Point Set Topology

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Week 02

Lecture 09

So, in the previous lecture we proved that three important maps, which we see very often are continuous. So, in this lecture let us continue with some general properties of continuous functions. So, we begin with a lemma: let X, Y and Z be topological spaces and suppose we are given maps $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ such that f and g are continuous. Then the assertion of the lemma is $g \circ f$ is continuous, and the proof is obvious and is left as an exercise. This is one very basic lemma and another very basic lemma is the following: Let $f : X \rightarrow Z$ be a continuous map, and let $Y \subseteq X$ be a subset with the subspace topology. Then the restriction of f to Y, that is the map $f \circ i : Y \rightarrow Z$ is continuous.

Proof: So, f is already given to be a continuous map and we had proved earlier that with the subspace topology, the inclusion $i : Y \rightarrow X$ becomes continuous. thus, f is continuous and i is continuous, applying the previous lemma, we get that $f \circ i$ is continuous. This proves this lemma. In order to prove the next lemma, which is also very important and very basic, we need some notations.

So, let us introduce that. So, here $f : X \rightarrow Z$ is a continuous map. Let us assume that the image $f(X)$ is a subset of Y, which is a subset of X. Let $i : Y \rightarrow Z$ denote the inclusion. What this means is that the map f factors as so we have $f: X \rightarrow Z$ and it turns out that the image of f is actually contained in Y, which means that there is this map $f_0 : X \rightarrow Y$, and what is f_0? $f_0(X)$ is simply $f(X)$.

So, with this notation (as above) the map f_0 is continuous. So, let me emphasize that Y has the subspace from Z. So, let us prove this lemma. Let $U \subseteq Z$ be open. Then $i^{-1}(U)$ is simply $U \cap Y$, is open, and every open subset of Y has this description.

So, to show that f_0 is continuous, it is enough to show that $f_0^{-1}(U \cap Y)$ is open in X, for every U open in Z. But $f_0\{-1\}(U \cap Y)$ can be written as $f_0^{-1}(i^{-1}(U))$, which can be written as $(i \circ f_0)^{-1}(U)$, and $i \circ f_0 = f$ and so, this is precisely equal to $f^{-1}(U)$, and as f is continuous, $f^{-1}(U)$ is open in X. Thus $f_0^{-1}(U \cap Y)$ is open in X and so f_0 is continuous. So, this proves three basic lemmas which are related to subspace topology and now we have a similar result, very basic result about continuous maps related to the product topology. Let us quickly come to that.

Proposition, or before we prove this proposition, let us introduce some notation. Let X be a topological space. Let I be a set and suppose that the size of I could be infinite. Suppose for each i in I, we are given a topological space Y_i. In addition to this, suppose we are also given maps of sets f_i : X -> Y_i.

Given this data, we can consider the map. We can define a map f : X -> product of Y_i's. So, what is F? F(x) just puts together all these maps f_i, for i in I. So, this is equal to (f_i(x))_{i in I}, this is an element of this product. So with this notation we have the following very useful proposition: The map F is continuous iff f_i's are continuous for all i in I.

Let us prove this. Obviously, to make sense of a continuous map, we need two topological spaces and X has a topology. This product is being given the product topology. So, let us prove this proposition. First let us assume that F is continuous.

Now, since this product has the product topology, the projection maps, recall we had seen the projection maps. So, what are these? We can look at the projection to the j^{th} coordinate. What this does is given a tuple (y_i)_{for i in I}, it sends it to y_j and we had denoted this projection map by p_j. So, these projection maps are continuous, and since composition of continuous maps is continuous. So, we are given F and here we are given p_j.

So, the composite is continuous, but the composite is precisely f_j. Therefore here x ->F(x) -> f_i(x) and then we are projecting (F) on to the jth coordinate. We just get f_j(x) as the composite map, it follows that f_j : X -> Y_j is continuous for all j in I. So, this proves one part of the proposition. Conversely, let us assume that each f_i : X -> Y_i is continuous.

So, recall that the product topology has as basis the sets product of U_i's where each U_i is open in Y_i, and the set J ={those indices j in I such that U_j is not equal to Y_j}, this is finite. And as we have seen many times now, to show that F is continuous, it is enough to show that the inverse image of basic open sets is continuous. If the topology has a basis, and the product topology is given by a basis. Thus it is enough to show that F^{-1} of any basic open set is open. But notice that f^{-1}(product of U_i's, where U_i = Y_i for all but finitely many U_i's), an easy set theoretic check which is left as an exercise, is equal to intersection of f^{-1}(U_i)'s for i in I.

This is an easy check, and so let us consider this right hand side. So we can divide the intersection into two parts : intersection over J, let us call this set J, (f_i)^{-1}(U_i) intersected with intersection over all i not in J of (f_i)^{-1}(U_i). But now, note that if i is not in J, then U_i = Y_i, and therefore (f_i)^{-1}(U_i) = X_i. So, this is equal to intersection of (i in J) (f_i)^{-1}(U_i) intersected with the intersection of (i's not in J) X_i. This is simply equal to intersection of i in J of (f_i)^{-1}(U_i).

Now as each f_i is continuous, this implies (f_i)^{-1}(U_i) is open in X, and as cardinality of J is finite and finite intersections of open sets are open, this implies that this intersection of (f_i)^{-1}(U_i) is open, which implies that f^{-1}(product of U_i's) is open, which implies that f is continuous. Let us just make a remark, and this remark is one of the reasons why we reject the box topology. So, the above proposition fails if this product is given the box topology. Recall the diagonal exercise given before. Let me say, OK, so having made this remark.

This proposition has some nice consequences. What are these? Let us see some consequences. So, using this proposition and the earlier theorems we had proved about addition and multiplication: Let f, g be two maps from X to ℝ which are continuous, then we have maps f + g. So, how do we define this map? This is a map from X to ℝ and it is defined as follows: (f+g)(x) := f(x) + g(x). Note that f(x) is a real number, g(x) is a real number, and therefore we can add them, and similarly we have another map fg : X -> ℝ.

This is defined as (fg)(x) := f(x)g(x). The content of this proposition is: these maps are continuous, and the proof is easy. First, using the previous proposition, we get a continuous map from X -> ℝ^2, given by x +->(f(x), g(x)), and this map is continuous as both f and g are continuous. So, for ℝ×ℝ, we have the addition map and we have the multiplication map. Since the standard topology, this was an exercise, since the standard topology on ℝ^2 is the same as the product topology, and we proved that 'a' and 'm', both these maps are continuous in the standard topology, this implies that they are also continuous in the product topology.

Therefore what we can do is, we can look at F: X -> ℝ^2, where F(x) = (f(x), g(x)), and we have the addition map, and we have the multiplication map. So, thus the composite of continuous functions being continuous implies aoF = f+g and moF = fg, both these are continuous. So, this completes the proof of the proposition. So, before we end today's lecture we will prove one final proposition which is similar to the above. So, this one says the following: let f, g : X -> ℝ be continuous functions such that g(x) is nonzero for all x in X.

Recall what ℝ* was, this is notation for ℝ\{0}. So, then we can define a function h(x) := f(x)/g(x) where h : X -> ℝ, this function is well defined because g(x) is not 0, and so h is continuous. So, the proof is very similar to the proof of the early propositions. Let us prove. First note that as the image of g(x) is contained in ℝ* using our earlier result, it follows that, Let me just mention what that earlier result is.

I am talking about this lemma. So using a earlier result, it follows that the function. So, g can be viewed as a function g_0 : X -> ℝ* is continuous when ℝ* has the subspace topology. In all these propositions and theorems, ℝ has the standard topology. So, we will abuse notation and continue to denote g_0 by g. So, g is actually a map from X to ℝ* and ℝ* being the subspace topology, g is continuous.

Therefore, now define a function (1/g) : X -> ℝ*. This is defined as First X -> ℝ*, we have g, and on ℝ* we have this continuous map which takes y to 1/y. So, g is continuous and maps y to 1/y which is continuous. Therefore the composition, which sends x to first g(x), and then this goes to 1/g(x).

So, the composition is continuous. This composite function shall be denoted (1/g): X -> ℝ*, but now we will just take the inclusion of ℝ* into ℝ, and this composition is also continuous because ℝ* has the subspace topology. Finally, we get that, once again we will abuse notation, the function (1/g): X -> ℝ, this is given by x +-> 1/g(x) is continuous. With this, now we define a map X -> ℝ×ℝ, which is given by x +-> (f(x), 1/g(x)). So, both the coordinate functions are continuous, and this implies that this function, say F, is continuous and then we have the multiplication map. So, the multiplication map is also continuous.

This shows that x +-> (f(x)/g(x)) is continuous. So let me end by making a remark. In the above proposition, we proved about, or in the theorem, we proved m is continuous when ℝ×ℝ, which is $\mathbb{R}^2$, has a standard topology. However, note that the standard topology on ℝ×ℝ is equal to the product topology, thus m is also continuous when ℝ×ℝ has a product topology. Over here, note that ℝ×ℝ has a product topology, and it is the property of the product topology that we have used that this function F is continuous, because both the coordinate functions are continuous.

Finally, we remark that the above results show that the set of real valued functions on a topological space X has nice algebraic properties. We can add them, we can multiply them, and if g is nonzero we can form the function f/g. So, we will end this lecture here.