Stochastic Processes
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Module - 3
Stationary and Auto Regressive Processes
Lecture - 1
Stationary Processes

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This is the course stochastic processes and in this we are starting module three. That is stationary and auto regressive processes. This is the lecture one on the stationary processes, and I will be taking the Auto regressive processes in the lecture two. In this talk, I am going to cover the introduction of the stationary processes and few definitions and the properties of the stationary processes.

Then there is a two important stationary processes. One is the strict sense stationary processes, the second one is wide sense stationary processes. After this I am going to give few simple examples of stationary processes. Introduction - a stationary processes is a stochastic processes, whose probabilistic loss remain unchanged through shift in times or in space. Stationarity is a key concept in the time series analysis has it allows powerful techniques for modeling and forecasting to be developed. What is the meaning of time series? Time series is a set of data ordered in time usually recorded at regular interval of regular in time interval.

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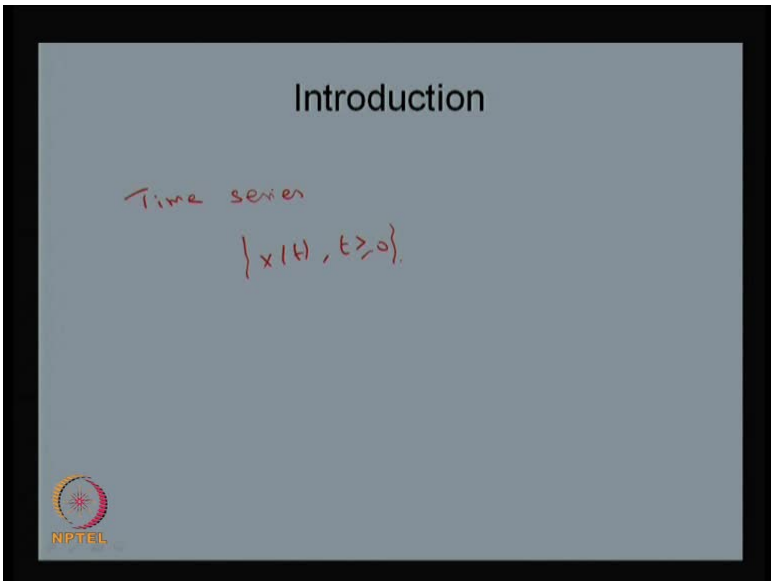

In probability theory a time series, if you make out the time series is a collection of random variable index by time. Time series is a special case of stochastic processes, one of the main features of time series is the interdependency of observation over time, this interdependency needs to be accounted in the time series data modeling to improve the temporal behavior and forecast of future of moment.

So, basically the stationary is used as a tool in time series analysis, when the raw data or often transformed to become stationary. That means if you collect the raw data and that raw data need not be satisfying the time series, need not satisfy the stationary property. But using the stationarity property the time series of that raw data is a transformed, so that you can model as well as you can forecast for the future moment by using the stationarity property.

There are different forms of stationarity depending on which of the stationary statistical properties of the time series are restricted. The most widely used form of stationarity are strict sense stationarity and weak sense stationarity. So, basically before we go to the two types of two important types of stationary property; that is weak sense stationary property and strict sense stationarity property, we will just see few definitions followed by these two important stationarity property.

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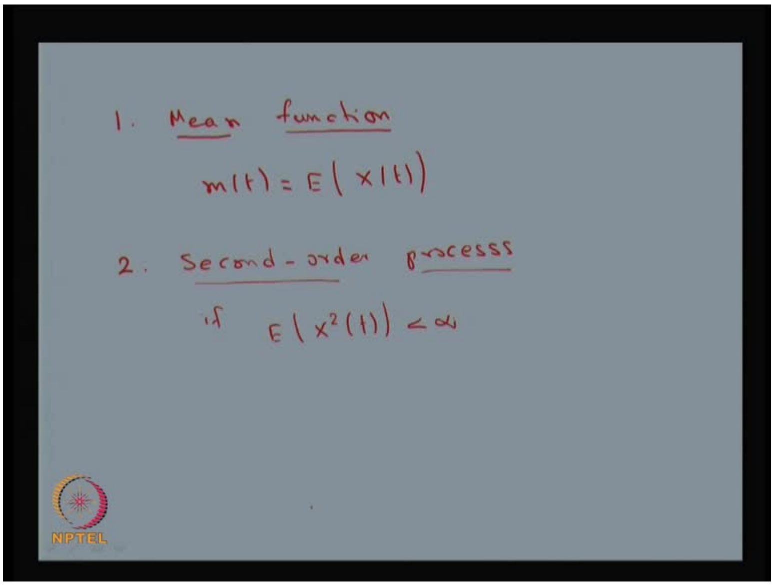

The first one is the mean function; mean function is defined has the with the notation m of t that is nothing but expectation of the random variable x of t. So, here is the stochastic processes is the collection of a random variable x of t over the t belonging to capital T, and you are defining the mean function has the function of t, that is the expectation of random variable x of t, sometimes this is going to be a function of t sometimes it is going to be a independent of t.

According to the function of a t or independent of t, we can classify the stochastic processes later, so this definition is going to be very important, that is mean function. The second one which has second order stochastic processes, when we say a stochastic processes is going to be a second order stochastic processes. If it is satisfies the condition, the second order moment it is going to be finite for all t.

If this condition is satisfied, that means if a random variables is a finite second order moment then that corresponding stochastic processes is called a second order stochastic processes. That means there is a possibility for stochastic processes may not satisfy, the second order moment may be infinite or it would not exists in that case it is not going to be call it has a second order processes, so whenever you collect the random variables form a stochastic processes and satisfying the second order moments are going to finite for all t, then you see that is stochastic process is going to be a second order processes.

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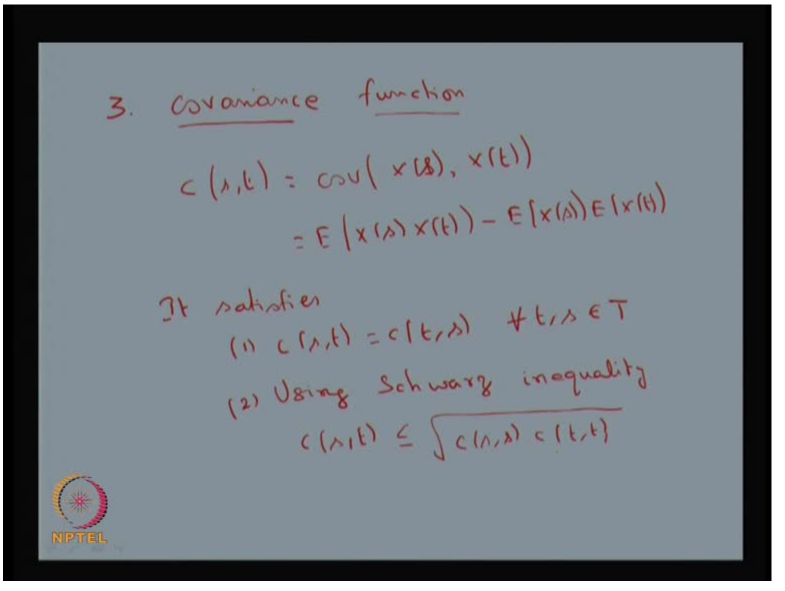

The third definition is a covariance function, how to define the covariance function; the covariance function in notation it is a c of x comma t, that is nothing but covariance of two random variables – x(s) comma x(t). Since, it is a collection of random variable, so for each t you will have one random variable, so that means here you have taken 2 s and t and you got the corresponding random variable and you are finding the covariance of these two random variables, that is nothing but the expectation of x of s x of t minus expectation of x of s, and the expectation of x of t. Obviously since you are finding the covariance of any two random variable, obviously this stochastic processes must be a second order stochastic processes, so the second order moments exists and you are able to find out the covariance of this one.

That means, the existence of the second order moments is going to be finite, that is assumative to be that is assumed and therefore you are getting the covariance of these two random variables, so using that we are defining c of x comma t, that is covariance. Since it is nothing it is a expectation of the product minus expectation of the individual one, it is going to satisfy satisfies the first condition the c of s comma t is same has c of t comma s for all t comma s belonging to capital T, where capital T is a parameter space.

From the parameter space if you take any 2 t and s then you find out the covariance function of s comma t is same has t comma s.

The second property using Schwarz inequality, you can always able to say the upper bound is going to be c of s comma s, and c of t comma t, this is going to be exists because this is going to be exists that the second order moments are finite, that was c of s comma s that is nothing but that is nothing but the variance of x of s. And this is going to be the variance of x of t, therefore this is nothing but the product of the variance and square roots. So, this is going to be a finite quantity therefore this has the upper bound of a c of s comma t has upper bound the square root of the product of the variance of x of s and x of t.

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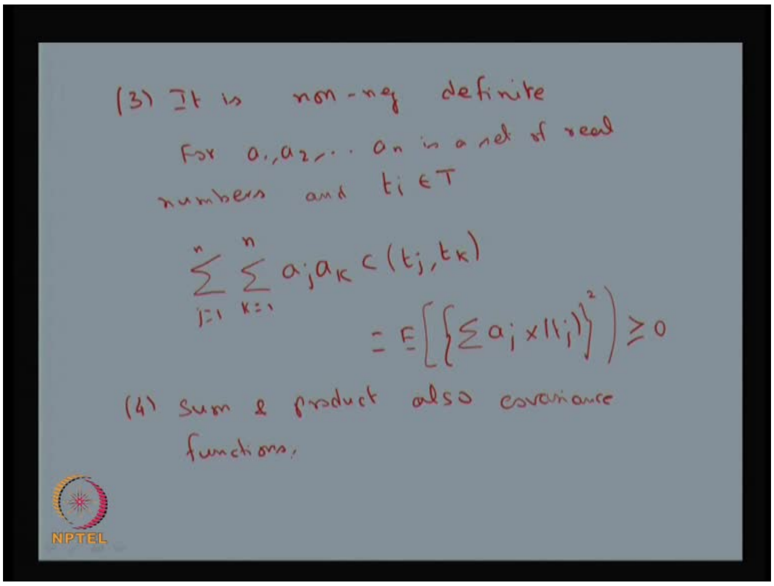

The third property, it is a the covariance matrix non negative definite clause, that means for a 1, a 2, a n, that is set of r that is a set of real numbers, and if you take t i is belonging to capital T. And if you find the double summation of a j running from 1 to n and k running from 1 to n a j a k, these are the real numbers with the covariance function of a t j comma t k, that double summation is nothing but the expectation of summation of a j is x of t j is the whole squared, these expectations quantity is always going to be greater than or equal to 0. Since, it is a whole squared, so the expectations of whole squared quantity is always greater than or equal to 0 for all the set of all real numbers a 1, a 2, a n and the t i is are belonging three, this is nothing but the expectation of this quantity, that quantity is always going to be greater than or equal to 0. So, you can conclude the covariance of the function is going to be non negative definite.

The fourth property the sum as well as the product of any two covariance functions also covariance functions, the sum and the products also going to be the covariance function. This property needs elaboration over we are seen this for this course. So, this four quantities are going to be use later whenever you like to crosscheck, whether the covariance function is going to be satisfied or how to find out the covariance function so this property will be used.

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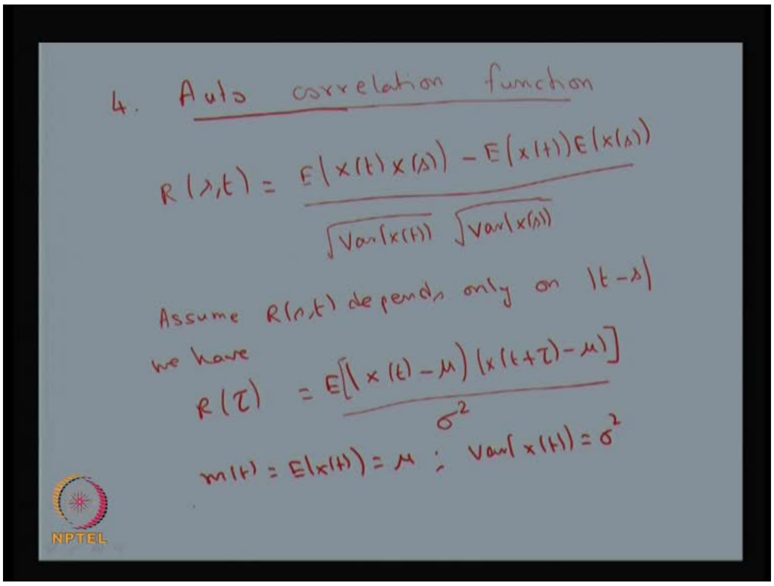

Now, we are moving into the fourth definition that is Auto correlation Auto correlation function. The way we have define the covariance function now you are defining Auto correlation function, it is define with the notation r of s comma t that is nothing but we can write in terms of expectation of x of t x of s minus expectation of x of t into expectation of x of s divided by the square root of variance of x of t, and the square root of variance of x of s. So, the numerator can be written covariance of x of t comma s divided by square root of variance of x of t by square root of variance of x of s.

So, this is going to be used within the notation of r of s comma t and this is going to be a Auto correlation function of the random variable x of t and x of s, so it is basically describe the correlation between values of process at the different time points s and t sometimes, we assume the we assume r s comma t depends only on absolute of t minus s. In the later case, then we are discussing the stationary pr it is going to be depends only on the interval length only on the interval length not the actual time. Therefore, the R of s comma t this is going to only depend only on the length of the t minus s we assume not the actual s and t, therefore by assuming a r of s comma t it is going to be a only depends on t minus s.

We can have r of instead of two variables, I can use the one variable has a R of tau, that is nothing but the expectation of x of t minus mu multiplied by x of t plus tau minus mu the expectation of that product divided by sigma squared. So, here I have made the one more assumptions the m of t that is nothing but the expectation of x of t, that is going to be mu and variance of x of t is going to be sigma squared, if that assumptions only the r of taw is going to be expectation of this product and divided by sigma squared, where the variance of x of t is going to be not a function of t. It is a constant that is sigma squared, similarly the mean function expectation of the x of t is going to be mu that is also independent of t.

Therefore, I can simplifies this R of s comma t the product expectation minus individual expectation that can be simplified has a expectation of the product, and so basically this is evaluated at x of t at x of t plus tau, and that difference is going to be tau, and this is also going to be a event function. That means a it has R of tau is same as r of minus tau, and this auto correlation function is used in time series analysis as well as signal processing; in the signal processing, we are assumed that the signal the corresponding time series satisfying the stationary property.

Therefore, the stationarity property implies the auto correlation function is going to be depends only on the length of the interval not the actual time. Therefore, this r of tau will be used in the signal processing as well as in general time series analysis also. The fifth definition, we are covering the different definitions, which we want the fifth definition. First you started the main function, second we started the second order stochastic processes, the third we started the third we given the covariance function, and fourth we have given the Auto correlation function. Now, we are giving the fifth definition that is independent increments. If for every t 1 less than t 2 less than t n the random variables of x of t 2 minus x of t 1 comma x of t 3 minus x of t 2, so on till x of t n minus t n minus 1 are mutually independent mutually independent random variables for all n, then we can say the corresponding stochastic process is having independent increment property.

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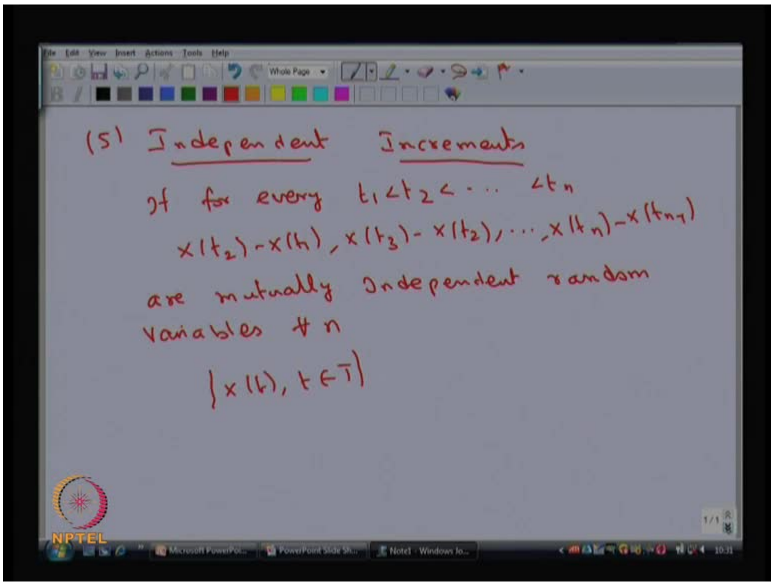

So, whenever you take few t is t 1, t 2, and t n and the increments that x of t 2 minus x of t 1 like that till x of t n minus x t n minus 1, so these are all going to be the increment, and each one is the random variable. Therefore, the increment is also going to be the random variable, and you have n such random variables, and suppose these n random variables are mutually independent random variable for all n. So, this is x for 1 n like that, if you go for all n this if this property is satisfies, then you can conclude the corresponding stochastic processes having the property of Independent Increments.

So the increment independent that does not implies, but some other properties but here what we are saying the increments are satisfies the mutual independent property; that means if you find out the c d f of the joint, c d f of this random variable; that is same has the product of individual c d f if the property satisfied by all the n, then you can conclude that stochastic processes has the Independent Increment.

The next property the next definition is a Ergodic property. what is the meaning of Ergodic property? It says the time average of a function along a realization for sample exit almost everywhere, and is a related to the space average. What it means whenever the system or the stochastic process is a Ergodic the time average is the same for all almost initial points, that is the process evolves for a longer time forgets it is initial state.

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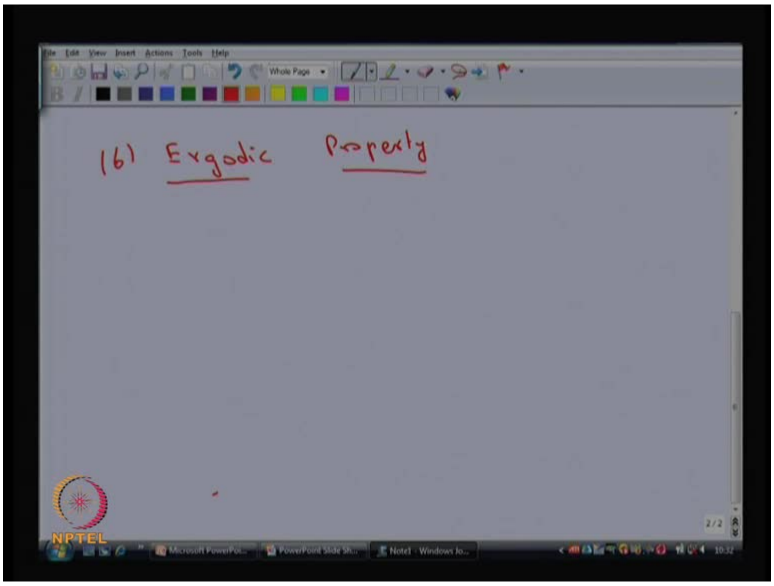

So statistical sampling can be performed at one instant across a group of identical processes or sampled over time on a single process with no change in the measured result. We will discuss the Ergodic property for the co process in detail later, but this Ergodic property is going to be very important, then when you study the Markov property or when you study the stationarity property. Therefore, this Ergodic property is always goes along with the stationarity property or it goes along with the Markov property therefore, the stochastic property is going to be in a different way in that we are going to discuss later .

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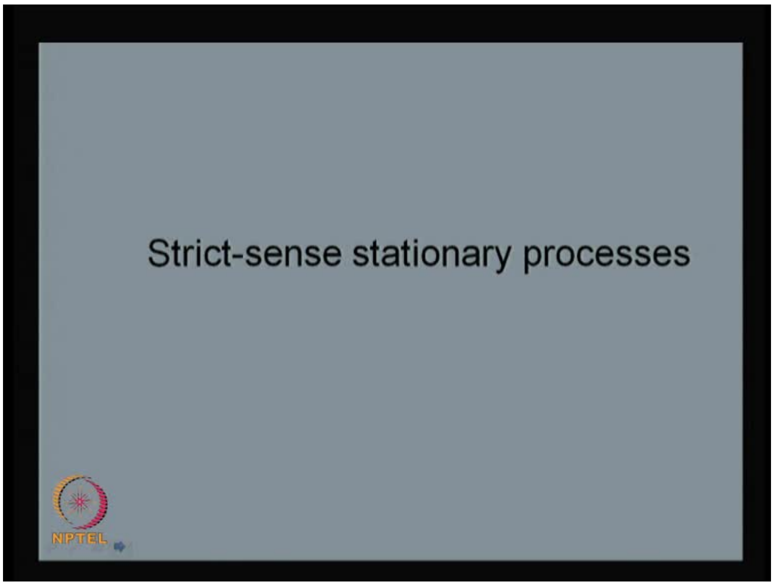

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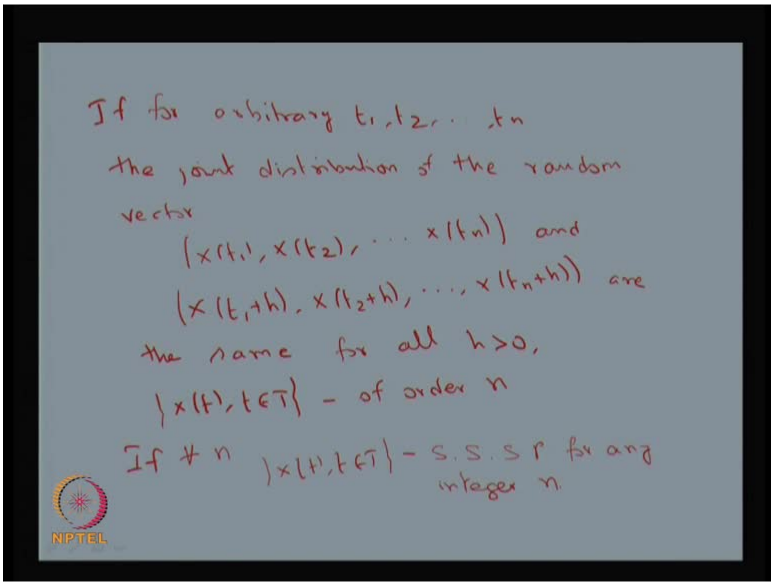

The most important stationary process, that is strict sense stationary processes. First let me start with this strict sense stationary process of order n, then I will define strict sense stationary process for all order n, then I will define strict sense stationary processes for all order n or strict sense stationary processes itself. If for arbitrary t 1 comma t 2, and so on, t n the joint distribution of the random vector; that is x of t 1 comma x of t 2, and so on, x of t n. And another random vector that is x of t 1 plus h comma x of t 2 plus h, and so on, x of t n plus h are the same for all h is greater than 0. Then we say the stochastic processes is a strict sense stationary of order n, because here we restricted with the n random variable, so we take a n random variable take the points t 1, t 2, and t n and find out the joint distribution of x of t 1, x of t 2, and x of t n.

So, we can find out what is the joint distribution of these n random variables, also you find joint distribution of n random variables shifted by h; that means a real random variable of x of t 1. Now, you have a random variable x of t 1 plus h, if the same shift h you do it with t 2. Therefore ,the random variable x of t 2 plus h, similarly the nth random variable is a x of t n earlier. Now, we have a random variable x of t n plus h, so you have a another random vector with n random variables, and find out the joint distribution of that if the joint distribution of this first n random variables, as well as the joint distribution of the shifted by h that random variable. If the both distribution are same this, they are identically distributed. The joint distribution are going to be identical, then you can conclude this stochastic processes is a strict sense, stochastic processes of order n, because it will use the n random variables, if a this is going to be satisfied, the other properties going to be satisfied for all n.

Then, you can conclude the stochastic process is going to be a strict stationary processes for any integer n, this is going to be a strict sense stationary process for any integer n, so we have to cross checking the joint distribution of n random variables. So, if it satisfying by only with the maximum some integers, then it is going to be a strict sense stationary processes of all that n, if it is going to be satisfy for all n, then for any integer n then that is going to be call it has a strict sense stationary processes.

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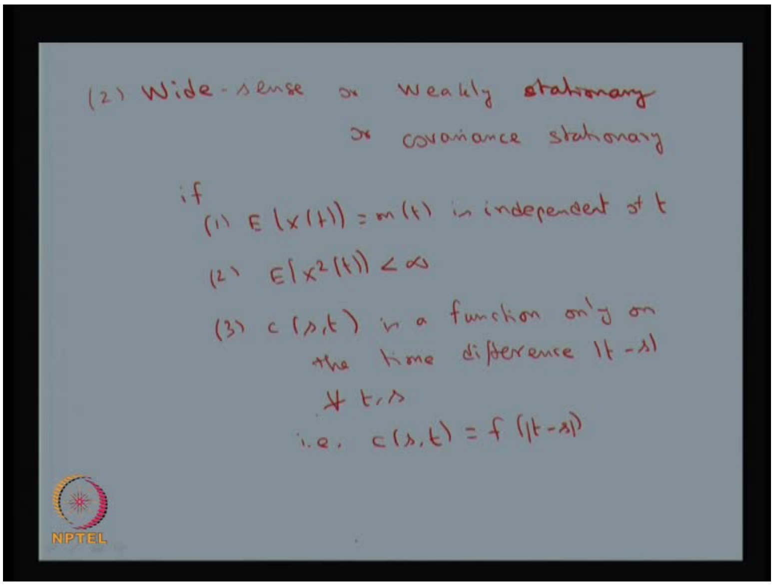

The next definition is a wide sense. The next one is wide sense or weakly stationary processes or another word covariance stationary processes, when we say the given stochastic processes is going to be wide sense or weakly stationary or covariance stationary. If it is going to satisfy the following properties following conditions mean function, that is m of t is a independent of t. The second condition the second order moment is going to be finite basically the stochastic processes is going to be a second order moments are going to be finite.

The third condition if you find the covariance function c of s comma t that is a function only on the time difference t minus s for all t comma s, you find the covariance function for any two random variables x of s and x of t. Then that is always going to be a function of the only difference t minus s not the actual t or actual s, that is in words in mathematically we can write c of s comma t; that is going to be a function of t minus s in absolute. If these three properties going to be satisfied by any stochastic processes, then we say that stochastic processes is a wide sense or weakly stationary or covariance stationary, this is entirely different from strict sense stationary, the strict sense stationary, you are finding out the joint distribution of the n random variables.

Then find the joint distribution of n random variables shifted by h for all h greater than 0 for all n if that property satisfies, then you say that is a strict sense stationary processes whereas here we check the mean function is going to be independent of t, and the second order moment is going to be finite value, and the covariance function is going to be a function of only the difference of t minus s. Therefore, the any stochastic processes is satisfying the strict sense stationary processes, strict sense stationary property that does not implies the wide sense stationary property as well as the wide sense stationary processes need not satisfied of all the strict sense property. Therefore, you cannot implies one stationary processes that does not implies the wide sense, and the wide sense stationary processes that does not implies the strict sense stationary processes.

So, in the strict sense processes what we are saying is it is stochastic process whose joint distribution does not change, when shift in time or space as a result the parameter such has the mean, and the variance if they exists also do not change over the time or passion in the strict sense stationary processes.

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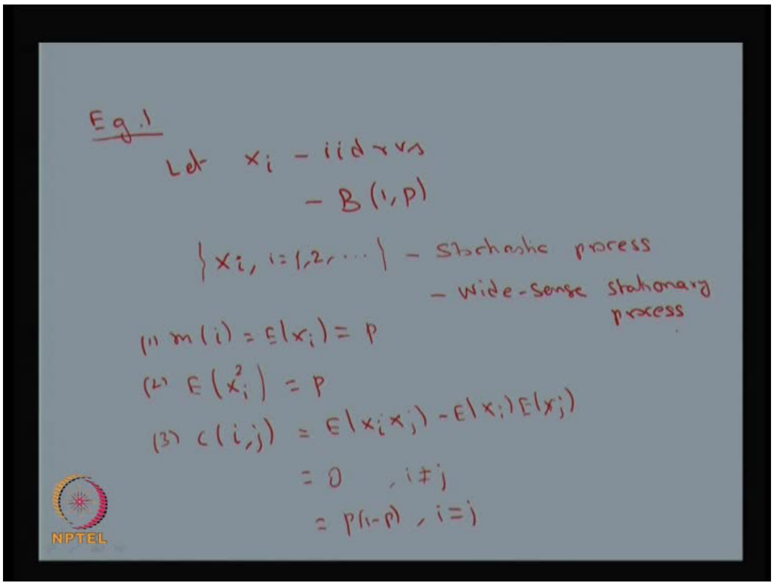

Now, I am going to give few examples for the stationary processes may be it could be strict sense stationary processes or it could be a wide sense stationary processes. The first example let x i is going to be a i i d random variables independent identically distributed random variables, and assume that each one is going to be Bernoulli distributed random variable with the parameter p in passion, it is a binomial distribution with parameters 1 and p, that is immersed that each excise are Bernoulli distributed random variable with the parameter p.

Now, I am creating a stochastic processes with the those, such i i d random variables in which each random variable is a Bernoulli distributed random variable, therefore this is going to be a stochastic process. Now, you can verify whether it is going to be a strict sense stationary processes or wide sense stationary processes, the assumptions are if all the random variables are mutually dependent and each random variable is identical distributed which is Bernoulli distributed. So, this is just for examples we taken and if you find out the mean function for each random variable, that is going to the expectation of x i and that is going to be the t naught of the Bernoulli distribution that is going to be p which is independent of i.

The second condition, if you find out what is the second order moment of second order moment of the second order moment is going to be 1 squared into p and 0 into 1 minus p, for that is also going to p, so if you find out c of some i comma j instead of s comma t you have i comma j that is nothing but the expectation of x of i x of j minus expectation of x of i expectation of x of j. If you find out this quantity this is nothing but x i and x j, and since they are independent random variable. Therefore, the expectation of x i into x j's nothing but the expectation of x i into expectation of x is j that is inverse this one.

Therefore, this is going to be 0 for all i is not equal to j for i is equal to j that is nothing but the expectation of x i square minus x i whole squared, that is nothing but the variance and the variance of the random variable Bernoulli distribution, that is going to be p q. Therefore, that is going to be p into 1 minus p for i is equal to j, and this value is independent of this value is going to be a function of i minus j, you can make out. Therefore, since these all three properties are weakly stationary property or wide sense stationary property satisfied, therefore this is going to be a wide sense stationary processes, in fact even if the random variables are simply i i d is then to we can check that the processes are wide sense stationary for illustration purpose, we have discussed Bernoulli processes.

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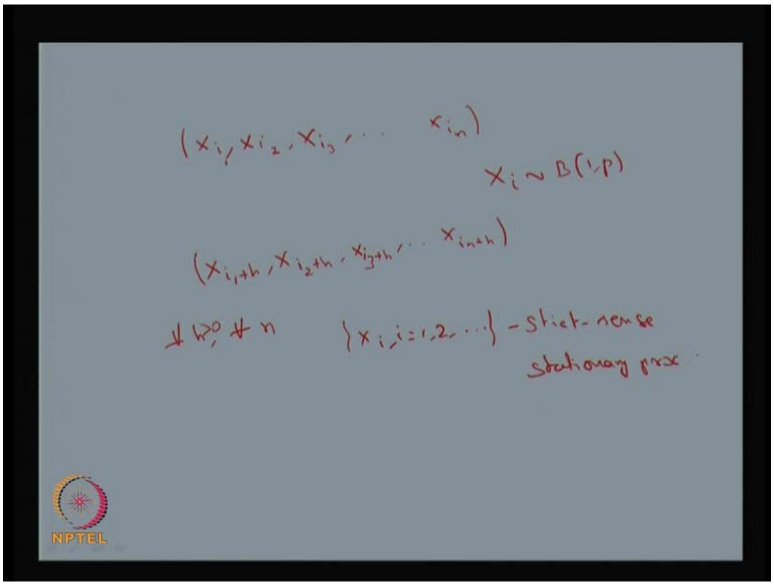

More examples on continuous time stochastic processes are discussed in the problem sheet. Now, we can cross check whether this is going to be a strict sense stationary processes also if you find out the joint distribution of suppose you take few random variables x of i 1, x of i 2, x of i 3, x of i n, so this is the n such random variables and each random variables are Bernoulli distributed with parameter p, and all are independent. Therefore, the joint distribution is going to the product of individual distribution, and if you shift this i 1 with some number h into x of i 2 plus h and x of i 3 plus h, and so on x of i n plus h, you shift those random variables with the h, if you find out the joint distribution.

And since each one is independent random variable, therefore the joint distribution by shifted by h, that is also going to be the product of those n random variable product. Therefore, the distributions are again going to be a identical, because they are because each random variable is going to be identical as well as mutually independent. Therefore, the joint distribution is going to be a product and all are going to be identical, therefore it is going to be power n of the distribution.

So, this is going to be satisfy the Stephens property, that is a joint distribution of this random variable, and joint distribution of this random variable are going to be same for all h, as well as for all n also. Since, it is satisfied by all h is greater than 0, and for all any integer n; therefore, this is going to be a same collection of random variables, the stochastic processes is going to be strict sense stationary processes. So this is the (( )) example in which this stochastic processes is going to be a strict sense stationary processes, as well as wide sense stationary processes, but there are many situations in which stochastic processes may be a strict sense not a wide sense. And stochastic processes may be a wide sense stationary processes is not a strict sense stationary processes, and how this particular stochastic processes become a strict sense or wide sense, because of each random variable is mutually independent as well as identical. Therefore, it is going to be a strict sense stationary processes, as well as wide sense stationary processes.

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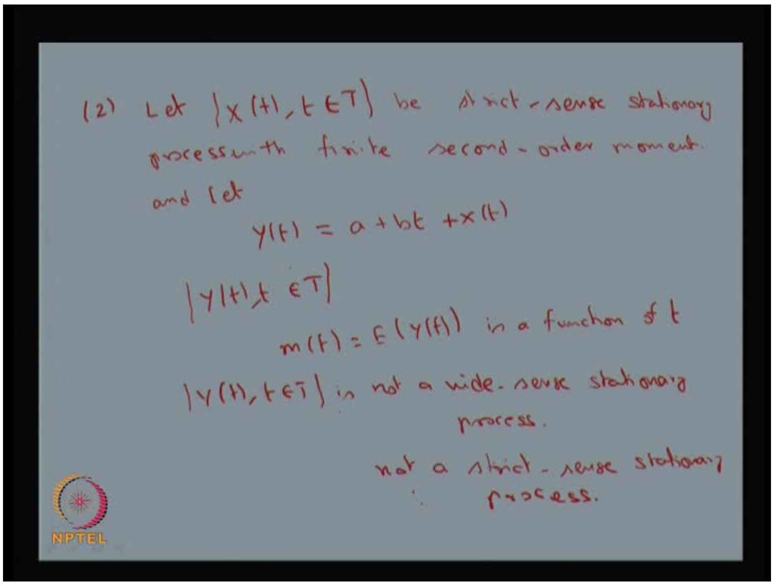

I have another example in which it is going to be a only the it would not be a strict sense itself, let me start with one example in which this stochastic processes is a strict sense stationary processes, the given x is a strict sense stationary processes with the finite second order moment. So, you do not want the finite second order moment are the strict sense stationary processes, but I have taken has a example the given x(t) is going to be strict sense stationary processes along with the finite second order moment.

Now, I am going to define another stochastic processes, the random variable y of t that is a plus b t plus x of t. So this is going to be a Stochastic processes. This a stochastic processes y(t), now you want to check whether y(t) is going to be stationary strict sense stationary process or not, as well as this is going to be a wide sense stationary or not the x is strict sense stationary processes. Suppose, you find out the mean for this random variable, if you find out the mean for the y of t, where a and b are constants. Therefore, this is going to be a function of t, since a and b are constants, the mean of y of t is the function of t. Therefore, this is a function of t since it is not satisfying the first property of the first condition to become a wide sense stationary processes therefore, the y of t is not a wide sense stationary process.

We started with the sixth stationary process, and we create the new stochastic processes y of t; that is a plus b t of x t of x t, where a and b are constants. Now, if you find out the mean of y of t mean function, that is going to be a function of t that is nothing but a depends on t. Therefore, y t is not going to be a wide sense stationary process whereas x t is a strict sense. Now, similarly you can cross check that the joint distribution of y of t and shifted by h t is shifted by h, we can conclude this is also not going to be a since it is a function of t.

Since it is the mean is going to be a function of t, and the y of t also involves the function of t as well as x of t, even though the x of t is a strict sense stationary processes the way you made a a plus b t plus x(t). You will land up the joint distribution are going to be a different by the t with a shifted the t plus h it would not be satisfied. Therefore, you can conclude y of t is not a strict sense stationary processes also that means from this example, we can conclude whenever you have a strict sense stationary processes, if you make a plus b t plus x t definitely the y of t is not going to be wide sense stationary processes as well as strict sense stationary processes.

We go for the third examples, in this third examples will start with the stochastic processes be a here, this which random variables are uncorrelated random variables with the mean of each random variable is going to be some constants k which may be assumed it to be 0. In some situation in general you keep mean of each random variable is going to be constants k, and you make a x of m x of n that is going to be variance for m is equal to n, and for all other quantity you make it 0, not only these each random variables are uncorrelated random variable.

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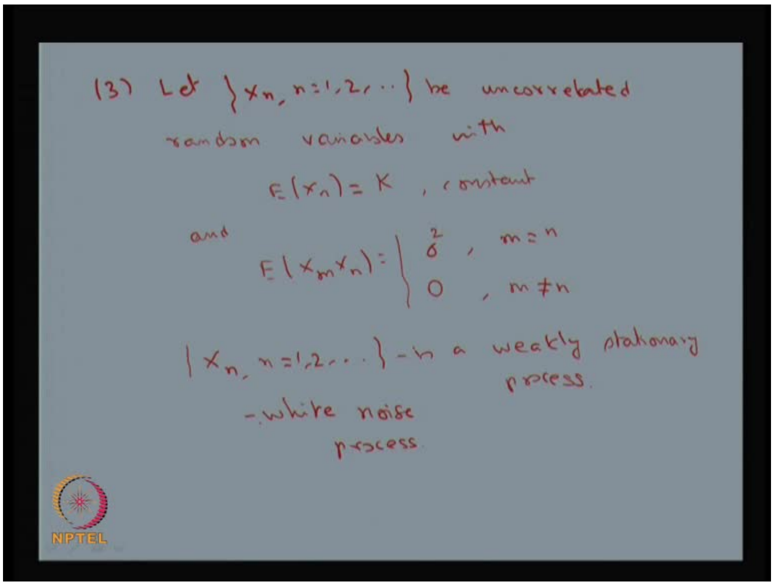

That means, you find out the correlation coefficients, that is going to be 0 and the mean is going to be constants. And the expectations of the product of any two random variable, if the difference is 0 and obviously if they are saying the you make the assumptions, therefore this is going to be a variance sigma squared. If you cross check all the properties of all the condition of the wide sense stationarity property is starting with the mean function, and second order moment exists; that is finite, and the covariance function of any two random variable is going to be the function of difference, there all those three conditions are going to be satisfied.

Therefore, you can come to the conclusions, and I am not working out here this is going to be a weakly stationary processes or wide sense stationary process or it is going to be a called as covariance stationary process also, and this stochastic process is also called white noise process, this is very important in the signal processing. You keep the uncorrelated random variable with these assumptions, the mean is going to be a constant which will be 0 and the product of the expectation of this is going to be this values, in this we going to be a weakly stationary processes. In the sense it satisfies the all three conditions of the weak sense or wide sense stationary process, and this stochastic process is called a white noise process.

Note that this stochastic process, we did not make the distribution of each random variable x n, what is the distribution of x n not defined here without that we will give the all assumptions of mean and the variance.

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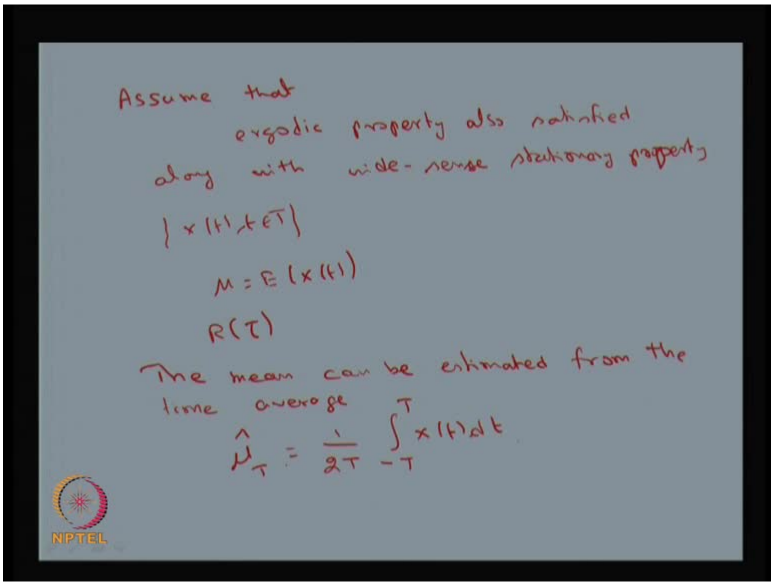

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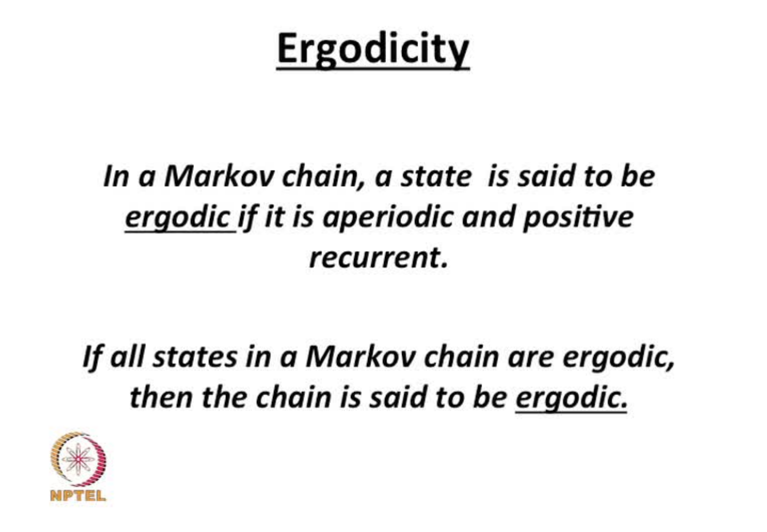

Therefore, this is going to be very useful in the time series analysis as well as the signal processing, and this particular stochastic process is called the white noise. In sometimes we make the assumptions the x n are going to be normally distributed random variable also, but in general we would not define, we would not give what is the assumptions? What is the distribution of x n without that if these stochastic processes is going to be call, it has a white noise process. Addition to the quite sense stationary process one can assume one can assume that Ergodic property also satisfied along with wide sense stationary property for illustration purpose, we discussed Bernoulli process; that means the given stochastic process is a wide sense stationary process as well as the it is the Ergodic properties also satisfied.

In that case the mean function is going to be some independent of t, that you can make it has a mu and the auto covariance function is going to be R of tau only, because it is a wide sense stationary process, the mean is independent of t and the Auto correlation function is going to be a function with only tau. And we have a Ergodic property therefore you can find the mean can be estimated from the time average, so this is possible only if the Ergodic property, that is why the mean can be estimated with the up arrow. That means the estimator estimation of a mean that same has 1 divided by 2 times t and minus t to t of x of t dt, therefore it is possible as long as the stochastic process is in general I define t belonging to capital T, that t is different from this t.

So, here you have the time interval of length to t within that 2 t, if you find out the time average and that time average quantity is going to be a estimation of the mean, that means if u t converges the squared mean to mu has a t tends to infinity, then this process is going to be a Ergodic, that stochastic process is going to be a call it has a mean Ergodic process. Similarly, one can estimate other higher order moments also provided the process is Ergodic with respect to those (( )). So, here I have met the Ergodic mean therefore you have estimating the mean with their Ergodic property. Similarly, if the given stochastic process is satisfying the Ergodic property the higher order moment then those measures can also be estimated in the same way.

So here the mu t converges in mean squared mean to mu has a t tends to infinity, so that is the conclusion, we are getting with the Ergodic property along with the white sense stationary property with these. Let me stop the today’s lecture and some more examples, for the stationary process, so those examples may be a wide sense stationary process or strict sense stationary process that I am going to give the next lecture.

So today's lecture, I covered is what is the stationary process and to conclude or for given stochastic process is going to be stationary process for that we have given few simple definitions, so with those definitions. So, we can come to the conclusions given to the stochastic processes is going to be a wide sense stationary process or weak sense stationary process, and I have given three examples in today's lecture, and that I will give two more example of stationary process in next lecture. Then I go for simple stationary process that is Auto regressive process and moving average process, and some more stochastic process for the stationary process example I will give it in the lecture two with this today's lecture is over.

Thanks.