We just picked up a k. In fact, we could be picked up k greater than n, but we tried to take up j to be that particular number for which n j minus k lies between 1. An n now what happens in that case what is my f to the power n j of y, right. I can say that this is nothing but f to the power n j minus k of f k y. Now where is my f ky? My f k y belongs to we right. So, this basically belongs to f to the power n j minus k of v. And where is this contained? So, we can say that this was basically contained n now I am looking into this part because this is my v, right. My v is the intersection of all this inverse images, right. And now I am saying that f n j minus k times I am again moving mj minus k on v. So, this would be basically contained in a ball of f of n j minus k times q, right. This is basically would be contained here, and vds delta.

So, this is basically contained in this particular ball, but we also know that f to the power n j of p will be same as p. So now, what is the distance here. So, we try to look into what is the distance between f n j of p and f nj of y. So, we can say that this is nothing but this is basically distance between p and f nj of y. Now we can say that this is greater than equal to, right. I am writing it as d of x f nj y sorry, f nj q minus d of f nj q minus f nj y minus d of p and x, right. Just using triangular inequality, now what is the distance between x and f nj q.

Now, we know that this is the orbit of q and x, right. They are at least 4 delta apart right. So, this distance is greater than this is this greater than 4 delta, right. What is this distance f nj of q f and f nj of y we see that f nj of y belongs to a delta ball around f nj minus k of q right. So, what we find here is that f k of y belonged over here right. So, f nj of y, right will be somewhere delta to this part. Sorry, this is f nj minus k I am taking f nj minus k of q here so that it remains in the delta ball here. So, let me take here f nj minus q of q and f nj y. So, we know that f nj y is just at a distance delta here. So, this is minus delta, and if I take p and x they came from the same ball right. So, this is again minus delta.

So, this becomes basically twice delta. Now what do we have, right. In the neighborhood of x, right. We have 2 points those 2 points where p and y, such that at some instant their orbits are at a distance twice delta apart; that means, I can conclude that and this since my x was arbitrary. Like this construction can be made for any arbitrary x. So, given any arbitrary x in the neighborhood of x you can find 2 points, such that there orbits are at a distance 2 delta apart; that means, our system is sensitive, right with sensitive sensitivity constant delta.

So that means, the system is sensitive with sensitivity constant delta, right because any 2 points in the neighborhood are at distance to delta apart. So, that means, I can say that here the system is transitive. So, we just we can conclude.

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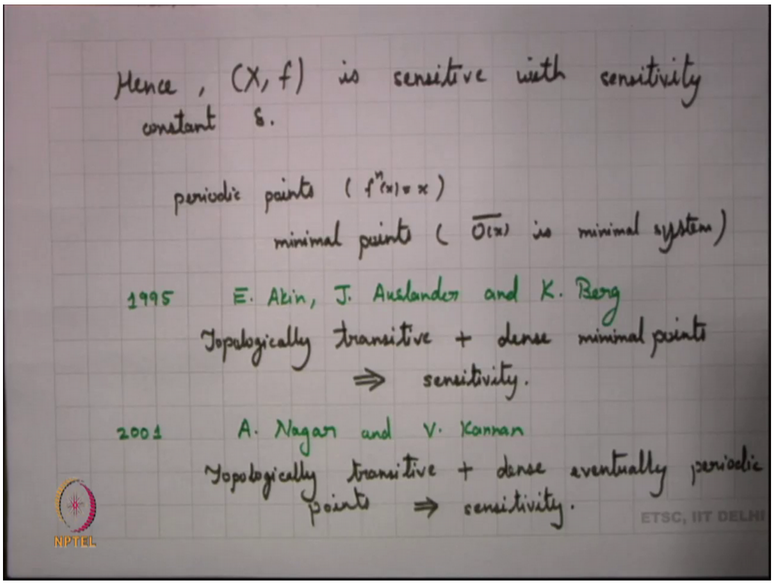

Hence, x f is sensitive. This result gave a very I mean this was basically a surprising result and of course, this also gives us the concept that this definition of chaos happens to be a dynamical invariant under any circumstances, but this was a very interesting result. And more than the fact that this was a very interesting result this has a very, very simple proof.

So, that means, now one could relate transitivity to many other dynamical many other chaotic properties. And that led people to think on what more could be there is whether we have transitivity and sensitivity leading to dense set of periodic points. Is it possible that sensitivity and dense set of periodic point leads to transitivity? Is it possible that transitivity plus something else leads to sensitivity? So, there were a lot of other possibilities that came up in mind and then people started looking researching into all these concepts.

So, what became important is now that if we have a dynamical property, what other dynamical properties can it imply. On under what circumstances what are all the properties that in can imply. So, that means, we are looking out for a very sort of a very compact definition of chaos, once we said that this property is there; that means, almost all properties are satisfied. So, people tried various hands into looking into studying into properties once again and studying into the relation between properties once again. So, will try to study this relation in the course of this course definitely, but right now here I just want to mention 2 results in this direction.

So, one of the results came up where which uses the concept of minimal points, now we know that we have this concept of periodic points and we know that the periodic points are both those points whose orbit, right. X comes back to itself after finitely many times. So, that means, fn x is equal to x, right. These are this is basically our concept of periodic points. Similar to periodic points or basically I should say somewhat analogous to periodic points is the concept of minimal points. Now what do we mean by minimal points? So, I want to look into points x such that if I take the orbit of x, right. If I take the orbit closure of x, then this happens to be a minimal system. So, this is a minimal system.

So, we have this concept of minimal points, and one can simply see that all periodic points are minimal points. Because a periodic orbit is anyway incase minimal, right. Minimal system. So, all periodic points are minimal points, but we have this concept of minimal points, where we say that a point is minimal if the orbit closure of the point happens to be a minimal system.
So, this is minimal points such some kind of minimal points are generalization of this periodic points. And it was in 1995 that we had this set of mathematicians E Akin, J Auslander and K Berg. So, they were they were also looking into the same proof in the same sort of definitions here. And they came up with this result that topological transitivity plus dense minimal points implies sensitivity. Again, there was a generalization to this result saying that fine if we have topological transitivity and you have dense set of minimal points, because this does not have much to do with chaos, but it has to do with lot of dynamical properties here.

So, this implies sensitivity. And then it was later in 2001, that we had another set here myself and V Kannan. And what we proved was that topological transitivity, eventually dense eventually periodic points. So, the periodic points may not be dense it is quite possible that there is only one periodic point, but if there is a dense set of periodic point in dense set of eventually periodic points then that implies sensitivity. And we will look into this for the properties later also, but today we stop here.