Chaotic Dynamical Systems
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Lecture - 14
Recurrence Cont'd

Welcome to students. Today again we are going to continue with the same things. So, let us have a quick recall of what we have already done. So, for us x is always a metric space.

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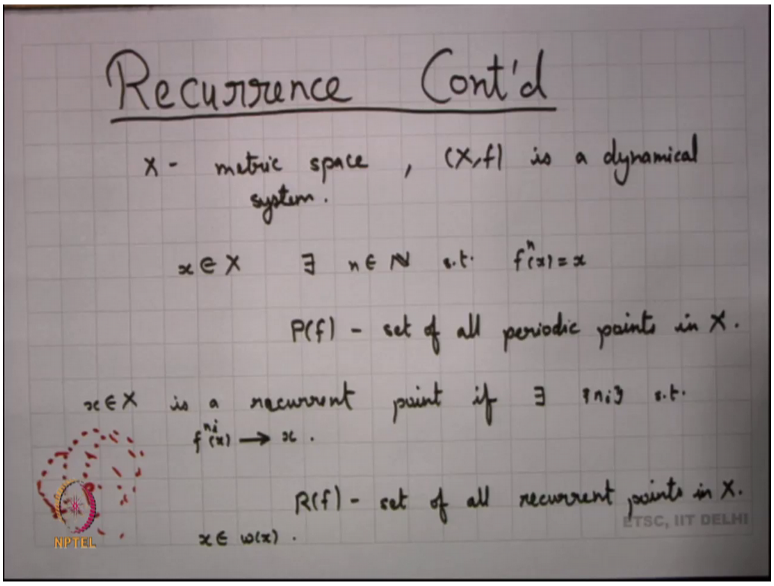

We will see whether x is compact or not it depends and x f is a dynamical system.

So, with these preliminaries we have already seen the concept of a periodic point. So, we say that if there is a point x in x, for which there exist an n in N such that fnx is equal to x, then we say that x is a periodic point and n is if the least such n which satisfies this is called the period of x. And we denote by P of f the set of all periodic points in x, but naturally f denotes that we are considering the periodic points of f.

We had seen a bit generalization of this concept of periodic points, and that was our recurrent point. So, do what do you mean by a recurrent point. So, we said that x in X is a recurrent point, if now we have seen many equivalent conditions of recurrent point and one of them which I want to state here is that you have a sequence there exists a sequence n i such that f to the power n i of x is converging to x right. This is one of the equivalent conditions which we had seen yesterday, and we had seen that the set of all the recurrent points happens to be which we had denoted it as Rf. So, Rf is the set of all recurrent points.

There is one more concept which we have seen yesterday, that is the omega limit set which is the limit set of a point x, and we had seen that for a recurrent point right x belongs to omega x right. So, this is what we had seen. Now again let us recall here what we see over here the difference between periodic point and recurrent point, that every periodic point is a recurrent point, in the sense that what happens in a periodic point whatever be the period suppose n is the period of a periodic point, then after n iterations the point comes back to itself right.

So, again we have this constant sequence n i becomes the period, basically it becomes a multiple of the period right there is the constant and then this becomes a constant sequence, which converges to the periodic point. So, every periodic point is a recurrent point, but every recurrent point need not be a periodic point and we had seen this example also. So, we look into this concept and you see that this recurrence gives us some kind of a generalization of periodicity.

Now, what happens here is that, the point does not return back to itself. For a periodic point maybe we can say this concept that supposing I have my p to be a periodic point, then after finitely in a many iterates right comes back to itself, but if my x is a recurrent point, then the point x does not return back to x itself, but it returns back arbitrarily close to x. So, maybe in the next step in return max returns that arbitrarily close to x and so on.

So, what happens is a point is returning its arbitrarily close to itself. Now we want to approximate this point, that we do not want to look into the point itself, but we are saying that fine there is approximation of returning back to itself right. So, instead of x, if you try to say that no we are not interested just in the point x returning back to itself, may be a neighborhood of x returns back to itself. So, a slight approximation where we are basically kind of blurring out the existence of x or basically blurring out the position of x right.

So, we are blurring out the position that gives us the concept of what? If you think of a point and we think of its neighborhood coming returning back to itself. So, that gives us another notion of recurrence and we call such a point.

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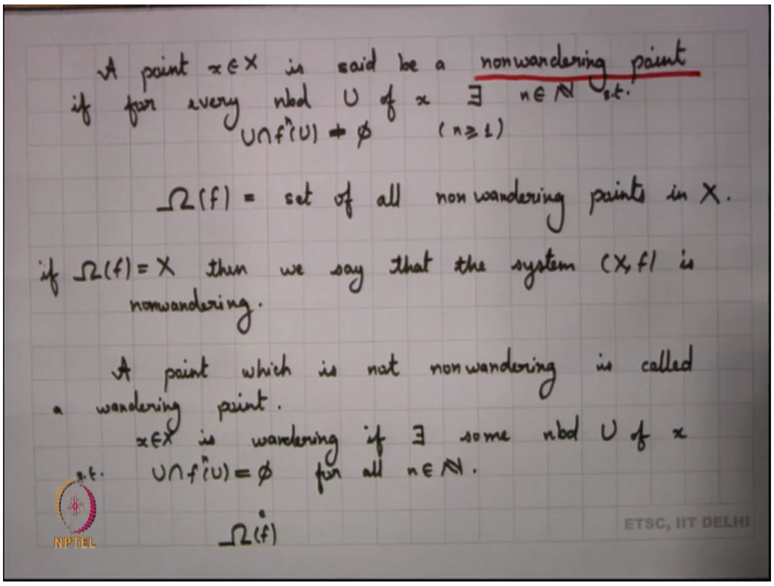

So, we say that a point x in x is said to be a non-wandering point, if for every neighborhood now when I talk of neighborhood, we are only talking of open neighborhoods right. If for every neighborhood U of x, there exists say a natural number N such that U intersection fnU is non-empty. Now when I am taking n definitely what I mean here is that my n happens to be greater than equal to 1 right.

So, what we want is that, for take a point x take any neighborhood of U right we are not sure what happens to x, we are not sure what happens to the iterates of x whether they return back to themselves or not. But what happens is that if you take the neighborhood U, there will be some point in U which returns back to itself after some time and if this is true for every neighborhood U of x, we say that x happens to be we call x to be a non-wandering point. We denote the set of all non wandering points as omega f . So, now, we are looking into the capital omega. So, this is basically the set of all non wandering points, if my omega f turns out to be equal whole of; that means, every point is non wandering then we say that the system is non wandering. So, if omega x equal to x then we say that the system xf is non-wandering.

Now, we want to typically recall what all can be wandering points and what all can be non wandering points. So, the concept of non wandering also gives us what happens to its complement means, not all points can be wandering. So, what are the points which are non wandering right. So, we said that a point a point which is not non wandering is called a wandering point.

Now, what is the characteristic of a wandering point? So, we say that x in X is wandering, if there exists some neighborhood U of x such that U intersection f nu is empty right for all n in N; that means, there exists a neighborhood U of x such that this neighborhood never returns back to itself. It just keeps on roaming wherever it wants to roam it just keeps on roaming, but never comes back to itself.

So, such points are called wandering points and we are not going to denote the wandering points by specific name. So, since we know that a point can be either wandering or non-wandering right, we can say that then set of non wandering points will be omega f complement right. So, the complement of omega f is the set of all wandering points.

Now, we will come back to say some kind of examples here, since we talked about non wandering points and wandering points and I want to recall the examples that we did previously. So, I want to recall all those examples here again, and we start with this first example which we have already done earlier.

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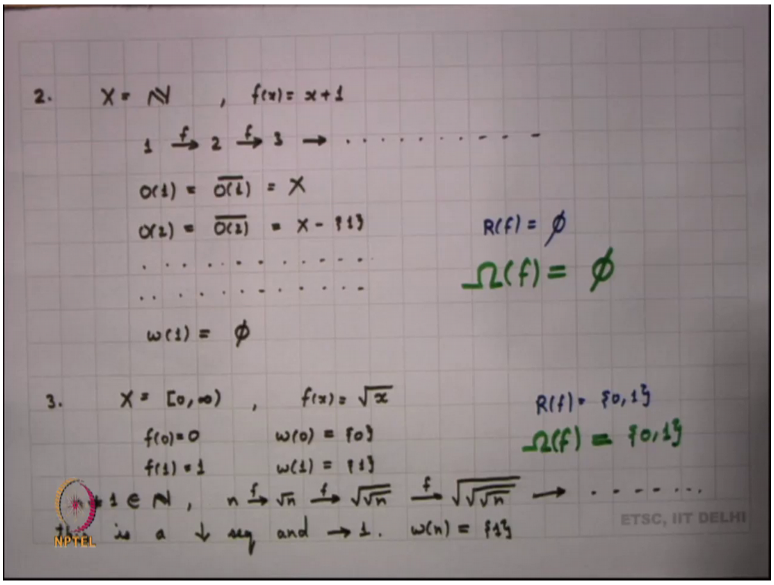

So, our space x is the set of all 1 by n n in n union 0 and our mapping f was x by 2, and we have already studied this example and we had looked in to the set of recurrent points here, which happens to be just singleton 0.

What is the non wandering point here what is this equal to again 0 right because no other thing because all see if you look in to all other points of x they are all its there is a discrete topology right on each of on the rest of the x accepting 0 what happens there it only keeps on wandering right. So, basically if we look into the rest of x all the points accepting 0, they are all wandering points only 0 is the non-wandering point here.

Let us look into the next example. So, the second examples that we had seen was the set of all natural numbers right and then we were looking into the map x plus 1, the translation here. So, what happened here we had seen that the set of all recurrent points was the empty set, what can you say about the non-wandering points here this will also be empty.

The third example we had seen was the positive half real line and the system that we had consider the map on it. We had considered was root of x we had already seen that the recurrent points here are just two points right 0 and 1, what happens to the non wandering points here? Just want you to look into it, 0 is non-wandering anything else one is also non wandering what happens to the rest of the points right? They are all wandering points right. So, the rest of the points are all wandering, you have only two points here which are non wandering 0 and 1.

Now, typically we go back to another example right and here we have the example of the rotations on the circle.

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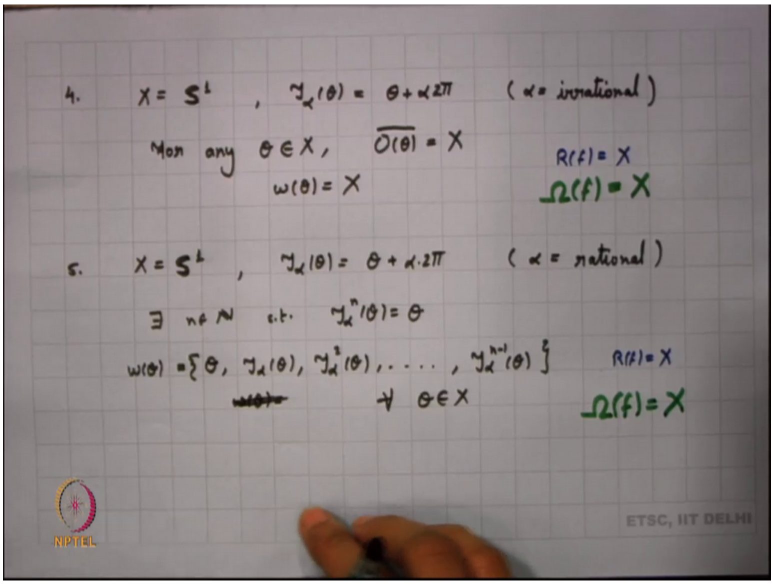

And we first looked into the irrational rotation, and we found that in an irrational rotation we do not have any periodic point right, but every point is recurrent. This is a minimal system. So, every point is recurrent here, what can you say about the non wandering points here? That will also be the whole of x right.

And what happens now for a rational rotation? Now the difference here is in the rational rotation we have every point to be a periodic point of the same period n, its a periodic point of the same period n and a periodic point is a recurrent point. So, we had that the set of recurrent point is whole of x, what happens to the non wandering points here will be whole of x. So, with this examples right we can now state a preposition here we will try to prove that also. So, let us try to state a proposition here.

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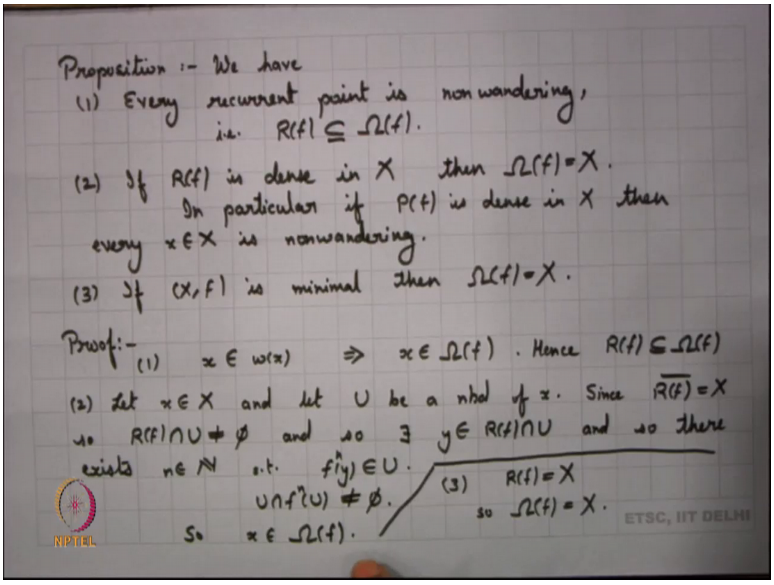

Now I am already considering that whatever we consider is for our system xf. So, I may not repeat xf again and again right. So, we have the first thing we have is, every recurrent point is non-wandering, in other words I can say that Rf is basically a subset of omega f. The second part we can say here is that if my Rf is dense in x, then omega f is whole of x the non wandering point is then the whole of x.

And in particular if P of f the set of periodic points is dense in x, then every point is non-wandering right. So, we can say that the system is non wandering if it has a dense set of periodic points right. The third thing is if my system is minimal, then again non wandering points is the whole of x right and we try to look into the proof of this although the proof here is really very very simple.

Now, think of that I am looking into a recurrent point right . So, if i try to took look into the proof of one right we are looking into a recurrent point. So, recurrent point means my x is a recurrent point if I can say that x belongs to small omega x right. So, it belongs to its limit set now since x belongs to its limit set right what does it mean? There is a sequence converging to that right. So, we have already discussed this this implies that x belongs to capital omega f right difference between small and capital, and hence we can say that Rf will be always a subset of omega f, taken right it could be equal also.

Let us look into the proof of second one. Now we know that Rf is dense in x we are not sure of anything else, but we know that Rf is dense in x. So, let us assume that then we want to show that omega f is the whole of x. So, let us start with any point x in x and we take and neighborhood. So, we are starting with a neighborhood U of x take any point x in x and start with the neighborhood U of x, then we know that Rf is dense in x right since Rf closure is whole of x right. So, Rf intersects is non-empty right and in fact, I can take some point y in Rf intersection U so; that means, there is a point y in U which is recurrent and what does that mean? Y is in U right. So, there will be another point there will be any iterate of y will also which is in U right and so, there exists fny will also belong to U and hence I can say that U intersection fn of U is nonempty. So, my x belongs to omega f. So, take any point in x it belongs to omega f. So, I can say that my omega f will be whole of x.

Now, comes the proof of the third part right. So, maybe let me write it somewhere here. So, what happens to the part third part? We know that what is minimal we know that if the set is minimal if the system is minimal, then every point is a recurrent point right. So, since the system is minimal my Rf is whole of x, already the first part we have seen that Rf is a subset of omega f right and. So, I can say that my non wandering points is the whole of x.

Let us now see we had seen some examples to very trivial examples, let us try to look into some more examples now here.

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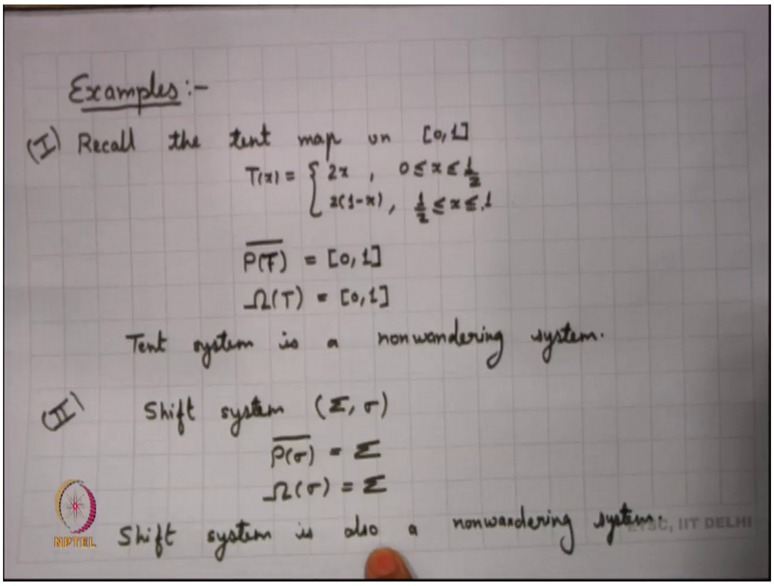

So, let us look into some more examples here, specifically want to recall. So, I am writing it the first example here, we recall the tent map. I think we have already studied the tent map right and 0 1. So, the tent map is given as Tx equal to 2 x right for 0 less than x less than or equal to half, and it is twice 1 minus x when you have half less than or equal to x less than or equal to 1 right.

So, recall the tent map on 0 1 and we recall some of the properties of tent map that we had studied. And one of the properties that we had studied was that in the tent map the set of periodic points is tents right. So, we had studied here that and for the tent map right P of T right closure was the whole of 0 1.

Now, since the set of periodic points is dense right periodic points comprise of like these can be considered recurrent points. So, recurrent points are dense right since recurrent points are dense, we know that in this particular case set of non wandering points will be the whole of 0 1 right. So, again our tent map seems to be its a non wandering system right.

So, you can conclude that the tent map the tent system I should say, is a non-wandering system. Now I want to recall again one more example which we had discussed earlier and that was the shift system right. So, the shift system which we have denoted by capital sigma small sigma right which is the set of all sequences of zeros and ones we had considered and this was just the right shift on the set of sequences and we have seen that the shifts a sequence was basically topologically conjugate to the horseshoe attractor. So, write the dynamic some horseshoe right.

So, this was the same stuff now I am looking into the same system here again, and again over here we had seen that the set of periodic points of sigma right was dense in sigma right. So, the set of periodic points was dense and. So, in this case also we can say that the non wandering points are the whole space here and so, we can say that this shift system is also a non-wandering system.

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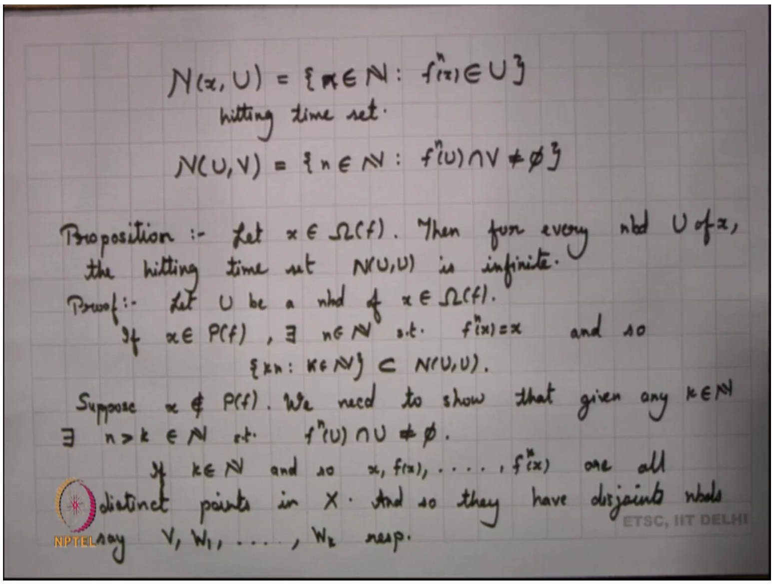

Now, there is one more thing which we had seen yesterday was this notation NxU, this is the hitting time set and this was the set of all x sorry this was the set of all n in N such that fnx belongs to U. Now the set of all n in n for f which fnx belongs to U now I can think of generalizing. So, this is the hitting time set, now we can slightly generalized this to accommodate something more. So, what do we mean by n I am looking into U, and I am looking into v what is that equal to?

So, now I can say that, yes I can say that there is a point x nu right which hits v right. So, and since there can be many x nu which hit v at different times right, what we can say that this is the set of all n in n such that fnu intersection v is non empty right. So, basically we are looking into all the instances n for which, there is some point in U which hits v right. So, we are look into this stuff and we can just look into this definition again.

So, this is a notation Nuv and again we call this is the hitting time set right and here we have a preposition here. So, let U be a neighborhood of. So, let x belong to omega f right and then we can take then for every neighborhood U of x the hitting time set Nuu right is infinite. This is interesting you can look into the proof of this, which means that if I am starting with U then there is a pointing U which intersects U again right which hits U again right and that happens infinitely often.

So, the proof here is actually very simple. So, we start with a neighborhood. So, let U be a neighborhood of x, but naturally my x is in omega f. Now there are two faces here it could be possible that this x in omega f could be a periodic point right since a set of periodic points is containing omega f right.

So, it could be possible that x is a periodic point. So, if my x belongs to p of f then i know that there exist an n in n such that fnx is equal to x right and. So, what happens here is that if i take the set of all k times n n in n sorry k in n right if i take the set of all k times n then that will always be contained in Nuu . So, my Nuu is infinite right

So, in that case if my Nuu happens to be infinite. So, I am particularly supposing now that x does not belong to P of f. So, suppose x does not belong to P of f. Now what happens in this case? So, all we want to show is that, we want to show that Nuu is infinite. So, what we want to show is that, we need to show given any k in n right there exist an n greater than k in N such that fn of U intersection U is non-empty right.

We can show that any k I can get a larger one for which this is non empty right we are done. So, that is what we propose to show here now. So, let us assume that k. So, let us assume that we are already fix our k. So, supposing we have our we fix our k. So, if k belongs to n we have fixing 1 k.

Now, we know that this point x is not a periodic point right and. So, these are all distinct points right these are all distinct points right. So, since my x is not a periodic point we are all distinct points.

Now, since they are distinct points in x, starting with a metric space then we know that they will all have disjoint neighborhoods and I can think of disjoint neighborhood everything h neighborhood disjoined from the other, you can always assume that part. So, we said that they have they have disjoint neighborhoods say I am saying that I am talking of disjoint neighborhoods respectively. So, say I have a neighborhood of x which I call as v right then neighborhood of fx which I call is w one neighborhood of fkx which I call as w k respectively. So, all these neighborhoods are disjoint from each other right.

Now, what happens after that? So, I am.

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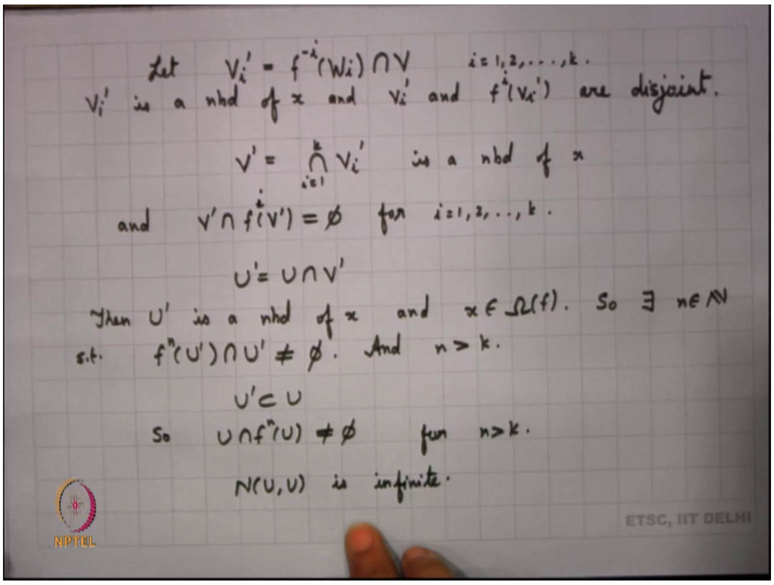

Now, taking defining V i prime to be equal to f minus i of Wi intersection V. Now I know that my point x belongs to V minus i Wi right and definitely it belongs to V also. So, this V i prime happens to be a neighborhood of x. So, I am saying that V i prime happens to be f minus i Wi intersection v for all i 123.

Let us quite possible. So, I have V i prime is a neighborhood of, but it is quite possible that each V i prime right need not be disjoint from Wi its quite possible here stills we have taking intersection with v it will be disjoint from W i. So, what we are starting is V i prime is a neighborhood of x right, and if I am looking into V i prime and I am looking into f to the power i of V i prime right these two sets are disjoint by construction these two sets are disjoint. So, these are disjoint.

Now since these two sets are disjoint right I can think of another definition here i will take v prime to be the intersection of all V i prime right I going from 1 to k, then what we observe here is that v prime is a neighborhood of and furthermore. If I take v prime it is disjoint from V i prime V i of V prime right for all I going from 1 to k. So, it is not just that these two are disjoint right now we have that this v prime is designed from each one of them right. So, we have the set V prime which is a neighborhood of x and this is disjoint from each one of them.

And now we define U prime. So, we started with if you recall what we started with was that we wanted to show that we started with the neighborhood U right of x and we wanted to show the existence of an n greater than each and every k right such that fn of U intersection U is non-empty.

So, now what we have landed up is just v prime. So, what we take is we take our U prime to be the same U intersection V prime. So, we take U prime to the U intersection V prime. Now when U is U intersection V prime we define this part, then my U, U prime is a neighborhood of x and my x belongs to omega f so; that means, that should exist since x is an non wandering point, and this is a neighborhood of x that should exist an n such that f n of U prime intersection U prime is non empty right.

So, there exists its non empty, but what is this n? This n cannot be one this n cannot be 2 this n cannot be 3 right this n cannot be k, because of the construction of U prime and. So, my n has to be greater than k now what was my U prime? My U prime was anyway a subset of U, my U prime was a subset of U because this U intersection U v prime. So, U prime is a subset of U and so, what we have is and this gives us that our Nuu is infinite.

Now, there is another aspect that we should know about our non wandering points and that is we take up the next proposition.

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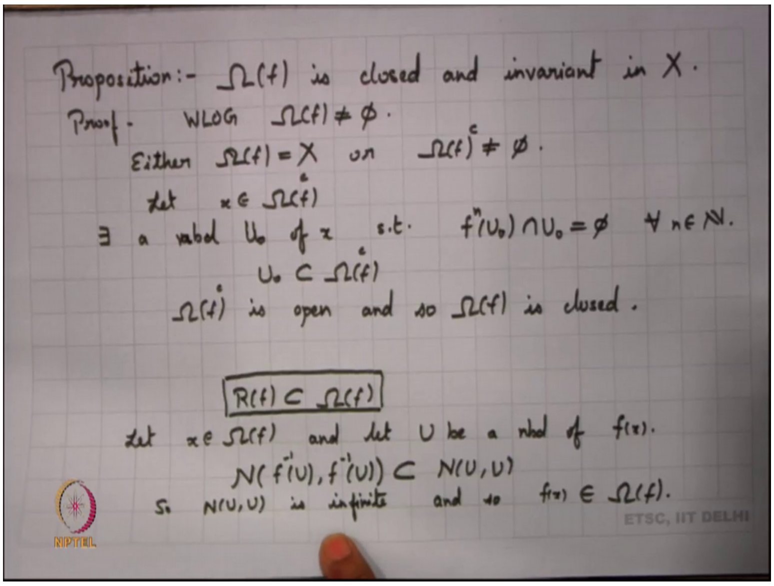

Now the prep here it is very simple. Now without loss of generality we can say that my omega f is not an empty set. Because if omega set omega f is empty set anyway its recklessly true right. So, just assume that omega f is non empty. So, since omega f is non empty we know that this set could be the whole of X right. So, we have either again if it is x it is closed and invariant right or we have let the set of wandering points is non empty right. So, this is non-empty.

Now, in the first case right again it is recklessly true it is closed an invariances. So, we look into the second case. So, let x belongs to the complement of omega f; that means, x is a wandering point. Now since x is a wandering point we know my definition there exists a neighborhood U naught such that fn of U naught intersection U naught is empty right for every n in n.

Now, think of that we have assumed that our U naught is an open set right and. So, U naught is not only a neighborhood of x, right its a neighborhood of all the points contained in U naught. And for all these points what do we have; we have a neighborhood for all those points such that fn of U naught intersection U naught is empty right for every n in n and what does that mean? My U naught itself should be a subset of right. So, U naught itself is a subset of the complement of set of non wandering points or basically its it consists of all wandering points, U naught itself consists of wandering points and; that means, that omega f complement is open right typical way of showing that as it is closed.

So, we know that the set of wandering points happens to be open and so, the set of non wandering points is a closed. Now we want to show that it is invariant; now we have already seen in one of the prepositions here that the set of recurrent points right is a subset of the set of non wandering points, and we had seen in one of the propositions previously one of the previous lectures that the set of recurrent points happens to be invariant.

Now, let us look into the other part, the set of recurrent points happens to be invariant right. So, what can we say about this part or let us look into this from a different angle, maybe i still need to define something more about the closure of recurrent point. So, let me try to look into another part here, let my x be a non wandering point and let my U be a neighborhood of fx. Now U is a neighborhood of fx you simply note that the hitting set f inverse U will be a subset of Nuu its very small observation, can say see that right. So, f inverse U f inverse n f inverse U fn inverse U is a subset of Nuu.

Now, what do we know about f inverse U? F inverse U is a again a open set and its a neighborhood of x right. So, you are n in f inverse U f inverse U should be infinite right and hence Nuu is infinite right. So, Nuu is infinite, and basically we just wanted one n right. So, we can say that f of x belongs to omega f, and hence omega f is invariant right.

So, we know that omega f is closed and invariant right. So, my f of x belongs to omega f. now here we just looked into this concept. So, let me try to define a little bit more on this particular concept, my set of recurrent points is contained in the set of non wandering points right.

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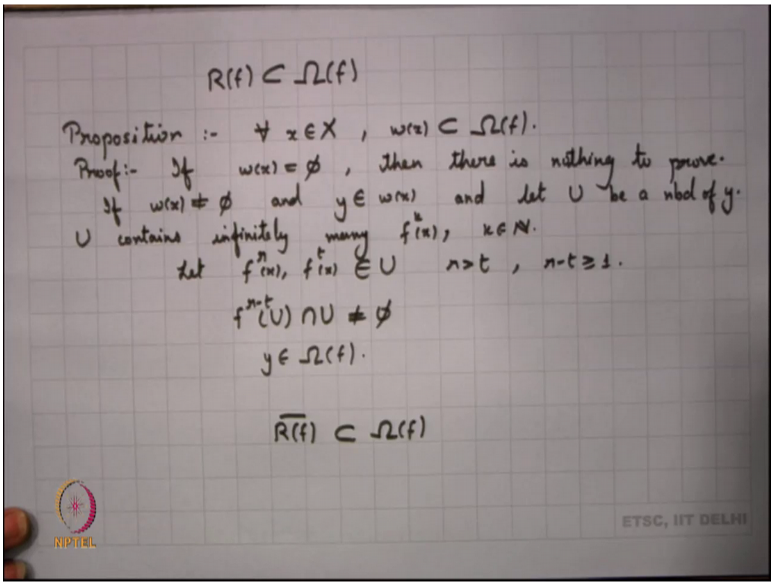

Just let us look into this concept, the set of recurrent points is contained in the set of non wandering points let us look into this part once again.

Now, for this let getting a little bit more into details into this let us look into this particular proposition which explains something more you take any x in x right. Then the limit set of x is contained in non wandering. So, any point in the limit set happens to be non wandering right and that clearly gives you why we have the capital and the small omegas over here.

So, the proof here is not very difficult. So, we start with limit set can be empty also right then there is nothing to prove. So, we start with limit set to be non empty. So, since the limit set is non empty we can choose a point y in the limit set. So, let y belongs to the limit set and let U be a neighborhood of y

Now, since U is a neighborhood of y we know that y is a limit point right so; that means, that U contains right infinitely many points of the orbit of x right. So, U contains infinitely many fkx for k belongs to N So, in particular I am choosing two such case right. So, let f r of x in ft of x belong to U and I can always choose my r to be greater than t r is greater than t which would mean that my r minus t will be greater than or equal to 1.

Now, what happens in that case now frx belongs to U, ftx belongs to U right I can simply say that f of r minus t of U intersection U is non-empty. So, take any point in the limit set right every neigh for every neighborhood you find some natural number such that fn of U intersection U is non-empty right and. So, my y belongs to omega f right. So, the limit points all limit points are basically non wandering and this gives us a little bit more details about what we can say about recurrence.

Now, we have seen that our recurrent set right Rf is an invariant set, but we did not know that it is a closed set. So, the set of recurrent sets need not be closed. So, what happens in that sense now. We know that Rf is a subset of omega f right now omega f is closed. So, all i can say is that Rf closure should be a subset of omega f now Rf is not necessarily closed so; that means, Rf closure is a set which is bigger than R f. So, for that we define say another definition right and I think I will push it up in the next page.

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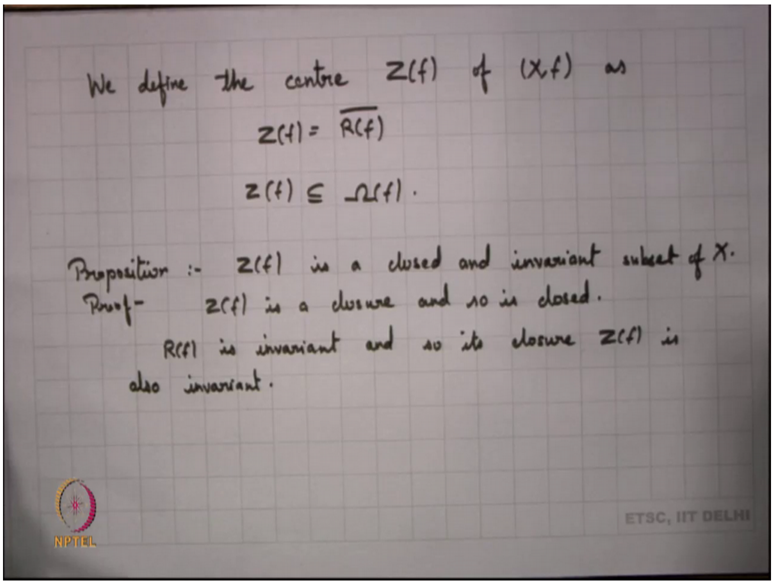

So, we define the center of a dynamical system which I am denoting a Zf the center Zf of a dynamical system xf. So, this is not just this is basically the closure of all recurrent points, and we know that this center will always be subset of non wandering. So, all points in the center have to be non wandering points. And we end up today with a small observation that the center is a closed set subset of x.

So, the proof again here is quite simple, the center its a closed set because its the closure of the recurrent points right. So, this is a closure and. So, it is closed is it invariant all we need to check out is that invariant. So, all we know is that Rf is invariant right, and closure of invariant sets as a invariant. Now when we try to look into the properties of chaos it is very very important to look into what is the center of the system.

So, once you identify the center of the system there are many properties, which can easily follow from that particular part. And that is what we shall be looking into the subsequent lectures, but today we stop over here.