Chaotic Dynamical Systems
Prof. Anima Nagar
Department of Mathematics
Indian Institute of Technology, Delhi

Lecture – 12
Dynamics of the Horseshoe Attractor

Welcome to students. So, last time we had looked into the construction of horseshoe attractor and today what we are going to see is; what is the dynamics of the horseshoe map on the horseshoe attractor. So, last time we had seen this construction the horseshoe attractor and we have the horseshoe map f acting on it.

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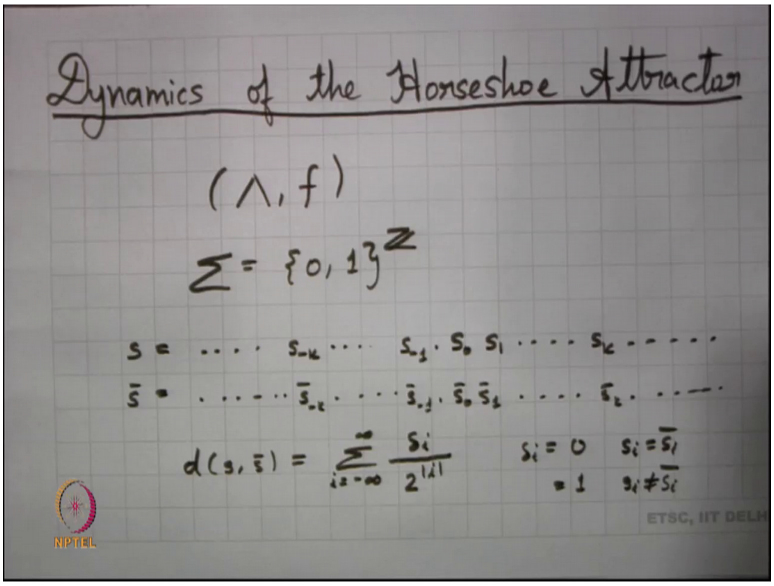

Now this resulted into coming up with another the space of all sequences of 1s and 1s; and we will now look into that space once again.

So, we have this space sigma and if you look into this, this is anyway this is a topological space given the product topology, but we can also think of giving a metric to this particular space. So, let us look into 2 sequences here. So, my sequence s happens to be say s minus k, s minus 1, s 0, s 1, S k and so on and s bar happens to be the sequence s bar minus k, s bar minus 1 s bar 0 s bar 1, s bar k and so on.

So, we have 2 such sequences and we can define a function d on sigma cross sigma. So, we defined this particular metric I should say, s s bar which happens to be summation over i from minus infinity to infinity, I look into delta i upon 2 to the power mod i where my delta i is 0 if my si is same as si bar, and it is equal to 1 if my si is not equal to si bar.

So, now if we look into this d, this d clearly defines a metric on this particular space sigma, and the topology that d gives is same as the product topology, which we are considering on the space. So, we can say that this particular sigma is a metric space. Again this is a compact metric space since we know that 0 1 is compact. So, our sigma happens to be a compact metric space under this particular metric d, and now we shall try to look into the shift map that we had seen last time.

So, what is this shift map?

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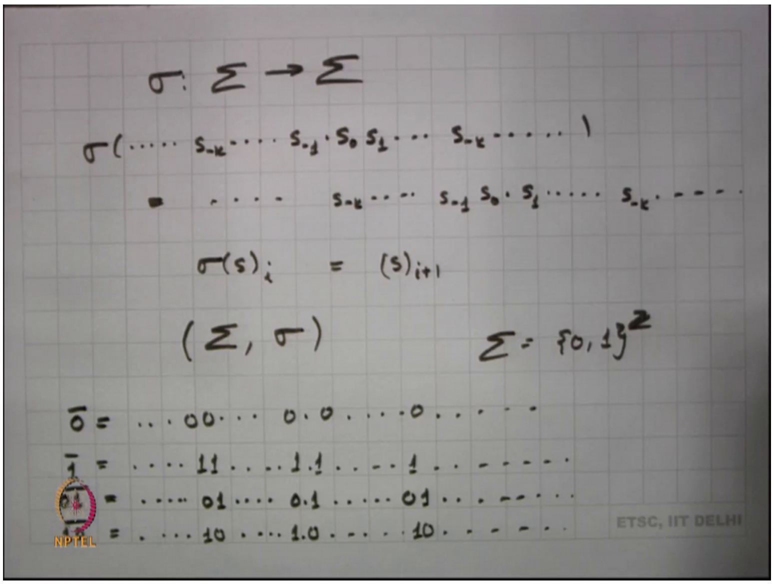

So, we can think of sigma from this capital sigma to sigma, the shift map defined as is given as. So, shifting all the symbols one places to the right or I can say that sigma of s right at I will be actually equal to what is s at i plus 1. So, this is the shift map defined on sigma and for this particular shift map now, it is very easy to see that this will be a continuous mapping.

So, sigma is a continuous mapping on your space of sequences, and we should now look into the dynamics of this system sigma sigma. What does the dynamics that you can observe over here? So, if I try to analyze this part right, now since we know that very well that our sigma is just by infinite sequences comprising of 0 and 1, we can think of many sequences arising from here and one of the sequence that you can observe over here is this sequence constant 0, and we call it we just denote it as 0 bar. So, 0 bar is the sequence of all constant zeros, similarly you will have 1 bar which is the sequences of all constant ones. So, you have all constant ones.

Now, I can think of a sequence where I am alternating my 0s and 1s. So, I have a 0 and 1 I have a 0 and 1 I have a 0 and 1, I am just alternating between the sequence 0 and 1 and I can denote this as 0 1 bar, and similarly I can have a sequence where I am alternating 1 and 0, and I can call it a 1 0 bar right these are alternating sequences, and now what we find over here is that.

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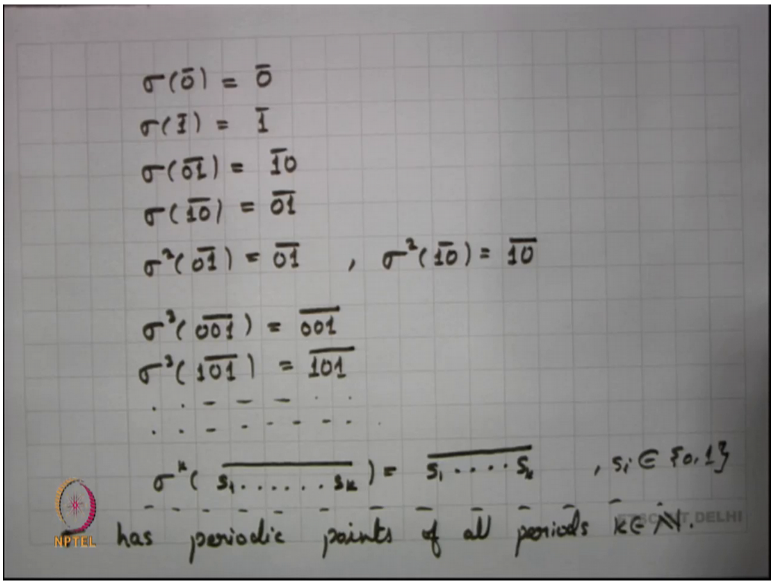

My 0 bar remains fixed under sigma, my 1 bar remains fixed under sigma, if I take the image of 0 1 bar, what I get here is 1 0 bar and if I take the image of 1 0 bar under sigma, I get a 0 1 bar.

So, the essential part here is that if I am looking into the sequence 0 1 bar or 1 0 bar, we find that under sigma square right 0 1 bar remains as it is, and 1 0 bar also remains as it is. We do have periodic points sigma and thus have fixed point at 0 and 1, 0 1 0 bar and 1 bar are fixed points, and then these are period 2 points.

Similarly, you can look into this part that, if I am taking sigma cube of since 0 0 1 bar right, which is again like the sequence 0 0 1 repeating itself, we find that this is going to be equal to 0 0 1 bar, similarly if you look into sigma cube of 1 0 1 bar right that also remains as 1 0 1 bar. So, you do have periodic points of period 3 and we know that there are total 8 possibilities right of such sequences such triplets and. So, you can say that there will be something like 8 periodic points right or 8 fixed point or 8 points, sigma cube equal to x 8 fixed points of sigma cube x equal to x right.

Now, similarly you can go forward and for any sequence of length k, which is repeating itself. So, supposing I have a sequence of length k repeating itself then I find that sigma k of that particular sequence of length k repeating itself right. So, maybe I am looking into in terms of s 1 up to S k right I find that this is nothing, but this is my s 1 S k bar, where my si is belong to 0 1 . So, we find that this constitutes right this will give me periodic points of period k right and hence there are periodic points of period k for all k in n right.

So, what we can conclude from here is, if we pass on like this that sigma has periodic points of all periods K in N. Now the interesting part comes up here is, then fine we do have periodic points, but can we have non periodic points. So, let us again go back to say our interval 0 1.

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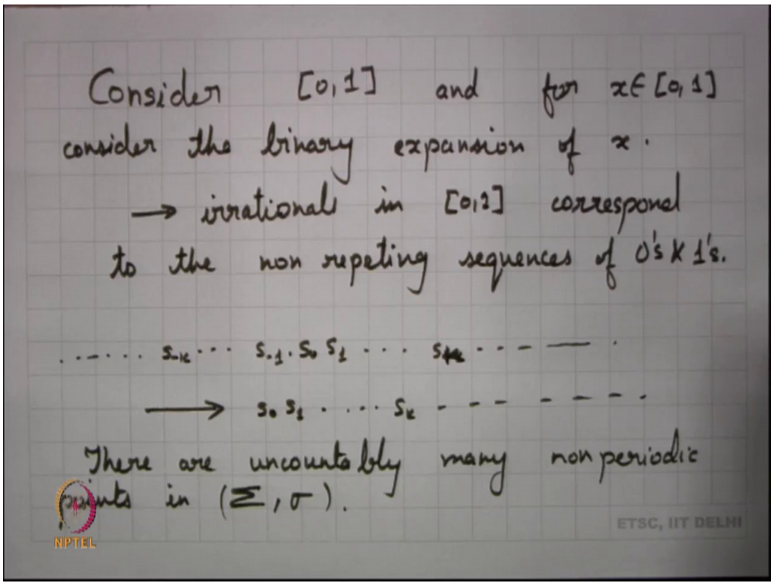

So, consider this interval 0 1 and for x in 0 1, consider the binary expansion. Now if I look into the binary expansion of x any x in 0 1, then again what we get is a sequence of zeros and ones. And we all know that the irrationals right the irrationals in 0 1, they correspond to the non repeating sequences of zeros and ones right normally correspond to the irrationals in 0 1 and we know that there are uncountably many irrationals.

Now, let us go back to our sequence space right in our sequence space, we have many sequences of this particular form and I can always associate just forget what was the past part right and I can associate this sequence to just looking into its present and its future, so; that means, I am just looking into the 1 half of the sequence right S k and so on and every sequence can be associated to such a sequence.

Now, again look into the irrationals in 0 1 right, the irrationals in 0 1 are giving us non repeating sequence of zeros and ones, nothing can come back to itself and if I try to apply shift map over here right. So, if I look into the non repeating sequences of zeros and ones, and if I say that my s naught s 1 S k etcetera, this was a sequence which is non repeating sequence of zeros and ones right, then basically that corresponds to a sequence that corresponds to some kind of sequence in my space sigma, which is non repeating or which cannot be periodic.

So, the non periodic points in our space sigma correspond to basically I can say it can be corresponded to the non repeating sequences of zeros and ones, and hence what we can conclude is that there are uncountably many non periodic points in sigma or I should say the system sigma. So, this system is very special because it has periodic points of all periods as well as it has uncountably many non periodic points, what other kinds of points can we look into here?

So, let us try to again take up all the sequences, and now we try to give some kind of order to the sequences. So, I am trying to look into say all sequences here. So, sequences of length k.

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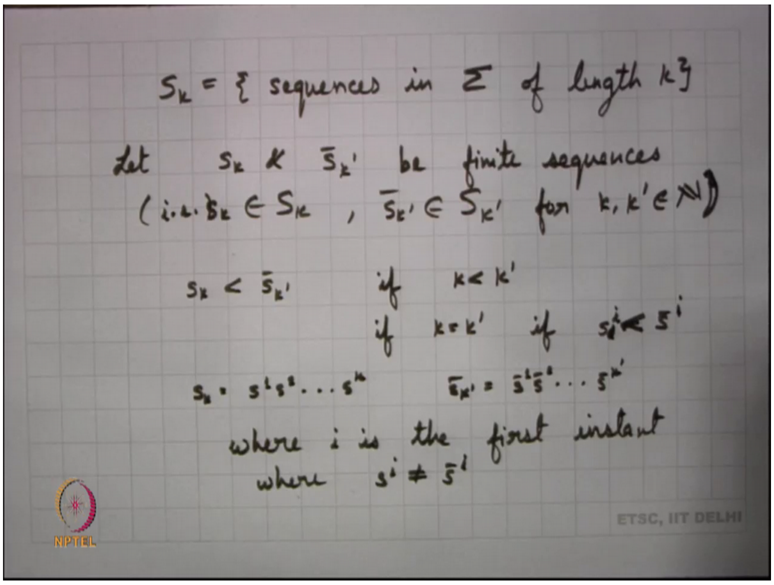

So, I want to say that S k right is basically the set of sequences of length k, and now we want to give some kind of a order to all finite sequences. So, because these are finite sequences now, we want to give some kind of order to the finite sequences. So, we say that let say I want to say S k and s bar k prime be finite sequences, what I want to mean here is that S k belongs to S k for some k and S k prime k prime bar belongs to s of k prime bar right for some kk prime.

Now, what we are interested in looking into is some kind of an order. So, we define an order on these sequences. So, we say that S k is less than S k prime s bar k prime is less than S k bar prime, if first thing is k is less than k prime. But if k is equal to k prime right then what happens and if k is equal to k prime, then we know that this will consist of n sequences, this will consist of say my S k happens to be of the form right s 1, s 2 right S k and my s bar k prime happens to be of the form say s bar 1 s bar 2 right s bar k prime.

And now since k and k prime are same; that means, these 2 are the same right then we say that S k is less than S k prime if my I am looking into si of sorry si right is less than s bar i right, where i is the first instant where si is not equal to s bar i. We try to now compare these sequences say s 1 is same as s bar 1, s 2 is same as s bar 2, we try to compare this part and then we see what happens to the i th instant, supposing this first place where they are not equal right then if my si is less than this then this is less than this factor.

So, we have an order on all finite sequences of zeros and ones, and now we will try to use this particular order to formulate something. So, now, what I here I have here is say I am looking into s 1 right.

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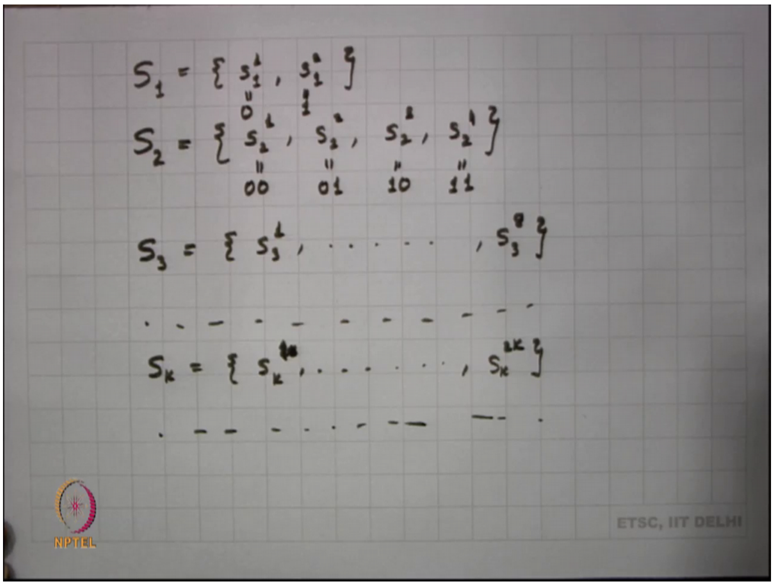

Then we know that sequences of length 1 right there are only 2 sequences of length, one is 0 and 1 is 1 right. So, I can say that this is s 1 and I am writing 1 here to denote that this is length 1 right and s 2 right this is length 1 right, I just have 2 sequences here. If I am looking into s 2 I know there are 4 sequences here of length 2.

So, I am saying that this is S 2, I have first second third fourth right and its needless to say this will be my 0 right, this will be my 1, this will turn out to be 0 0 this will turn out to be 0 1, this will be turn out to be 1 0 and this is my 1 1. Similarly I can think of my S 3 right to be equal to s 3 1, now I know that I will have 8 such sequences. So, s 3 8 for length k I know that there will be 2 to the power k such sequences.

So, this will be of the form we can write this as S k 2 k sorry S k 1, S k 2 to the power k right we have all these distinct sequences now I want to define another sequence s using these sequences.

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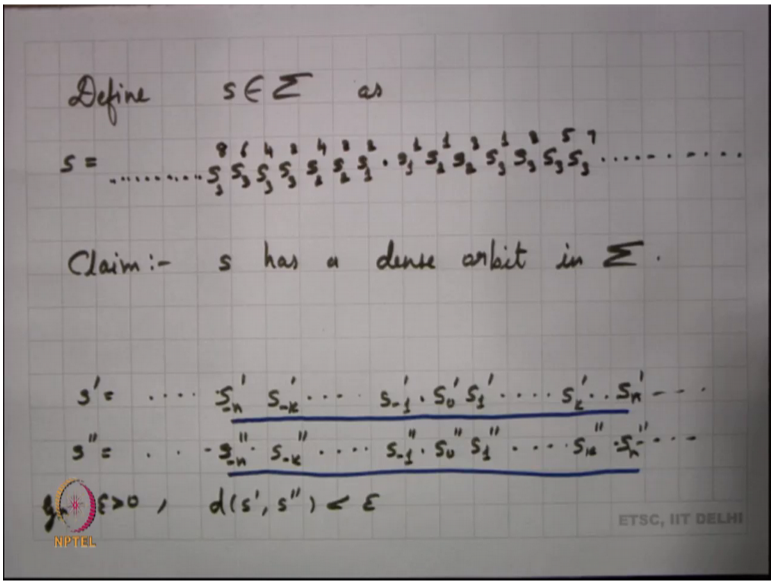

Define s belonging to sigma as now how do I define my s. So, what I do is, I start with a midpoint here the decimal point here. So, I having my s 1 1 here s 1 2 here then s 2 1 here s 2 2 here s 2 3 here s 2 4 here, then I should have my next because I have exhausted all my sequences of length 2 I am now going on to sequences of length 3.

So, we have s 3 1 here and I have s 3 2 here I have s 3 3 here, I have s 3 4 here right similarly s 3 5 will be here and s 3 6 will be here, s 3 7 is here, s 3 8 is here now we have exhausted all sequences of length 3 right we now go up to length 4. So, like that we can keep on adding sequences on both the sides of the decimal places and the sequence that we defined in this manner is our sequence s.

Now, my claim here is s has a dense orbit in sigma. We try to look into this particular claim and. So, we start with any sequence say s prime. So, I start with the sequence s prime and my sequence s prime can be written as say s minus k prime right s minus 1 prime, s naught prime s 1 prime S k prime and so on. So, we have this sequence s prime and for this particular sequence s prime, we want to say that there is some point in the orbit of s which is close to it.

So, we try to look into now under this metric. So, what happens if some other point has to be close to this one. So, supposing now my s double prime, which I am writing it as sequence s minus k double prime s minus 1 double prime what is the meaning of saying that d of s prime and s double prime is less than epsilon. So, I can say that given epsilon some its have some positive epsilon, what is the meaning of saying the d of s prime and s double prime is less than epsilon.

So, they again go back to the definition of the metric d, which says that it is basically the summation of maybe i should say delta i upon 2 to the power mod i, and what happens in that case. So, for this summation to be arbitrary small as possible right, what is needed is that in the middle part right somewhere in the middle part, this sequences should have the same value. Because only then and then you can make them you can make the summation as small as possible.

So, if we try to look into this metric this means that under some say there should be some value here, such that this middle portion is same as this middle portion. So, maybe there is some n here. So, I can say that this is s minus n prime s minus n double prime, and I have s n prime and I have sn double prime. So, I should have some such portion middle portion which is the same in both the sequences, for this to be made arbitrarily small right may be as small as epsilon.

So, we can say that this means that.

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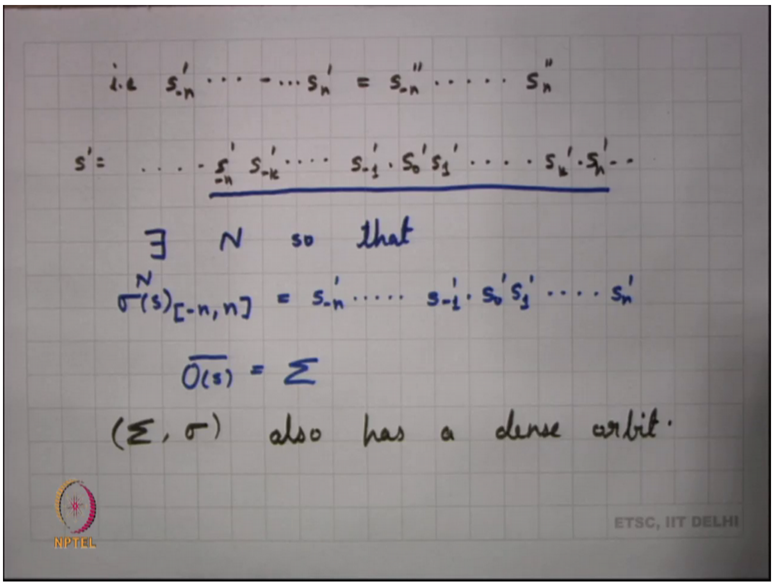

So, this is saying that if I look into this block middle block right. So, I am looking into s minus n prime right up to sn prime, this middle block is same as the block s minus n double prime and sn double prime. So, this 2 blocks are the same we want to look into.

So, now, we have started with our s prime. So, we start again with our s prime. So, our s prime was. So, this is our s prime and we wanted to show that this orbit the orbit of our sequence s comes very close or maybe comes epsilon close to s prime.

Now, so for a given epsilon positive right we find that all the sequence which are epsilon close to s prime right should agree on the middle block so; that means, that if I am looking into the middle block here. So, somewhere in the middle block right maybe let me write here is minus n prime minus n prime right so; that means, that this all the sequences which agree with s prime on this particular block are epsilon close to this block. And now when they are epsilon close to this block right we are interested in this particular block.

Now, if I look into this block this is a block of length 2 n plus 1, and if I look into my sequence s right. My sequence s is basically consisting of all blocks all blocks of all possible lengths so; that means, that this particular block of length 2 n plus 1, would be lying at some place right we do not know what that place value is, but at some place in s this block particular block is also their existing. So, we find an n. So, basically we can say that there exists an n. So, that if I am taking sigma s right and I am looking into this particular blocks sorry sigma n of s and I am looking into this blocks, the block of sigma n of s from minus n to n right.

So, this block will turn on to be nothing, but s minus n prime, we find an n maybe the n is positive or negative depend on how we have placed the sequences, but we find that there is some place here where you will find that this block appears and that is all we needed right. So, in the epsilon neighborhood of s prime, we find a point from the orbit of s which means that the orbit of s the orbit of s is dense in sigma.

So, all that we can conclude here is that sigma this particular system when I am looking into sigma sigma this also has a dense orbit. So, let us try to summarize this particular system here. So, I can summarize this in form of say a theorem.

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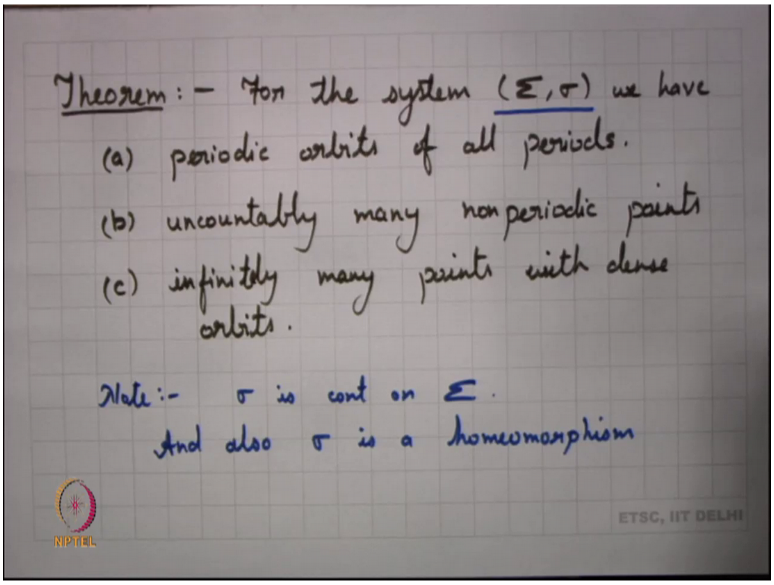

So, for the system sigma sigma, we have periodic orbits of all periods right, the next thing what we have observed here is that we have uncountably many non periodic points, and the third is we have infinitely many points with dense orbit.

Now, why do we say infinitely many points with dense orbit, because if s has a dense orbit right sigma s also has a dense orbit what about sigma inverses, we did not discuss this earlier all we have seen is that sigma is continuous is it 1 to 1 is sigma 1 to 1, because if you are shifting your shift it by 1 right. The sequence remains the same; that means, your original sequence was also the same right. So, sigma is 1 to 1 sigma is also on 2 right because anything you can look into fact right.

So, can I conclude that sigma is a homeomorphism? So, my sigma inverse is also continuous and that is what we are using for the fact is, that we our block could be on the left hand side of the decimal point also and when it is in the left hand side of the decimal point we can always traverse backwards and then we can reach we have this orbit which is into the neighborhood of s prime right. So, this sigma is also homomorphism, and hence if I am looking in to see if s has a dense orbit right, sigma s is also dense orbit and sigma inverse s also has a dense orbit. In fact, sigma square s also has a dense orbit right.

So, you can have infinitely many points which have dense orbit. So, the dynamics of this is very clear now our goal was looking into something more we will come back to the dynamics of this once again, because this is a very interesting space, but our goal here was to look into the dynamics of the horseshoe attractor. So, let us get back to the horseshoe attractor. So, we were interested in the horseshoe attractor.

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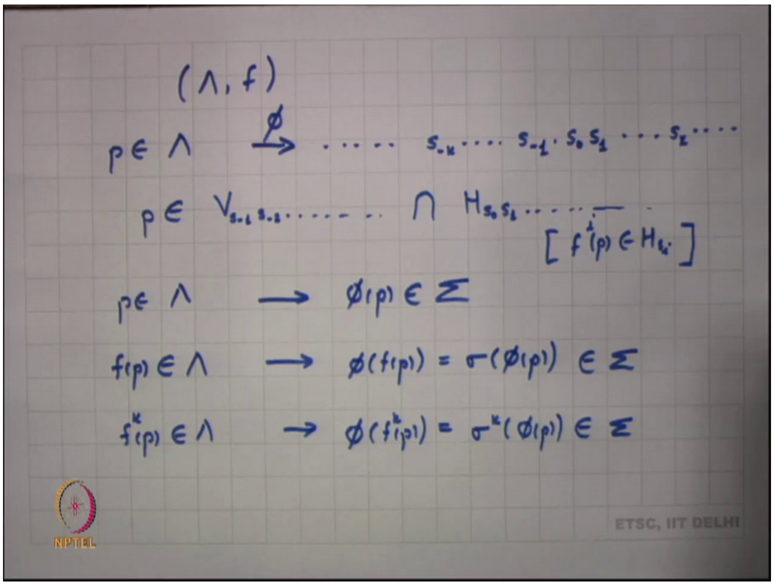

And we found that any point p in lambda was corresponding by phi to this sequence s. Let us try to look into this part this point this point p right was the unique point right with this vertical line V intersecting with this horizontal line, and this corresponded by phi to this particular sequence and this is basically a point of sigma.

So, we can say that for every p belongs to we get phi p belongs to sigma. Now we had also seen this, but it is quite easy to see what happens here is, that if my fp. Now I am looking into fp my fp because sigma is an invariant this lambda is an invariant set. So, fp also belongs to lambda and for fp belonging to lambda right what we get here is correspondingly is phi of fp, but if we try to look into what is phi of fp we have seen this last time this is nothing, but this is my sigma of phi p right because what is fp? Now since we are taking we are just trying to take 1 part to the right. So, we get fp lying in as a point of intersection of a different vertical line and different horizontal line and what does that turn out to be right the turns out to be nothing, but under the summation sigma I am just moving 1 point apart right because my p happened to be the set of all points such that if you remember that this was like fip belonging to h of si right.

So, if we try to look into that if I am taking fp it is nothing, but we are just shifting everything to the right. So, phi of fp is same as sigma phi p, and I can simply say that my fkp belongs to lambda is same as saying it will give correspond to the image which is phi of fkp and what is that equal to right belonging to sigma. Now I want to recall a definition which we have done earlier. So, I am recalling this definition. So, this is the definition of a property called topological conjugacy.

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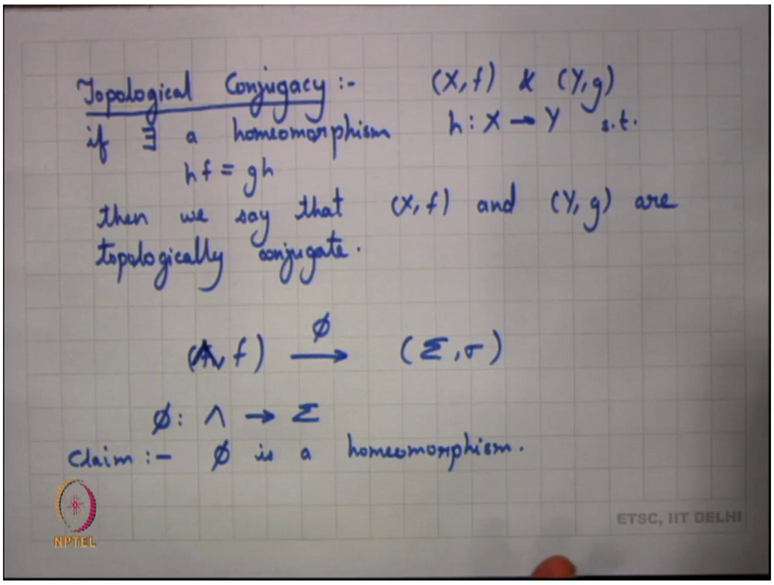

So, I am given systems xf and yg right and for this systems x and yg if there exists a homeomorphism h from X to Y such that h of f equal to g of h right, then the systems xf and yg are conjugate topologically conjugate and though we had discussed this earlier right I will again try to recall what happens here. What happens, when 2 systems are topologically conjugate orbit of 1 corresponds to the orbit of other right.

So; that means, that if my xf has a dense orbit there will be dense orbit in yg also. If my yg has a periodic point of certain period right my x will also have a periodic point of the same period so; that means, that orbits of both the space system first of all since there is a homomorphism right, each of the points here are identified with oneanother right. So, you have the same number of points here. So, every point here is an image of some point here every point here is also can be considered as an image of point here. So, the points are identified, not only where the points identified there orbits are also identified.

So; that means, the dynamics of Xf in the dynamics of Yg are the same, and that is what we are going to use over here. So, now, in our case we have this system right say lambda f and we have another system right sigma sigma. We have these 2 systems right and we have this mapping phi, which takes every point in sigma lambda to every point in sigma.

So, now let us look into this map phi once again. So, phi maps lambda 2 sigma and how does it map we have seen that part also. Now my claim here is this phi is a homomorphism. So, phi is a homomorphism; that means, my phi should be continuous my phi should be 1 1 should be on 2 that is enough because since both my spaces are compact that is enough for me to conclude that it is a homeomorphism.

So, the first of all observe that phi is 1 1 its injective.

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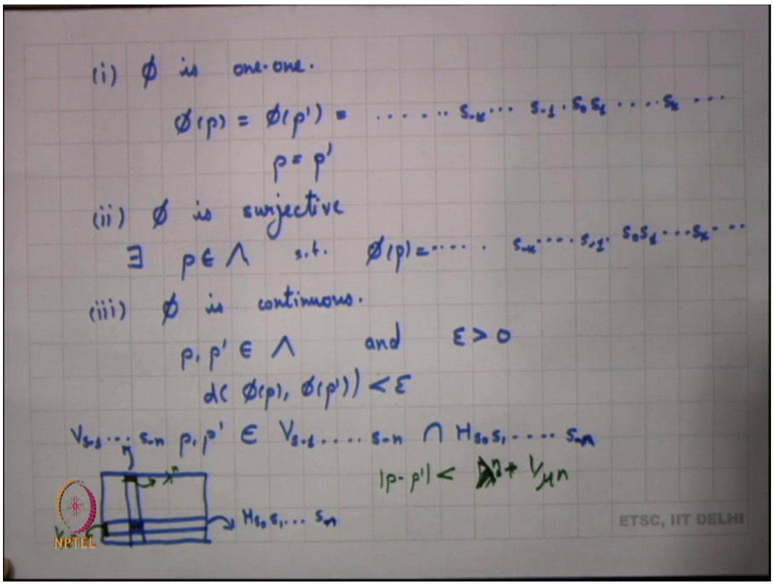

Why is it injective? Supposing now I have 2 points p and p prime such that phi p is same as phi p prime right and that corresponds to the same sequence what does that mean? This point p corresponds to the sequence, if it is the unique point of intersection between this vertical line right which is indexed by this factor and the horizontal line which is indexed by this factor this is a unique point of intersection, but if my p prime is also corresponding to this particular sequence; that means, p prime is also the unique point of intersection of the vertical line which is corresponding to this factor and the horizontal line which is corresponding to this factor and; that means, that p should be equal to p prime there is no other option right.

So, the next observation here is that phi is on two; that means, phi is surjective and that is always possible; because always any vertical line and any horizontal line will always intersect and it will always give me some point p in lambda right. So, this particular phi is always surjective. So, there always exists a p in my lambda such that, phi p can be corresponds to this particular sequence right, which comes up from the intersection of a vertical line in a horizontal line. So, this is always surjective; a little involved part would be to see that this is continuous.

So, I want to say that phi is continuous and in order to see that phi is continuous, all when need to look into what happens. If I am saying that there exist points P and P prime right. So, supposing for pp prime belonging to lambda, what is the meaning of saying that the distance between phi p and phi of p prime is less than epsilon. So, let me take an epsilon positive; what is the meaning of saying that the distance between phi p and phi p prime is less than epsilon. We have already seen this earlier right that would mean that there exists some n such that the middle block of phi p and phi p prime should be the same; that means, we can say that now at the finite stage right I can say that my p and p prime they both belong to say V s minus 1 up to s minus n right this vertical triangle this vertical rectangle and I should say intersection with h s naught h s 1 say s minus n plus 1 or maybe I can take s minus n does not matter.

So; that means, that my p and p prime they both belong to the intersection of this vertical rectangle with this horizontal rectangle, what does that mean. So, let us go back to our figure once again right we try to look into this figure once again, I had this vertical rectangle right which maybe I am claiming it as Vs minus 1 s minus n and maybe we have this horizontal rectangle, and I am claiming this to be my H s naught s 1 s sn.

Now; that means, now my p and p prime, they both belong to this region this intersection of this 2 rectangles and what happens to this rectangles if I see that the width of this happens to be lambda to the power n. So, if I am trying to look into what is this particular region what is this particular width right. We know that this particular width is lambda to the power n and if I look into what happens to this particular width right we know that this particular width will be 1 upon mu to the power n so; that means, if I want to look into what could be the possible distance because my pn p prime they both belong over here.

So, what could be the possible distance between p and p prime here, since this is less than epsilon means that this has to be right; that means, that at somewhere p and p phi p and phi p prime they agree on a middle part right, which means that they belong to the same intersection, which means that the distance cannot be greater than say the distance between p and p prime. So, that would basically mean that mod of p minus p prime right it cannot be greater than or maybe it is always less than 1 upon lambda to the power n, plus sorry lambda to the power n right plus 1 upon mu to the power n. The distance cannot be greater than this, right.

So, we can always choose a delta right such that and delta which is less than this part right. So, that gives us that our mapping phi happens to be continuous right. So, this map phi is continuous and that gives me that phi happens to be a homeomorphism. Now since phi is a homeomorphism we have another say observation here that my phi from lambda to sigma is a homeomorphism.

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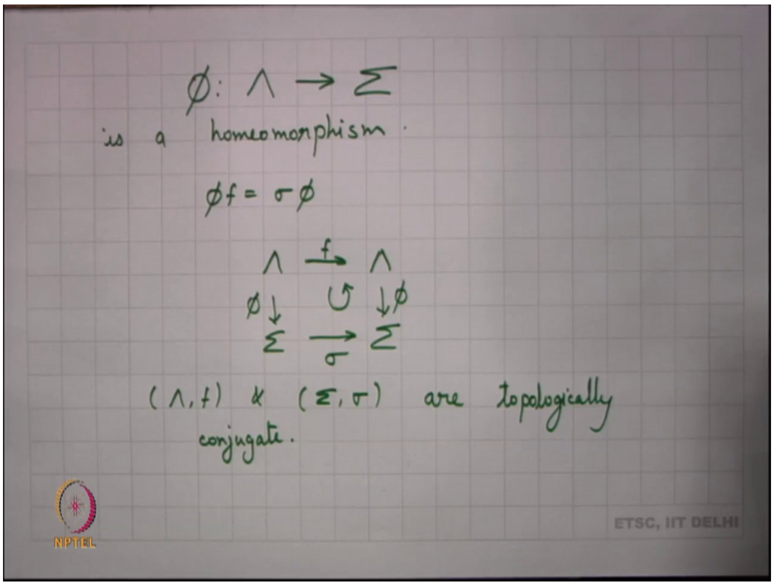

We have already observed that you have phi of f to be same as sigma of phi right at all points p of lambda.

So, our phi is an observe is an homeomorphism for which, this relation holds or in other words I can say that if I take this diagram this particular diagram commutes right, which means that these systems right lambda f and sigma sigma, right these are topologically conjugate and what does that give. So, let me recall this once again, we knew something about our system sigma sigma, we have periodic orbits of all periods we have uncountably many non periodic points and we also have infinitely many points with dense orbit.

So, what we can conclude is that again I should write it in form terms of the theorem here right, we can conclude what we conclude here is.

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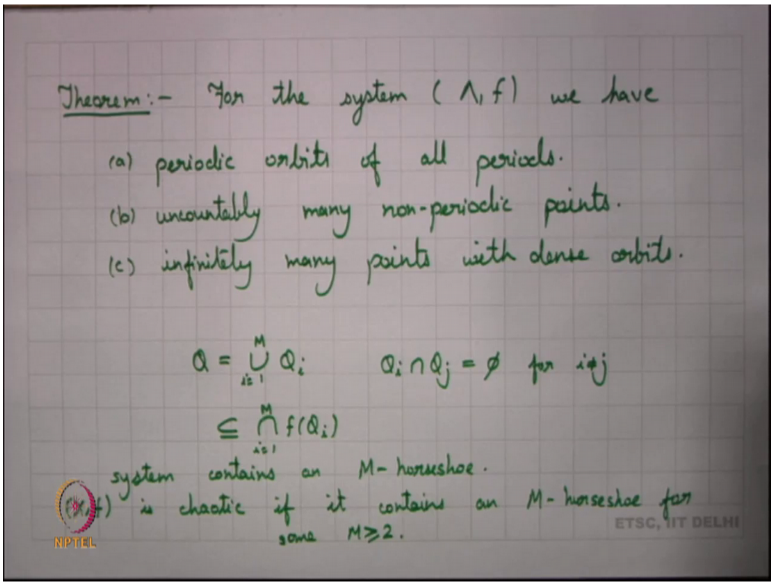

For the system; that means, for the horseshoe attractor, we have periodic orbits of all periods, uncountably many non-periodic points and infinitely many points with dense orbits.

So, the system the horseshoe attractor has a very interesting kind of dynamics, but instead of looking into the horseshoe attractor because again that is a very difficult system to study, what we have found out is an easier way to study is looking into the space of sequences and we will come back to the space of sequences sometimes again later also, but right now we try to conclude this by one more observation.

Now, we found an horseshoe, what was a horseshoe right. I can typically say that horseshoe was some kind of a compact set Q right, such that I can write q as a finite union of some other compact sets. So, let me say that Q can be written as a finite union of say Qi say 1 to m for something. So, now, I had Q written as finite union of Qi, such that what could you say about my fq right? When my qi was such that of course, my Qi intersection Qj was empty. So, these are pair wise disjoint and what we can say about q is again that Q is contained in right I am looking into the intersection of all i going from 1 to m f of Qi .What we observe is that whenever you have a system right which consist of or which contains a compact set Q, such that Q can be written as a disjoint union of some finitely many compact sets which satisfy the property that Q is contained in the intersection of the image of each of this Qi, then we say that the system contains an m horseshoe now this is what we mean by a topological horseshoe.

So, then we say that the system contains an m horseshoe and the existence of horseshoe in a system is good enough to tell me that the system is chaotic. So, we say that the system Xf is chaotic, if it contains an m horseshoe for some m greater than equal to 2 of course, we need equal to 2 1 does Not suffice you. So, for some m greater than equal to 2 if it contains it is called an m, it this particular construction this particular Q is called an M horseshoe and if it contains an M horseshoe then we say that the system is chaotic. So, this is Pnemore definition of chaos, and this is one more observation that such systems would be chaotic, but we are not going to get into details into horseshoe later they do have their own looking practically seeing whether there is a horseshoe or not is a very difficult exercise.

But that gives us some kind of symbolic dynamics which occurs from this existence of M horseshoe because M horseshoe will give you that the system can be splitted into a symbolic system or basically you can have a sequence with m symbols right and again you have the same dynamics which gives you chaos. So, existence of a m horseshoe is enough to say that the system is chaotic, although realizing the chaos within is much easier when you get into the symbolic system. So, I think this is where we end today.