Point Set Topology
Prof. Ronnie Sebastian
Department of Mathematics
Indian Institute of Technology Bombay
Week 02
Lecture 07

So, in the first part of this course we saw several examples of topological spaces. Now, we will see the notion of continuous maps between these, which will help us study their topological properties later on. So, this lecture is about continuous maps. Definition of continuous map: So, let X and Y (the definition is very simple) be topological spaces and let f : X -> Y be a map of sets. We are just given a map of sets between X and Y, and X and Y happen to be topological spaces, they have topologies. So, we say that f is continuous if for every open subset V of Y So, recall what this means.

Y is a topological space and therefore it has a topology $\mathcal{T}$. So, we say that V is open in Y if V belongs to that topology $\mathcal{T}$, the set f^{-1}(V), which is contained in X, is open in X, or in short given an open set in Y, its inverse image should be open in X. So, recall the meaning of the definition of f^{-1}(V), this is those x in X such that f(x) belongs to V. Let us see some very simple examples of continuous maps.

Let X be a topological space, and consider the identity map "f". This f : X -> X is given by the identity map. So, it is f(x) = x. We claim that f is continuous. See that we need to let U be an open subset.

So, what we need is f^{-1}(U) should be open. But f is the identity map. So, this is simply equal to U. f^{-1}(U) is just equal to U. Therefore, as U is open in X, this implies f^{-1}(U) is open, and this implies f is continuous.

So, this is a really silly example. Let us see a slightly more complicated example. Let X be a topological space. I do not have to write this always. Let X be a topological space and let Y ⊆ X, and we give Y the subspace topology.

Let us denote this inclusion map by i. We claim that, now Y now has a topology which is a subspace topology coming from X and of course, X is has a topology. So, we can ask if this inclusion map is continuous. We claim that I is continuous. What do we need to do? So, let U ⊆ X be an open subset.

We need to show that this i^{-1}(U) is open in Y, in the subspace topology. But i^{-1}(U) is simply U∩Y by definition. I mean what is i^{-1}(U)? i^{-1}(U) is those y in Y such that i(y) belongs to U. But i simply the inclusion map. So, this is the same as saying that y belongs to U.

Therefore, $i^{-1}(U) = U \cap Y$. So, this shows that i is continuous, the inclusion map is continuous. Let us take a third example. So, this is a same example as above. Let X be a topological space and let $i: Y \to X$ be inclusion of a subset.

Let $\mathcal{T}$ be a topology on Y. So, here we are saying $\mathcal{T}$ is any topology on Y, it need not be the subspace topology. For instance $\mathcal{T}$ could be the discrete topology on Y for example, or it could be any other topology on Y, but we have this condition that the map i is continuous. We are given that this map i is continuous. Then we claim that $\mathcal{T}$ contains the subspace topology $\mathcal{T}_Y$ on Y.

We are given that $\mathcal{T}$ is a topology on Y, such that the inclusion map is continuous and we need to show that $\mathcal{T}_Y$ is contained in $\mathcal{T}$. How do we see this? This is easy. So let U be an element of $\mathcal{T}_Y$ and recall what the subspace topology was. Let us call this V. This (the topology) is all $U \cap Y$, where U is open in X.

So, given that the inclusion map is continuous when Y is given the topology $\mathcal{T}$. So, what this means is, and we have to show that $\mathcal{T}_Y$ is contained in $\mathcal{T}$. We just we start with any V in $\mathcal{T}_Y$ then there is $U \subseteq X$ that is open, such that $U \cap Y$ which is precisely equal to $i^{-1}(U) = V$ This is by the definition of the subspace topology, and now as i is continuous this implies that $i^{-1}(U)$ is open in $\mathcal{T}$, that is $i^{-1}(U)$ is an element of $\mathcal{T}$. But $i^{-1}(U)$, as we have seen, is just equal to V. So, this implies that V belongs to $\mathcal{T}$.

Thus we have proved that $\mathcal{T}_Y$ is contained in $\mathcal{T}$. So, the point of this example is, Let us make a remark: this example shows that the subspace topology on a subset Y is the smallest topology which makes the inclusion maps continuous. We have proved that if we put any topology on Y, for which the inclusion map is continuous, then that topology contains the subspace topology. So, in other words the subspace topology is smallest topology which makes the inclusion continuous. Let us see a similar example.

So, this is example 4. Let X_i (for i in I) be a collection of topological spaces, and consider their product with the product topology. So, recall what the product topology was, recall that this is the topology generated by the basis $\mathcal{B}$ = {product of U_i's such that (a) U_i is open in X_i, and (b) when we look at the cardinality of those i's for which U_i is not equal to X_i, this cardinality is finite} Now from this product, we have natural projection maps from the product to X_j So, we have an element over here that looks like (x_i)(for i in I) and we just send this element to the jth coordinate x_j. So, these maps we shall denote by p_j. We claim that these projection maps p_j's are continuous.

What we have to do to prove continuity, we have to take an open subset in X_j, and show that the inverse image of that open subset is open in the product topology. So, for this, let $U \subseteq X_j$ be an open subset. So, then $(p_j)^{-1}(U)$. So, what is $(p_j)^{-1}(U)$? It is a subset of the product of X_i's. So, this is $(p_j)^{-1}(U)$ and it is exactly equal to those elements $(x_i)(i \in I)$, such that the jth coordinate is in U.

So, note that we can write $(p_j)^{-1}(U)$ to be equal to, it is actually equal to a product U_i, where $U_j = U$ and U_i is equal to X_i for i not equal to j. If i is not equal to j then X_i can be anything, and if i is equal to j then we need that x_j should belong to U. So, that is a description of this set $(p_j)^{-1}(U)$ So, you can convince yourself that this is correct, that is an easy exercise. Clearly this is in $\mathcal{B}$. What is $\mathcal{B}$? $\mathcal{B}$ is this correction, because each of the $(p_j)^{-1}(U)$ each U_i is either X_i ,or it is U where U is open in X_j, and X_i is obviously open in X_i, and this collection i in I such that U_i is not equal to X_i, this contains only one element which is j.

So, in our case this collection of i's such that X_i is not equal to U_i, it has just this one j, which has finite cardinality. So, therefore this $(p_j)^{-1}(U)$, it is in $\mathcal{B}$, and which is contained inside $\mathcal{T}_{\mathcal{B}}$ right the product topology generated by $\mathcal{B}$. So, therefore this implies that the projection maps are continuous. Now, similar to the earlier example, let us show the following. So, let $\mathcal{T}$ be a topology on this product of X_i's such that all the projection maps are continuous right.

So, then we claim that that $\mathcal{T}$ contains the product topology. Let us denote the product topology by $\mathcal{T}_{prod}$. How do we see this? So first note that if $\mathcal{B}$ is the basis that we define for the product topology and product of U_i's is an element of $\mathcal{B}$ right. So, for this element, let J be equal to the subset of those i's for which U_i is not equal to X_i. So, then cardinality of J is finite by our definition of this set $\mathcal{B}$.

So, notice that what we can do is so this product. So, note that this product of U_i's, it can be written as the intersection of $(p_j)^{-1}(U_j)$ for j in J. So, I will leave this as an exercise. So for instance, if we have $U_1 \times X_2 \times X_3 \times$.

.., let us say our I is $\mathbb{Z}$, and we have topological spaces X_i, not $\mathbb{Z}$, let us say natural numbers, for each i in $\mathbb{N}$. So, then this, intersected with $X_1 \times U_2 \times X_3$ cross $\times$ ..., this is equal to $U_1 \times U_2 \times X_3 \times$.

... So, this is the basic idea. So using this we can easily prove this assertion in the square. So, we will use this. Now our aim is to show that, We need to show that the product topology is contained in our topology $\mathcal{T}$. Using the earlier result we have proved, or let me just say using the earlier result, we have proved it suffices to show that this basis $\mathcal{B}$

for the product topology is contained in $\mathcal{T}$. But the basis $\mathcal{B}$ every element so now let us pick up an element of $\mathcal{B}$.

If this product U_i is an element of $\mathcal{B}$, then using this identity we can write it as U_i is equal to the finite intersection, because the cardinality of the subset J is finite. Now each $(p_j)^{-1}(U_j)$ is open in $\mathcal{T}$, as the projection maps are given to be continuous. So, each of these sets is open, and we know that in our topology finite intersection of open sets is open, since a finite intersection of open sets is open, this implies that this intersection for j in J, which is a finite set, this is open in $\mathcal{T}$ which implies that this product of U_i's is in $\mathcal{T}$. So, this implies that $\mathcal{B}$ is contained in $\mathcal{T}$, this implies that $\mathcal{T}_{prod}$ is contained in $\mathcal{T}$. Therefore, in view of the above, we see that the product topology is the smallest topology we can put on this set product X_i, for which the projection maps, all the projection maps are continuous.

So, this product topology has this nice property. Let us make some more remarks now. We will explain why the box topology is not such a great topology. Let us see a natural map which is not continuous when the product is given the box topology.

So, let me explain. We take $X = \mathbb{R}$ with the standard topology and for each natural number, so, by natural numbers, I mean the set of positive integers. Let X_n be the topological space $\mathbb{R}$ with again with the standard topology. So, consider the diagonal map so this is the map from x to product of X_n's over $\mathbb{N}$. What is this map? It sends x to (x, x, x, .

..) and so on. So this is the diagonal map. Let us give it a name. Let us call it Δ. So, we claim that if this product is given the box topology, then Δ is not a continuous map. So, this is just one example so for instance How do we see this? We have to construct a set which is open in the box topology, but whose inverse image is not open in X. So, for this consider the set: So we can look at this set U, let me just write it as product of U_n's contained in the product, where each U_n is the set $(-1/n, 1/n)$.

Then one checks easily that this is open in the box topology. In fact, it is in the basis $\mathcal{B}_1$, which we define and check that Δ^{-1} of this set is just equal to $\{0\}$, which is not open in X. So, that is the map Δ is not continuous, if this product is given the box topology. I will end this with an exercise. So let X be a topological space and let $X_i = X$ for i in I, i is any set, and then consider the diagonal map this is the diagonal map and give this the product topology now.

Then show that Δ is continuous. In order prove this exercise we will need the following lemma which we will prove in the next class, but you can try it if you want to. So, let f:X -> Y be a map of sets between topological spaces X and Y. Let $\boldsymbol{\mathcal{B}}$ be a basis for the topology on Y. If f^{-1}(V) is open for every V in $\boldsymbol{\mathcal{B}}$, then f is continuous. We will stop here.