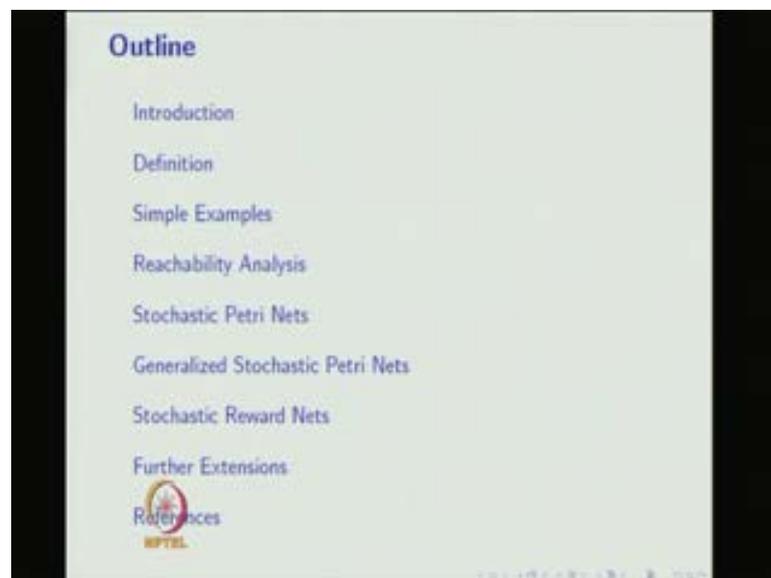


**Stochastic Processes**  
**Prof. Dr. S. Dharmaraja**  
**Department of Mathematics**  
**Indian Institute of Technology, Delhi**

**Module - 5**  
**Continuous time Markov Chain**  
**Lecture - 8**  
**Stochastic Petri Nets**

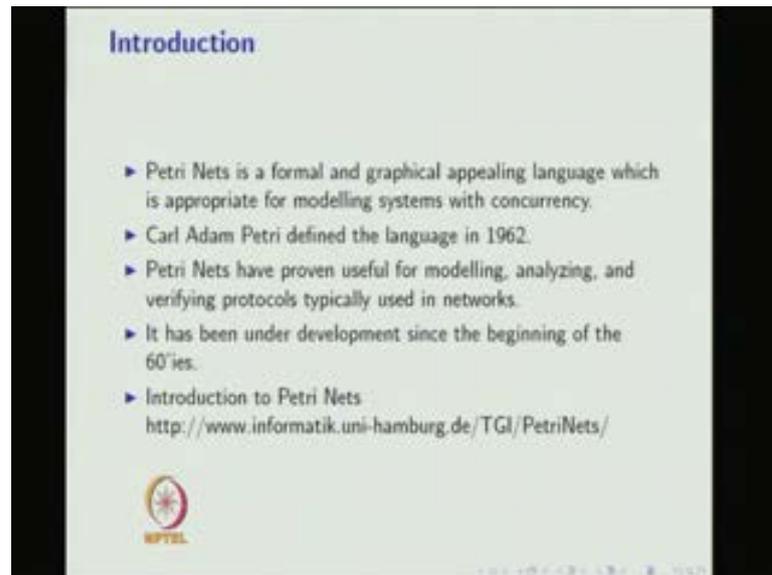
This is stochastic processes, module 5, continuous time Markov chain. In the lecture one, we have discussed the definition, kolmogorov differential equations, infinitesimal generator matrix, with examples. In the lecture two, we have discussed birth death processes; in the lecture three, we have discussed a Poisson processes; lecture four, we have discussed M M 1 queuing model. Simple Markovian queuing models with examples is discussed in lecture five. Queuing networks is discussed in lecture six; applications in communication networks, simulation of simple Markovian queuing models with examples are discussed in lecture seven. This is lecture eight, Stochastic Petri Nets.

(Refer Slide Time: 01:39)



In this lecture, I am going to cover the definition, simple examples, followed by that Stochastic Petri Nets, then Generalized Stochastic Petri Nets, then finally Stochastic Reward Nets.

(Refer Slide Time: 01:52)



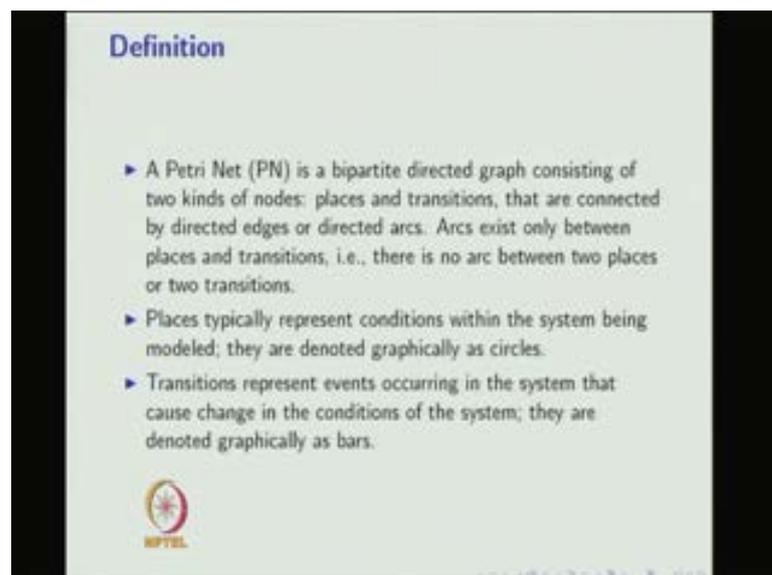
**Introduction**

- ▶ Petri Nets is a formal and graphical appealing language which is appropriate for modelling systems with concurrency.
- ▶ Carl Adam Petri defined the language in 1962.
- ▶ Petri Nets have proven useful for modelling, analyzing, and verifying protocols typically used in networks.
- ▶ It has been under development since the beginning of the 60's.
- ▶ Introduction to Petri Nets  
<http://www.informatik.uni-hamburg.de/TGI/PetriNets/>



Petri Nets is a formal and graphical appealing language, which is appropriate for modelling systems with concurrency. Carl Adam Petri defined the language in 1962. Petri Nets have proven useful for modeling, analyzing and verifying protocols, typically used in networks. It has been under development since the beginning of the 60s. You can find the introduction to Petri Nets in this website.

(Refer Slide Time: 02:43)



**Definition**

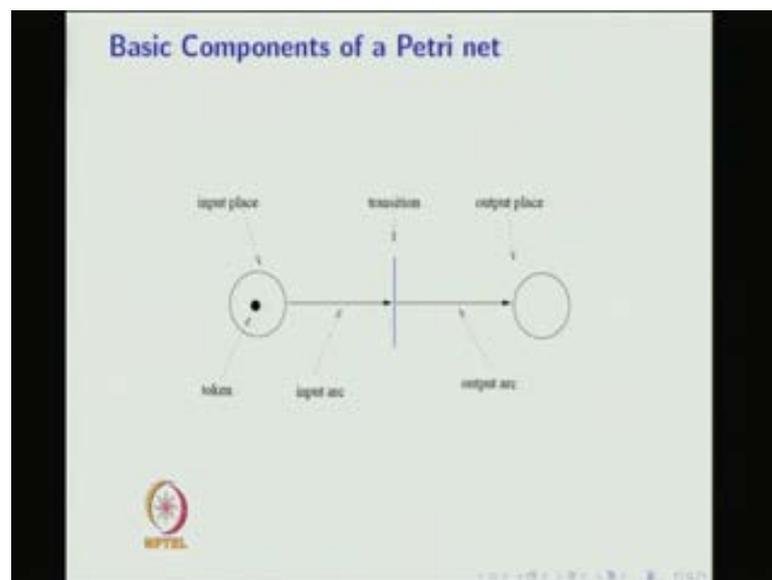
- ▶ A Petri Net (PN) is a bipartite directed graph consisting of two kinds of nodes: places and transitions, that are connected by directed edges or directed arcs. Arcs exist only between places and transitions, i.e., there is no arc between two places or two transitions.
- ▶ Places typically represent conditions within the system being modeled; they are denoted graphically as circles.
- ▶ Transitions represent events occurring in the system that cause change in the conditions of the system; they are denoted graphically as bars.



Now, we are going in to the definition of Petri Net. Petri Net is a bipartite directed graph, consisting of 2 kinds of nodes. The first one is the places; the second one is, transitions;

transitions that are connected by directed edges, or directed arcs. Arcs exist only between places and transitions; that is, there is no arc between two places, or two transitions. Places, typically represents, represent, conditions within the system being modeled. Places typically represent conditions, with the system being modeled. They are denoted graphically as circles. Transitions represent events occurring in the system that cause change in the conditions of the system; they are denoted graphically as bars.

(Refer Slide Time: 03:58)



See this diagram; here are the basic components of a Petri Net. This is called input place and this arc is called a input arc, because the arc is a connecting from the place to transition; and this is called, the line is called a, this bar is called a transition. The transition to the place, that arc is called the output arc, and the corresponding place is called output place. The number of dots inside the place, called a tokens.

Here, input place has 1 token, whereas, this place has no token. The definition continues. Input arcs, directed arcs drawn from places to transitions. They represent the conditions that need to be satisfied for the event to be activated. Output arcs are directed arcs drawn from transition to places. They represent the conditions resulting from the occurrence of an event. In the previous diagram, this is the input arc, because it connects place to the transitions; and this is called output arc, because it connects transition to place.

(Refer Slide Time: 04:56)

**Definition ...**

- ▶ Input arcs are directed arcs drawn from places to transitions; they represent the conditions that need to be satisfied for the event to be activated.
- ▶ Output arcs are directed arcs drawn from transitions to places; they represent the conditions resulting from the occurrence of an event.
- ▶ Input places of a transition are the set of places that are connected to the transition through input arcs.
- ▶ Output places of a transition are the set of places to which output arcs exist from the transition.



Navigation icons: back, forward, search, etc.

Input places of a transition are the set of places that are connected to the transition through input arcs. Output places of a transition are the set of places to which output arcs exist from the transition. In this diagram, we have only one input place, because the transition has only one input arc. With respect to the same transition, we have only one output place, because we have one output arc.

(Refer Slide Time: 06:25)

**Definition ...**

- ▶ Tokens are dots (or integers) associated with places; a place containing tokens indicates that the corresponding condition is active.
- ▶ Marking of a Petri net is a vector listing the number of tokens in each place of the net.
- ▶ When input places of a transition has the required number of tokens, the transition is enabled.
- ▶ An enabled transition may fire (event happens) removing a specified number of tokens from each input place and depositing a specified number of tokens in each of its output places.

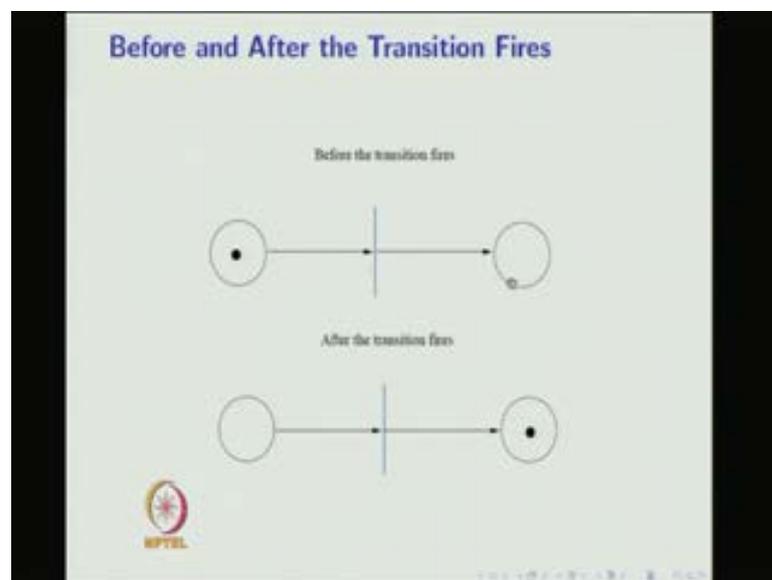


Navigation icons: back, forward, search, etc.

The definition continues. The tokens are dots, or integers associated with places. A place connecting tokens indicates that the corresponding condition is active; that means, in the

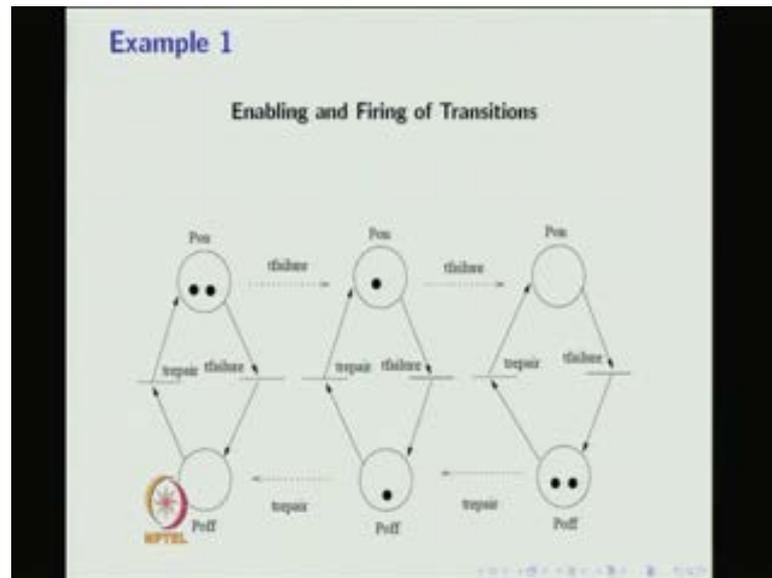
previous diagram, the 1 token deposited in the input place means that, this transition can be activated. The place containing a tokens indicates that, the corresponding condition is activated, active; the corresponding condition is active. Marking of a Petri Net is a vector, listing the number of tokens in each place of the net; that is called marking. When input place, when input places of a transition, has the required number of tokens, then the transition is enabled. An enabled transition may fire, event happens, removing a specified number of tokens from each input place and depositing a specified number of tokens in each of its output places.

(Refer Slide Time: 08:00)



In the same example, before the transition fires, one token in the input place; no token in the output place. Whenever there is a token in the input place, then the condition is active. Then the transition first enabled; after the transition enabled, it fires. After the transition fires, the token will be removed from the input place and the token will be deposited to the output places. So, here, we have only one input place and only one output place; therefore, after the transition fires, one token is removed from the input place and 1 token is deposited in the output place.

(Refer Slide Time: 09:00)



Now, we will see the, another example, enabling and firing of transitions. In this example, we have 2 places; one is called a...Here, we have 2 places; one is called a, the place P on, and the other place is called P off. Initially, 2 tokens deposited in the place P on; no token in the place P off. We have 2 transitions; one is the t failure, and the other one is t repair. Whenever the conditions are satisfied, then the transition will enable first, then it fires. Since 2 tokens are in the place P on, the transition t failure enables, whereas, no tokens is in the place P off; therefore, the transition t repair will not enable. If a t failure enables, then the required number of tokens will be removed from the place P on and the required number of, number of tokens will be deposited in the place P off. So, once the t failure enables and fires, enabling and firing of transitions...So, when the t failure transition enables and fires, 1 token will be removed from the place P on; 1 token will be deposited in the place P off. Now, the situation is this Petri Net.

Since 1 token is in the place P on, as well as 1 token in the place P off, both the transitions t repair and t failure enables. If t failure fires, then 1 token will be removed from the place P on; 1 more token, sorry, 1 token will be deposited in the place P off. Already, 1 token is in the place P off; therefore, 2 tokens in the place P off and 1 token is removed from the place P on. Hence, 0 token in the place P on and 2 tokens in the place P off. Suppose, at this stage, both the t repair as well as a t failure enable, but t repair fires; that means, at this stage, if t repairs fires, then 1 token will be removed from the place P off and 1 token will be deposited in the place P on, therefore when t repair fires,

then 0 token in the place P off; 2 tokens in the place P on. Similarly, in this situation, no token in the place P on; 2 tokens in the place P off; therefore, the t failure cannot enable. The only enabling transition is t repair. Therefore, if these fires, t repair fires, 1 token will be removed from the place P off; one token will be deposited in the place P on, hence 1 token in the place P off; one token in the place P on, by t repair fires. So, these are all the dynamics in the Petri Net, by enabling and firing of transitions. So, this is a very simple example. We are going to consider some more examples in this lecture.

(Refer Slide Time: 13:25)

**Reachability Analysis**

- ▶ Consider a PN containing  $M$  places and  $N$  transitions.
- ▶ A marking,  $M(t)$  of a PN is an  $M$  tuple,  $[m_1(t), m_2(t), m_3(t), \dots, m_M(t)]$  of non-negative integers, where  $m_i(t)$  denotes the number of tokens in place  $i$  ( $1 \leq i \leq M$ ) at any given instant of time,  $t$ . For a given reference time instant,  $t_0$ ,  $M(t)$  is called as the initial marking of the PN.
- ▶ A marking is reachable from another marking if there exists a sequence of transition firings starting from the original marking that result in the new marking.



Now, we are going to introduce another concept called a reachability analysis. Consider a Petri Net containing  $M$  places and  $N$  transitions. A marking  $M$  of  $t$  of a Petri Net is a  $M$  tuple, with  $m_1$  of  $t$ ,  $m_2$  of  $t$ ,  $m_3$  of  $t$ , and so on, till  $m$  suffix capital  $M$  of  $t$ , because we have  $m$  places of non-negative integers, where  $M_i$  of  $t$  denotes number of tokens in the place  $i$  at any time, at any given instant of time  $t$ . We have  $M$  places, therefore, if you take a  $M$  tuple and the number of tokens in each place, at any given instant of time, then that is called a marking. For a given reference time  $t$  naught,  $M$  of  $t$  naught is called a initial marking of the Petri Net;  $M$  of  $t$  naught is called as a initial marking of a Petri Net. A marking is reachable from another marking, if there exists a sequence of transition firings, starting from the original marking that result in the new marking.

Let us see the one simple example. For, we will have some more concepts, then we will go for the examples. The reachability set of a Petri Net is a set of all markings that are

reachable from its initial marking through any possible firing sequences of transitions. A reachability graph is a directed graph whose nodes are the markings in the reachability set, with directed arcs between the markings, representing the marking to marking transitions. The directed arcs are labeled with the corresponding transition, whose firing results in a change of the marking, from the original marking to the new marking.

(Refer Slide Time: 14:56)

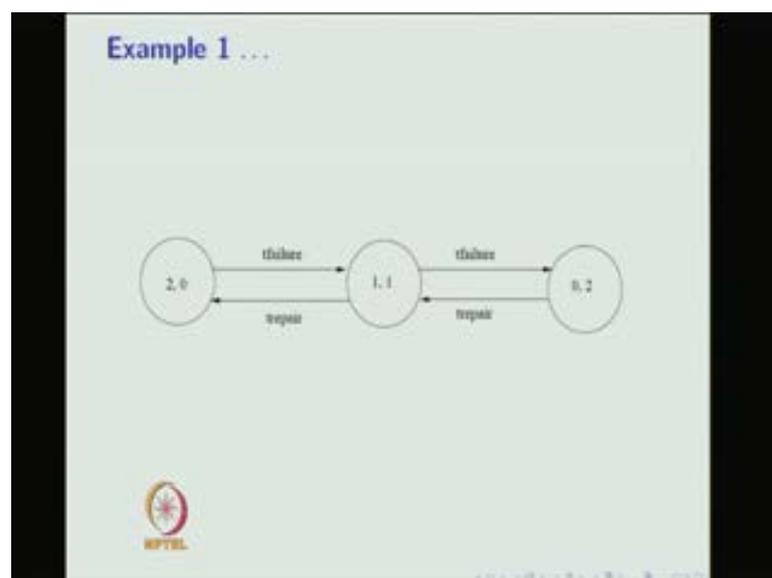
### Reachability Analysis ...

- ▶ The reachability set of a PN is the set of all markings that are reachable from its initial marking through any possible firing sequences of transitions.
- ▶ A reachability graph is a directed graph whose nodes are the markings in the reachability set, with directed arcs between the markings representing the marking-to-marking transitions.
- ▶ The directed arcs are labeled with the corresponding transition whose firing results in a change of the marking from the original marking to the new marking.



Navigation icons: back, forward, search, etc.

(Refer Slide Time: 15:49)



Consider this example. The example which we have considered earlier, example 1; that means, we have 2 places, place P on, the other place P off, initially two tokens in the place P on; no token in the place P off. Therefore, the marking will be a two tuple; number of tokens in each place form a marking. Therefore, the first place in P on; second place is P off; with that assumption, suppose, suppose the first place is P on, second place is P off, then the number of tokens at time 0 is 2 comma 0.

After t failure firings, the marking will be 1 comma 1. Again, the t failure firing, the marking will be 0 comma 2. From 1 comma 1, if t repair fires, then the marking will be 2 comma 0. From 0 comma 2, if t repair fires, then the marking will be 1 comma 1. Hence, the marking is a m tuple, number of tokens in the place i at any given instant of time and the marking is reachable whenever some sequence of a transition fires; and, the reachability graph is a directed graph, whose nodes are the markings in the reachability set, with directed arcs between the markings, representing the marking to marking transitions. Hence, in the same example, the markings are 2 comma 0, 1 comma 1, 0 comma 2. The marking is a, can be reachable from the marking can be reachable from 2 comma 0 to 1 comma 1 by the transition t failure fires. The marking 1 comma 1 is reachable to the marking 0 comma 2 with the t failure firing. The marking 0 comma 2 is reachable to the state, to the marking 1 comma 1 by t repair fires. Similarly, 1 comma 1 to 2 comma 0 by firing t repair transition. So, this is called a reachability graph.

(Refer Slide Time: 18:57)

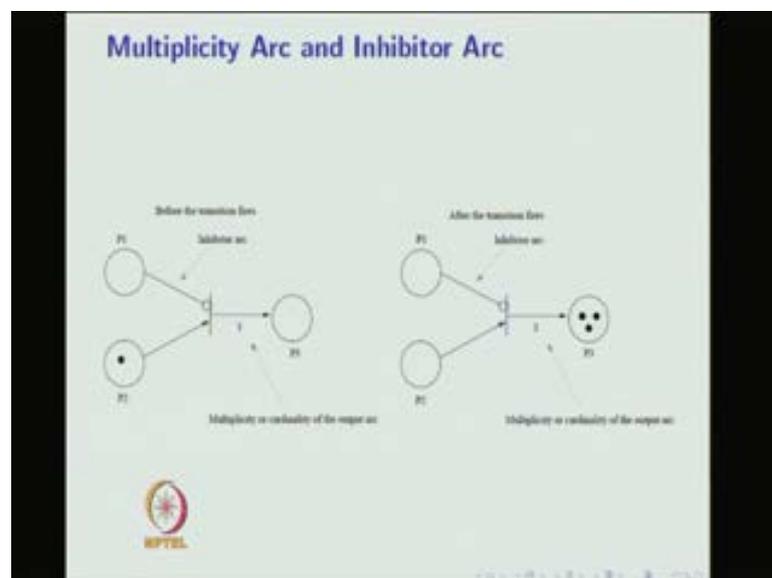
**Arc Extensions in Petri Net**

- ▶ Both input and output arcs in the PN are assigned a weight or a multiplicity (or cardinality), which is a natural number.
- ▶ If the multiplicity of an arc is not specified, then it is taken to be unity.
- ▶ An inhibitor arc drawn from place to a transition means that the transition **cannot fire** if the corresponding inhibitor place contains at least as many tokens as the cardinality of the corresponding inhibitor arc.
- ▶ If there exists an inhibitor arc with multiplicity  $n$  between a place and a transition, and if the place has  $n$  or more tokens, then the transition is inhibited even if it is enabled.
- ▶ Inhibitor arcs are represented graphically as an arc ending in a small circle at the transition instead of an arrowhead.

NPTEL

Now, we are moving extensions, arc extensions in Petri Net. Till now, we have considered very simple Petri Nets. Now, we are going for arc extensions. Both input and output arcs in the Petri Net are assigned a weight, or a multiplicity, or cardinality, which is a natural number. If the multiplicity of an arc is not specified, then it is taken to be unity. So, in the previous examples we have considered, has a, multiplicity is unity. The next extension in the Petri Net is an inhibitor arc. An inhibitor arc, drawn from place to the transition, means that, a transition cannot fire, if the corresponding inhibitor place contains at least as many tokens as the cardinality of the corresponding inhibitor arc. Usually, input arc and output arcs makes a transition, enabling and firing, removing the tokens, depositing the tokens with the multiplicity or cardinality. But the inhibitor arc, drawn from place to the transition, means that, the transition cannot fire, if the corresponding inhibitor place contains at least as many tokens as the cardinality of corresponding inhibitor arc. If there exists an inhibitor arc with multiplicity  $n$ , between a place and a transition, and if the place has  $n$  or more tokens, then the transition is inhibited, even if it is enabled.

(Refer Slide Time: 21:50)



The transition can enable, based on the conditions through the input arcs; but if there is a inhibitor arc, then it may be inhibited, based on the number of tokens in the corresponding input place. Input arcs are represented graphically as an arc ending in a small circle at the transition, instead of arrowhead. Usually, input arcs, as well as output

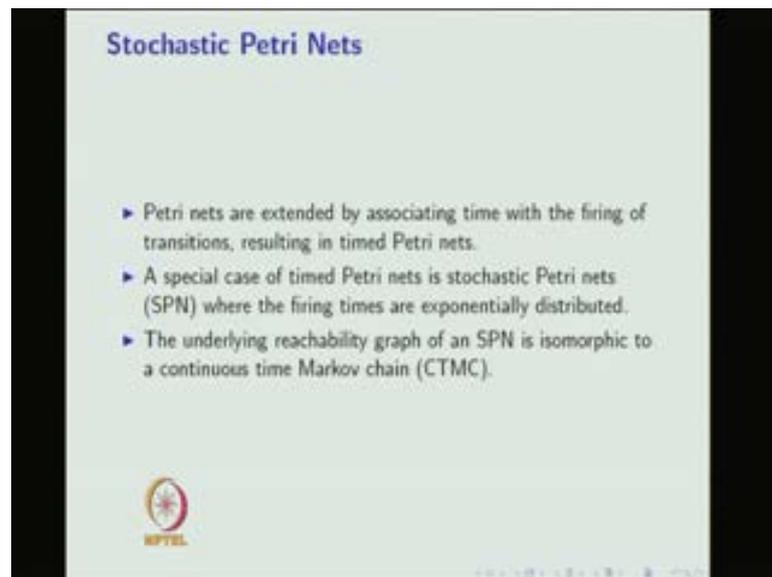
arcs, drawn with a arrowhead; but inhibitor arcs are represented graphically, as an arc ending in a small circle. We will see the example in this slide.

In this example, we have a, 3 places, P 1, P 2, P 3. We have 1 input place, 1, sorry, we have a 1 transition, 1 input arc, 1 output arc, 1 inhibitor arc. Here, whenever the number is written next to the arcs, that means, the multiplicity or cardinality of the output arc is number 3. If there is no number, natural number, written next to the arcs, that means, the default multiplicity is 1. Here also, the default multiplicity is 1; that means, if 1 or more token in the place P 1, even the transition is enabled by the condition through the input arcs for this place, input place, this transition may not fire, if 1 or more tokens in the place P 1. So, here, no token in the place P 1, whereas, 1 token in the place P 2; no token in the place P 3.

Hence, the transition enables and then fires, by removing 1 token in the place P 2 and 1 token deposited in the place P 3. Only the tokens will be removed from all the input places which are connected, the places connected with the input arcs to the transition and the tokens will be deposited to the, all the output places which are connected from transition to places through output arcs. So, in this example, after the transition fires, no token in the place P 1, no token in the place P 2 and 3 tokens in the place P 3, because the multiplicity of the output arc is 3. So, even though 1 token is removed from the place P 2, because the multiplicity of the output arc is 3, hence, 3 tokens will be, multiplicity is 3, therefore, 3 tokens will be deposited at same time in the place P 3.

Now, we are moving into the extension of Petri Nets into Stochastic Petri Nets. Petri Nets are extended by associating time with the firing of transitions, resulting in timed Petri Nets. A special case of timed Petri Nets is Stochastic Petri Nets, in other words, SPN, where the firing times are exponential distributed, exponential distribution. So, whenever the firing times of all transitions are exponential distribution, then the corresponding timed Petri Nets are called Stochastic Petri Nets. Every Petri Net, one can get the reachability graph. The underlying reachability graph of a Stochastic Petri Nets is isomorphic to a continuous time Markov chain. For a Stochastic Petri Net, the underlying reachability graph is isomorphic to a continuous time Markov chain.

(Refer Slide Time: 24:37)

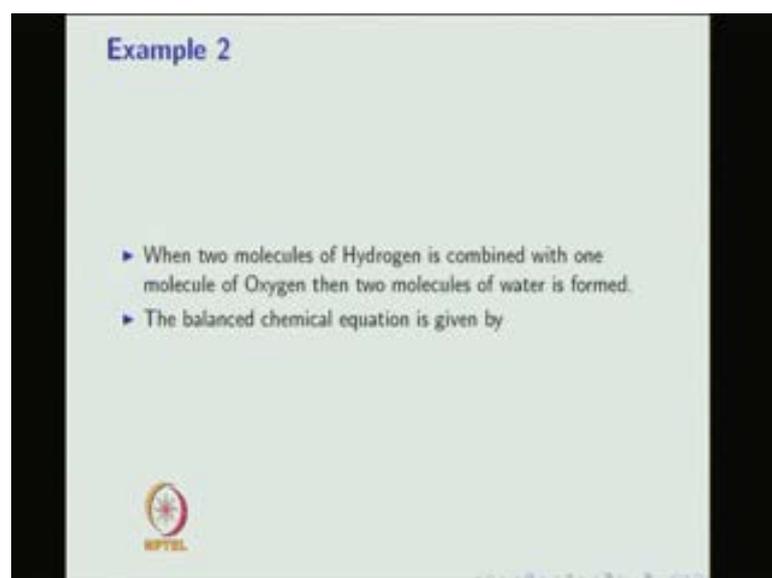


**Stochastic Petri Nets**

- ▶ Petri nets are extended by associating time with the firing of transitions, resulting in timed Petri nets.
- ▶ A special case of timed Petri nets is stochastic Petri nets (SPN) where the firing times are exponentially distributed.
- ▶ The underlying reachability graph of an SPN is isomorphic to a continuous time Markov chain (CTMC).

NPTEL

(Refer Slide Time: 26:04)



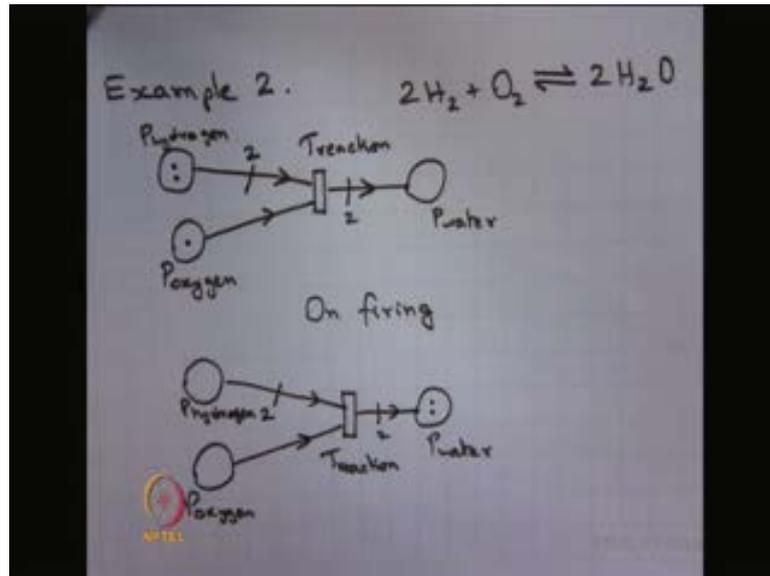
**Example 2**

- ▶ When two molecules of Hydrogen is combined with one molecule of Oxygen then two molecules of water is formed.
- ▶ The balanced chemical equation is given by

NPTEL

Let us see, through the examples. When 2 molecules of hydrogen is combined with 1 molecule of oxygen, then 2 molecules of water is formed. The balanced equation is written, the balance...

(Refer Slide Time: 26:31)



The balanced chemical equation is given by  $2\text{H}_2 + \text{O}_2 \rightleftharpoons 2\text{H}_2\text{O}$ . So, this is the balanced chemical equation. For this scenario, we have made 3 places, P hydrogen, P oxygen, P water; and, we have one transition, that is called a t reaction. The multiplicity of a input arc from the place P hydrogen to the t transition, that is 2; whereas, the multiplicity of the arc from P oxygen to the t reaction that is 1; whereas, the multiplicity of a output arc from the transition to the place P water, that is 2. On firing of the transition t reaction, 2 tokens, because the multiplicity is 2, 2 tokens will be removed from the place P hydrogen. The multiplicity is 1; therefore, 1 token will be removed from the place P oxygen. Since it is a rectangle bar, that means, the timed transition follows the time of a, timed transition follows exponential distribution.

So, after that, so 2 tokens will be removed from the place P hydrogen, 1 token will be removed from the place P oxygen. On firing of a, the reaction t reaction, 2 tokens will be deposited in the place P water, because the multiplicity is 2. And here, we made the assumption the reaction time is a exponential distribution. Hence, we made a Stochastic Petri Nets and we make the model using Stochastic Petri Nets of this chemical reaction. Now, we are moving to the, another example, that is, consider the M M 1 queuing model.

(Refer Slide Time: 29:11)

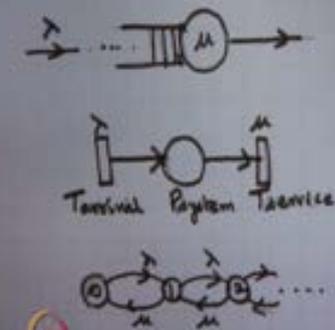
**Example 3**

- ▶ Consider  $M/M/1$  queueing system.
- ▶ Arrival follows Poisson process with rate  $\lambda$ .
- ▶ Service time is exponentially distributed with rate  $\mu$ .
- ▶ One server serving one job at a time.



(Refer Slide Time: 29:25)

**Example 3.  $M/M/1$  Queueing System**



Arrival System Service



Here, the arrival follows Poisson process, with the rate  $\lambda$ . Service time is exponential distribution with the rate  $\mu$ , only one server, only one server, serving one job at a time. Therefore, this is the queueing model of  $M/M/1$  queue. So, this can be represented by the Stochastic Petri Net with the 1 place called P system; and, we have 2 transitions; one is called the transition for arrival, and the other transition is for service. Both are timed transition. Assume that, each one is exponential distribution and both are independent and arrival follows the Poisson process with the rate  $\lambda$  and service follows exponential distribution with the rate  $\mu$ . Hence, the Stochastic Petri Net of the

M M 1 queuing model will be shown here. Whenever the transition  $t$  arrival enables, and then fires, 1 token will be deposited in the place  $P$  system.

Since there is no input place, that means, this will be enabling, not removing any tokens, but only depositing 1 token in the place, that is the output place  $P$  system. Whenever there is a token in the place  $P$  system, the transition  $t$  service will enable first, since this timed transition is exponential distribution with the parameter  $\mu$ , after random amount of time, this transition will fire; means, one token will be removed from the place  $P$  system, and since there is no output arc from this place, no token will be deposited to any places; because there is no place also, which connects the transition to place with the output arcs. So, this is a very typical, very important Stochastic Petri Net of a M M 1 queuing model, in which there is no input arc; there is no output arc also. So, here, the number of tokens in the place  $P$  system is same as number of customers in the queuing system; both are one and the same.

Now, we can go for creating the reachability graph. So, the reachability graph is nothing but the number of tokens in each place, that is, a marking; then connect the marking with the possible transitions. So, the markings will be, the possible number of tokens in the place  $P$  system will be 0, 1, 2 and so on. Now, each marking can be connected with the transitions. The marking 1 is reachable from the marking 0 by enabling the transition  $t$  arrival and then firing; that will happen with the rate  $\lambda$ . Therefore, instead of writing the transition name, we can write the corresponding rate, because each transition is a timed transition, as well as a, we assumed that, it is exponential distribution with some rate. Hence, the marking 0, the marking 1 is reachable from the marking 0 by firing the transition  $t$  arrival with the rate  $\lambda$ . Similarly, the marking 0 will be reachable from the marking 1 by firing the transition  $t$  service with the rate  $\mu$ . Similarly, the marking 2 will be reachable from the marking 1 by firing the transition  $t$  arrival with the rate  $\lambda$ . Similarly, 2 to 1 will be the, will be with the rate  $\mu$ .

Similarly, you can connect one marking with the other markings. Therefore, this is the reachability graph of M M 1 queuing model. So, this is the, this is the M M 1 queuing model and the corresponding Stochastic Petri Net is this, and the underlying reachability graph is this. And, if you see, this reachability graph is isomorphic to the continuous time Markov chain of M M 1 queuing system. So, whenever all the timed transitions are exponential distributions and are independent, then the underlying reachability graph is

isomorphic to the continuous time Markov chain. So, once we get the continuous time Markov chain from the Petri Net, then it is the easy way; instead of a designing a system, you can make a Stochastic Petri Net for the system, for any system, then you can visualize for the underlying reachability graph, which will be the, which will be the same as the continuous time Markov chain of the corresponding system.

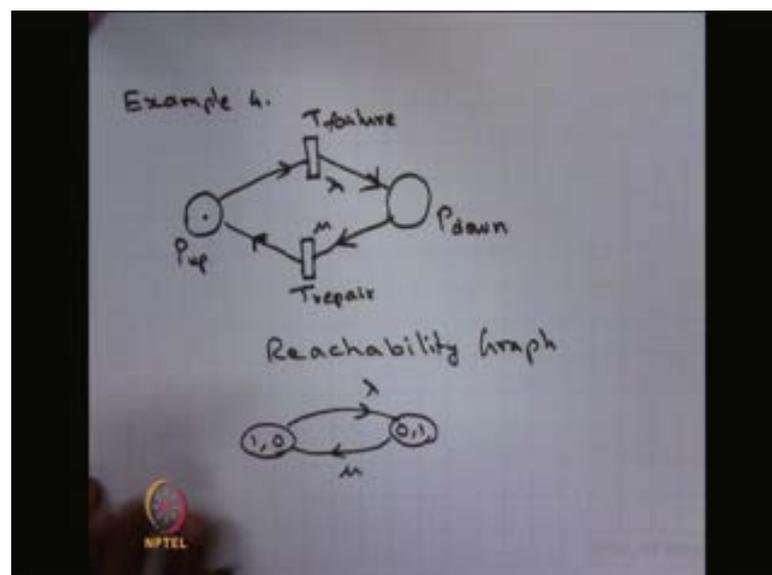
(Refer Slide Time: 36:41)

**Example 4**

- ▶ A working processor whose time to failure is exponentially distributed with rate  $\lambda$ .
- ▶ Time to repair is exponentially distributed with rate  $\mu$ .
- ▶ Let state up denotes the processor in working and state down denotes the processor in not working.
- ▶ Consider stochastic process  $\{X(t), t \geq 0\}$  with  $X(t)$  denotes the state of the processor at time  $t$ .
- ▶ Assume that, the system is in up state at time 0.



(Refer Slide Time: 37:08)

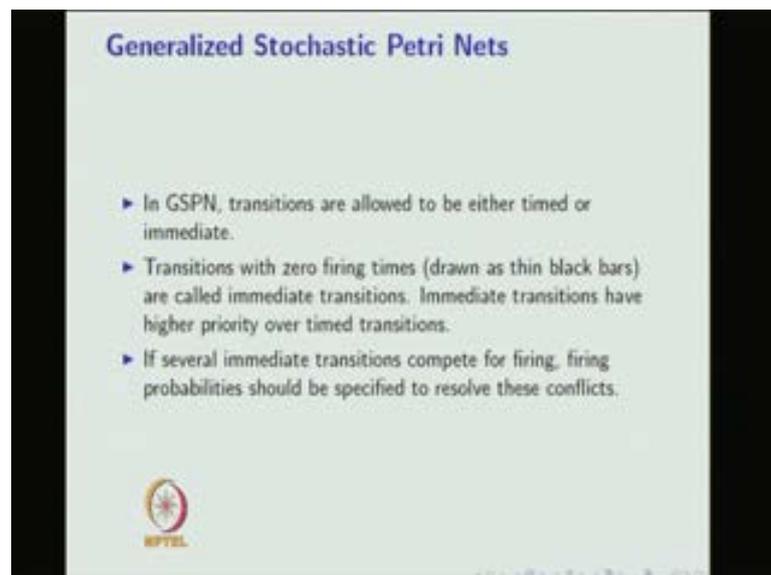


Now, we are moving in to the fourth example; a working processor, whose time to failure is exponential distribution with the rate lambda; time to repair is exponential

distribution with the rate  $\mu$ . Now, you consider this diagram. The state up denotes the processor is working; so that is, this place, the corresponding place is called P up, and the state down denotes the processor is not working. So, the corresponding Stochastic process is  $x$  of  $t$ , where  $x$  of  $t$  denotes the state of the process at time  $t$ . At time 0, we assume that, the system is in the up state. Therefore, we deposited one token in the place P up; at time 0, assume that, the system is in the up state. Hence, the transition  $t$  failure can enable, not the transition  $t$  repair; because to enable the transition  $t$  repair, we need one token in the place P down; and we assumed that, the  $t$  repair is exponential distribution with the parameter  $\mu$ ; and also, we assumed that, the failure is also exponential distribution with the parameter  $\lambda$ .

So, now, the  $t$  failure can enable and it fires, after the random amount of time, which is exponential distribution with the parameter  $\lambda$ ; then 1 token will be removed from the place P up; 1 token will be deposited in the place P down. Once 1 token is deposited in the place P down, now, the transition  $t$  failure, sorry,  $t$  repair can enable, after a random amount of time which is exponential distribution with the parameter  $\mu$ , 1 token will be removed from the place P down; 1 token will be deposited in the place P up. Hence, the possible markings are 1 comma 0, 0 comma 1.

(Refer Slide Time: 40:22)

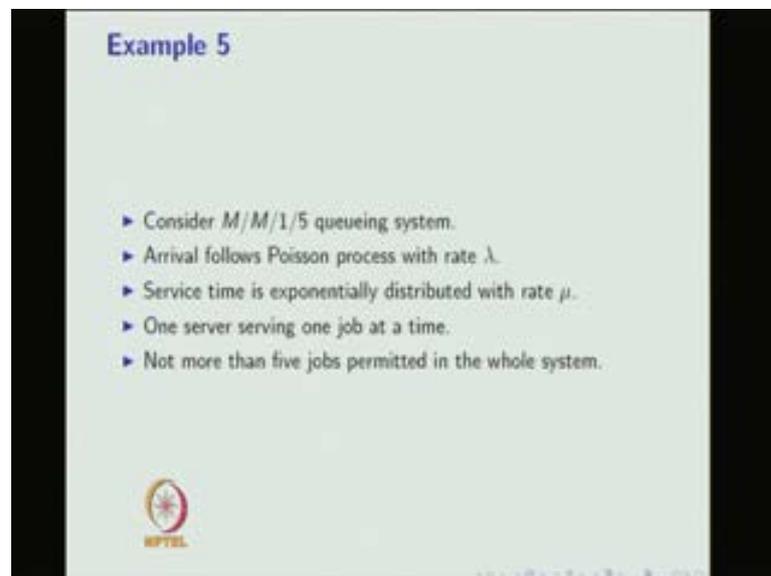


Therefore, the arcs between the marking 1 comma 0 and 0 comma 1 are a arc from 1 comma 0 to 0 comma 1 with the rate  $\lambda$ , 0 comma 1 to the 1 comma 0, that will be

the, with the rate  $\mu$ ; and, this is the continuous time Markov chain for this example, for this system. So, the system can be studied via, by developing first Stochastic Petri Net; then see the underlying reachability graph; once you get the reachability graph that is same as the continuous time Markov chain.

Now, we are moving in to generalized Stochastic Petri Nets. In other words, it is called GSPN. In GSPN, transitions are allowed to be either timed or immediate; if you recall, in the Petri Nets, the transitions are immediate; whereas, in the Stochastic Petri Net, the transitions are timed and we assumed that, each timed transition is exponential distribution. Whereas, in GSPN, the transitions are allowed to be either timed, or immediate; that means, you have the combination of, few transitions could be timed; few transitions could be immediate also. Transitions with zero firing times, that is nothing but the immediate transitions, drawn as the thin black bars are called immediate transitions. Immediate transitions have higher priority over timed transitions. If several immediate transitions compete for firing, firing probabilities should be specified to resolve these conflicts.

(Refer Slide Time: 41:57)



**Example 5**

- ▶ Consider  $M/M/1/5$  queueing system.
- ▶ Arrival follows Poisson process with rate  $\lambda$ .
- ▶ Service time is exponentially distributed with rate  $\mu$ .
- ▶ One server serving one job at a time.
- ▶ Not more than five jobs permitted in the whole system.



Now, let us consider the example for the GSPN. Consider  $M/M/1/5$  queueing system. Here, arrival follows the Poisson process with the rate  $\lambda$ ; service time is exponential distribution with the rate  $\mu$ ; one server serving one job at a time; not more than 5 jobs permitted in the whole system.



the possible number of tokens in the places, we have only 1 places; therefore, the number of tokens in the places will be 0, 1, 2, 3, 4 and 5. The markings can be possible with the transition firing with the rate  $\lambda$ , with the rate  $\lambda$  and so on; whereas, 5 to 4, 4 to 3, 3 to 2, 2 to 1, 1 to 0, the transition firing with the rate  $\mu$ . So, this is the reachability graph for the, for the above GSPN. This is same as the continuous time Markov chain of M M 1 5 queuing system. Suppose, your interest is to find out, what is the steady state probability that the system is empty; that means, what is the probability that, the system is empty; that is,  $\pi_0$  in steady state.

That is same as, what is the probability that, what is the steady state? Probability that the place P system has no tokens; both are one, and the same. In the CTMC, what is the probability that, the system is in the state 0, in steady state? That is same as, what is the probability that, in steady state, the place P system has no tokens. Therefore, whatever the measures you want it for the system, instead of making continuous time Markov chain, you can make a Stochastic Petri Net, or GSPN, and get the measures in the Petri Net level, instead of the state in the Markov chain, whenever you are able to get the underlying reachability graph is isomorphic to the continuous time Markov chain.

(Refer Slide Time: 48:18)

**Stochastic Reward Nets**

- ▶ A marking  $M(t)$  in an SPN is called vanishing if at least one immediate transition is enabled at time  $t$ .
- ▶ A marking which is not vanishing is called a tangible marking.
- ▶ Let  $\hat{M}(t)$  be the set of all possible markings at time  $t$ . A guard function  $g_T(t)$  associated with transition  $T$ , is a Boolean function defined over  $\hat{M}(t)$  (i.e.,  $g_T(t) : \hat{M}(t) \rightarrow \{0, 1\}$ ), such that the transition  $T$  does not fire if  $g_T(t) = 0$  (even if enabled and not inhibited), and fires if it is enabled, not inhibited and  $g_T(t) = 1$ .
- ▶ A reward is a non-negative weight associated with each marking.

NPTEL

Now, we are moving into the, another extension of Petri Net. We started with the Petri Net; then we discussed Stochastic Petri Net; then we have discussed Generalized Stochastic Petri Net. Now, we are going to discuss Stochastic Reward Nets. A marking

$M$  of  $t$  in a Stochastic Petri Net is called vanishing, if at least 1 immediate transition is enabled at time  $t$ . A marking which is not vanishing is called a tangible marking. Let  $\bar{M}$  of  $t$  be the set of all possible markings at time  $t$ . A guard function  $g$  of  $t$  associated with the transition  $T$ , is a Boolean function, defined over  $\bar{M}$  of  $t$  such that, the transition  $T$  does not fire, if  $g$  of  $T$  of small  $t$  is equal to 0. That means, even if it enabled and not inhibited; and it fires, if it is enabled and not inhibited, and  $g$  of  $t$  is equal to 1. A reward is a non-negative weight associated with each marking. So, we are first defining what is a reward and what is guard function. Along with the reward and guard function, we are going to define Stochastic Reward Nets.

(Refer Slide Time: 49:56)

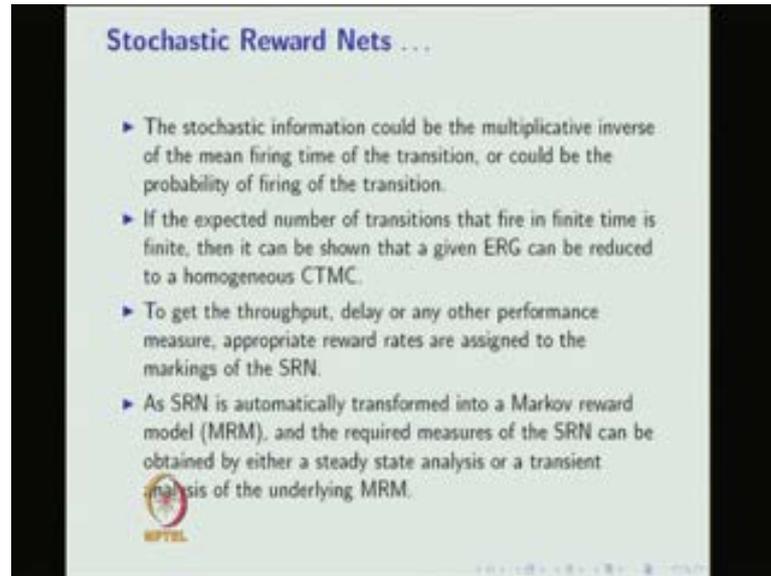
**Stochastic Reward Nets ...**

- ▶ A stochastic reward net (SRN) is an extension of an GSPN that allows extensive marking dependency and expresses complex enabling or disabling conditions for transitions (a) through guard functions, or (b) by assigning one or more reward rate(s) to each tangible marking.
- ▶ For a given SPN or SRN, an extended reachability graph (ERG) is a directed graph with the markings of the reachability set as the nodes and a directed edge from node  $M_1(t)$  to  $M_2(t)$  if  $M_2(t + T)$  can be obtained from  $M_1(t)$  by firing of a single transition.
- ▶ To each edge in the ERG, some stochastic information is attached.



A Stochastic Reward Net is an extension of Generalized Stochastic Petri Net, that allows extensive marking dependency and expresses complex enabling or disabling conditions for transitions through guard functions; or, by assigning one or more reward rates to each tangible marking. That means, you should know, what is tangible marking and you should know, what is guard function; and also, you should know, what is reward rates. By using these three concepts, the extension of GSPN will be called Stochastic Reward Nets. Or in other words, it is SRN. For a given SPN, or SRN, an extended reachability graph is a directed graph, with the marking of the reachability set as the nodes and a directed edge from node  $M_1$  of  $t$  to  $M_2$  of  $t$ , if  $M_2$  of  $t$  plus capital  $T$  can be obtained from  $M_1$  of  $t$ , by firing of a single transition capital  $T$ .

(Refer Slide Time: 51:22)



**Stochastic Reward Nets ...**

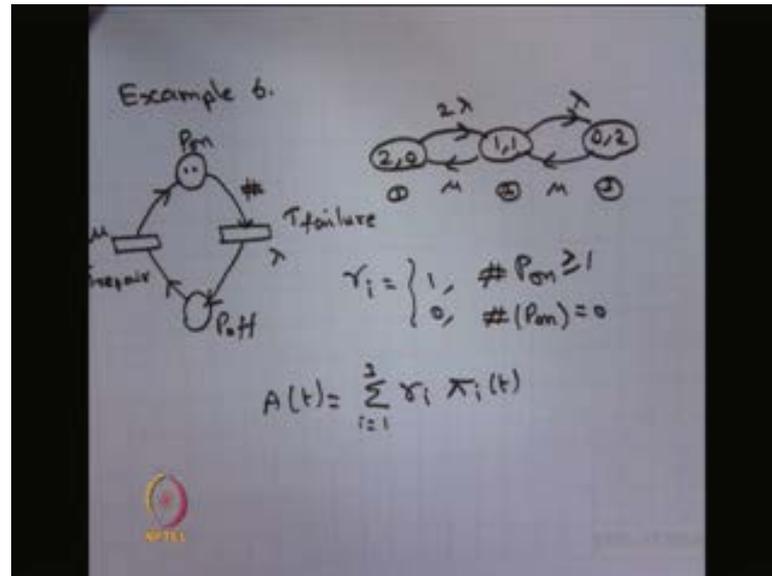
- ▶ The stochastic information could be the multiplicative inverse of the mean firing time of the transition, or could be the probability of firing of the transition.
- ▶ If the expected number of transitions that fire in finite time is finite, then it can be shown that a given ERG can be reduced to a homogeneous CTMC.
- ▶ To get the throughput, delay or any other performance measure, appropriate reward rates are assigned to the markings of the SRN.
- ▶ As SRN is automatically transformed into a Markov reward model (MRM), and the required measures of the SRN can be obtained by either a steady state analysis or a transient analysis of the underlying MRM.



To each edge in an extended reachability graph, some Stochastic information is attached. The Stochastic information could be the multiplicative inverse of the mean firing time of the transition, or could be the probability of firing of the transition. If the expected number of transitions that fire in a finite time is finite, then it can be shown that a given extended reachability graph can be reduced to a homogeneous continuous time Markov chain. Not all the extended reachability graph will be reduced into the homogeneous Markov chain; whenever the condition of expected number of transitions that fire in a finite time is finite, then only the extended reachability graph can be reduced to a homogeneous continuous time Markov chain.

To get the throughput, delay, or any other performance measure of the system which you are considering, the appropriate reward rates are assigned to the markings of the SRN. You know the meaning of marking. So, when you assign reward rates to the marking, then you can find the measures of your interest. As SRN is automatically transformed into the Markov Reward Model, the required measures of the SRN can be obtained, by either a steady state analysis, or transient analysis of underlying Markov Reward Model. So, by assigning proper reward rates, you can study the Markov chain with the rewards, called a Markov model, Markov Reward Model. The same way here, the Stochastic Petri Net, along with the, assigning reward rates, you can call it as a Stochastic Reward Nets.

(Refer Slide Time: 53:43)

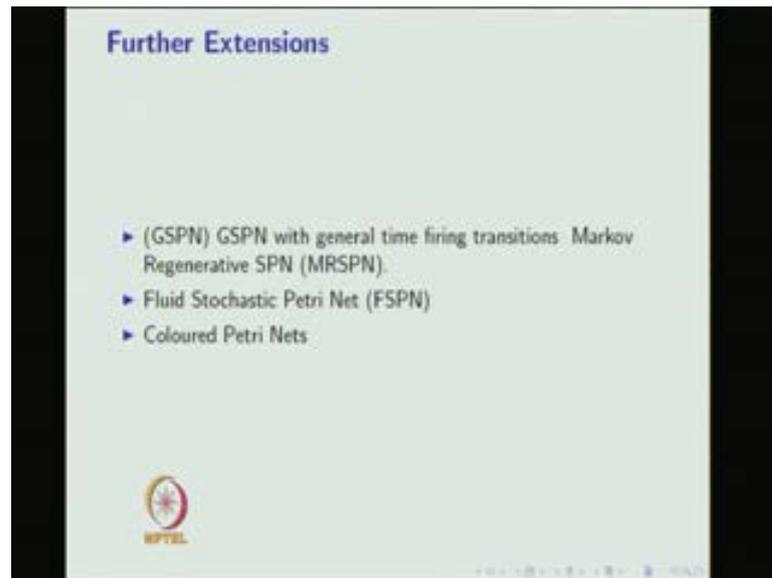


Let us see the simple example of a Stochastic Reward Net. Consider the example 6, with the 2 places P on and P off and 2 transitions, one is called T failure, and other transition is called T repair. This transition is a input arc and this is the output arc. Similarly, this is the input arc and this is the output arc. Assume that, a T failure is exponential distribution with the parameter lambda; the T repair, that is also exponential distributed with the parameter mu. We assume that, we have a two components. We have two components in the system. Each component failure is exponential distribution with the rate lambda. Therefore, there is a another symbol asterisk; if the 2 tokens in the place P on, then the rate will be 2 lambda; if one token in the place P on, then the rate will be lambda; for that we have used the asterisk; that means, the actual rate of this transition will be lambda times number of tokens in the input place, corresponding to the input arc.

So, here if two tokens in the place P on, then the transition T failure has a rate 2 lambda; otherwise, it is a lambda. Therefore, the reachability graph will be 2 comma 0, 1 comma 1, 0 comma 2; and the corresponding transitions are 2 lambda, lambda, 0 to 2 to 1 comma 1, that is mu, 1 comma 1 to 2 comma 0, that is mu. So, this is same as the time homogeneous continuous time Markov chain. Suppose, my interest is to find out the availability; whenever two tokens in the place P on or one token in the place P on, that means the system is available, so by assigning the reward rate r i is equal to 1 for the number of tokens in the place P on, if it is greater than or equal to 1, that is 1 or if it is 0, number of tokens in the place P on, that is equal to 0; that reward rate is 0; then the

availability is nothing but summation  $\sum_i r_i p_i$  of  $t$ , where  $i$  is nothing but the marking. So,  $i$  is nothing but, this is the marking; suppose, you treat this as the marking one, and you treat this as the marking two, and this is as the marking three. Therefore, here,  $i$  is running from 1 to 3. So, by assigning a proper reward rates  $r_i$ 's, you can get the availability.

(Refer Slide Time: 57:27)



The further extensions of the Stochastic Petri Nets are Markov reward Regenerating Stochastic Petri Nets, Fluid Stochastic Petri Nets and Colored Petri Nets. These are all the references.