Point Set Topology
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Week 01
Lecture 04

Hello and welcome to the fourth lecture in this course. In the previous lecture, we saw what a basis for a topology is. We are going to begin by proving a small lemma. Let us begin with this lemma. Let $\mathcal{T}$ be a topology on X and let $\mathcal{B}$ be a basis for $\mathcal{T}$. Then union of all the elements of $\mathcal{B}$ is equal to X.

So, let us prove this. It is clear that the union over all W in $\mathcal{B}$ is contained in X as $\mathcal{B}$ is a subset of $\mathcal{T}$, which is a subset of the power set of X. So W belongs to $\mathcal{B}$ implies W is a subset of X and we are just taking the union. So, we only need to prove the reverse inclusion.

So, let x ∈ X be an element. Then taking U = X, and using the defining property of the basis and the element x ∈ X we get that (note that U = X ∈ $\mathcal{T}$, therefore we can use the defining property of the basis) there is a W_x ∈ $\mathcal{B}$ (we will denote the set as W_x) such that x is in the set (W_x) which is in $\mathcal{B}$. Obviously W_x is contained in X which is equal to U. Thus, we get that this union contains x for all x ∈ X. Thus X is contained in the union.

This is because one of these W's is W_x. So, this completes proof of the lemma. So now we will prove this very important proposition which tells us when a subset of the power set can be used to generate a topology. So, what it means to generate a topology we will explain just now. Proposition: Let X be a set and let $\mathcal{B}$ ⊆ P(X) contained in the power set of X, be a collection which satisfies the following two properties.

So, the first is the union of all W's ∈ $\mathcal{B}$ should be equal to X. Note that we have not given X a topology right now. So, $\mathcal{B}$ is just a subset of the power set of X and the second condition is that given W_1 and W_2 ∈ $\mathcal{B}$ and any x ∈ W_1∩W_2, there is W ∈ $\mathcal{B}$ such that this point x ∈ W and W is contained in W_1∩W_2. So, note that W_1∩W_2 may not be in $\mathcal{B}$, but there is another W which is in $\mathcal{B}$ and which is contained in W_1∩W_2 and contains x. So, we have this collection $\mathcal{B}$ which satisfies these two conditions and we define a topology.

So, define a collection $\mathcal{T}$, which is a subset of P(X) as follows: $\mathcal{T}$ contains those subsets U of X such that given any x ∈ U, there exists a W ∈ $\mathcal{B}$ such that x ∈ W and W is contained in U. So, then $\mathcal{T}$ is a topology on X and $\mathcal{B}$ is a basis for $\mathcal{T}$. So, let us prove this proposition. In order to prove this proposition we have to check that $\mathcal{T}$ satisfies the three conditions which define a topology. So, let us check this one by one.

So, $\phi \in \mathcal{T}$. This is vacuously true since there is no point $x \in \phi$ and so therefore there is nothing to check. The full set $X \in \mathcal{T}$. So, why is this? As the union of W's from $\mathcal{B}$ is equal to X, and so given any $x \in X$, it is in one of these W's $\in \mathcal{B}$ such that $x \in W$ and obviously W is a subset of X. Thus, we have proved that both the empty set and X are in $\mathcal{T}$.

This proves the first condition. Let us look at the second condition. Here we want to say that finite intersections of elements of $\mathcal{T}$ are in $\mathcal{T}$. Suppose, U_1, U_2, .

.., U_n are in $\mathcal{T}$. Then we need to show that the intersection of U_i's is in $\mathcal{T}$. So, for this we choose x in this intersection of U_i's, then for each i, as U_i is in $\mathcal{T}$ and x is in U_i there is W_i in $\mathcal{B}$ such that x is in W_i and W_i is contained in U_i. So, we claim that from property (2). So, this property (2).

This implies the following (property) which we call (2'): If x is in the intersection of all the W_i's for finitely many $W_1, \ldots, W_n$ where each $W_i \in \mathcal{B}$, then there is $W \in \mathcal{B}$ such that $x \in W$ and W is contained in the intersection of W_i's. So, note that this property (2) is same as (2') when $n = 2$.

However we are saying that, using an easy induction argument, which is left as an exercise, the proof that (2) implies (2') is left as an exercise, the hint is use induction. So, then using (2') we get that there is W which contains x and W is contained in the intersection of W_i's which in turn is contained in the intersection of U_i. So, thus this intersection of U_i's is in $\mathcal{T}$ as it satisfies the property defining $\mathcal{T}$. Finally we have to check the third condition. So, given a set I and subsets U_i of X such that U_i is in $\mathcal{T}$, we need to show that the union of U_i's (for all $i \in I$) is in $\mathcal{T}$.

Once again we do the same so let x be an element in the union. then $x \in U_j$ for some $j \in I$ and since U_j is in $\mathcal{T}$ this implies there exists some W such that x belongs to W, and W is contained in U_j which in turn is going to be contained in this union. So, this shows that the union satisfies the defining property for $\mathcal{T}$ and so is contained in $\mathcal{T}$. So, this completes the proof that $\mathcal{T}$ is a topology on X. So, next we have to show that $\mathcal{B}$ is a basis for $\mathcal{T}$.

Which means that we need to show that for every $U \in \mathcal{T}$ and $x \in U$, we need to show there exists $W \in \mathcal{B}$ such that x is in W and W is in U. But this follows from the definition of U being in $\mathcal{T}$. So, this completes the proof of the proposition. So, later on we shall in the next lecture we shall use this proposition to generate topologies on various sets. So, this is a very useful proposition.

Let me give an exercise over here. Notice that in this proposition we started with a set and a collection $\mathcal{B}$ which satisfied two properties. Then we defined a topology $\mathcal{T}$ on X using this collection $\mathcal{B}$ and moreover $\mathcal{B}$ turned out to be a basis for the topology $\mathcal{T}$. But we can ask ourselves, what happens if we started with a topology and a basis for that topology. So, let X be a set and let, before this let us introduce some notation.

Let X be a set and $\mathcal{B}$ contained in P(X), $\mathcal{B}$ such that it satisfies the topology hypothesis in the proposition. Then we shall denote the topology we defined in the previous proposition by $\mathcal{T}_\mathcal{B}$ and note that $\mathcal{B}$ is a basis for $\mathcal{T}_\mathcal{B}$. So, with this notation here is an exercise. Let X be a set and let $\mathcal{T}$ be a topology on X.

Let $\mathcal{B}$ be a basis for $\mathcal{T}$. So, then by the lemma that we proved; using this lemma and the definition of a basis it follows that $\mathcal{B}$ satisfies the two hypothesis in the proposition. Thus $\mathcal{B}$ defines a topology on X which we will denote $\mathcal{T}_\mathcal{B}$. So, a priori this $\mathcal{T}_\mathcal{B}$ may be different from $\mathcal{T}$ and the exercises show that $\mathcal{T}$ is equal to $\mathcal{T}_\mathcal{B}$. Now we will put this aside for a few minutes and let us define topologies on subsets of X, given a topological space X. Definition of subspace topology: before we define subspace topology, from now on, when we say "let X be a topological space", we shall mean that X is a set and it has a topology $\mathcal{T}$.

So, with this let us define subspace topology. Let X be a topological space, and let us say with topology $\mathcal{T}$. Let Y be a subset of X. Then using $\mathcal{T}$, we can define topology on Y in a natural way. So, define a subset $\mathcal{T}_Y$ contained in P(Y), as follows: it is the collection of $U \cap Y$, where U belongs to $\mathcal{T}$.

So, we simply take all open subsets in $\mathcal{T}$. So, recall that if I said U belongs to $\mathcal{T}$ then we agreed to use the notation U is open in $\mathcal{T}$. So, we just take all open subsets in $\mathcal{T}$ and intersect that with Y and that gives a topology on Y. So, claim is that this gives a topology on Y. So, this is being left as an exercise check that $\mathcal{T}_Y$ satisfies the three conditions for being a topology.

And so defines a topology on Y, which we call the subspace topology. So, let us see some simple examples of subspace topology. First, let X have the trivial topology. So then recall that $\mathcal{T}$ just consists of the empty set and X, and in this case clearly the subspace topology $\mathcal{T}_Y$ also just consists of the empty set and Y. And so the subspace topology on Y is the trivial topology.

Similarly, let X have the discrete topology. So, recall that the discrete topology meant that we are going to take $\mathcal{T}$ = P(X). So, then clearly $\mathcal{T}$_Y is equal to the power set of Y right because given any V which is a subset of Y, and so it is also a subset of X. So V belongs to $\mathcal{T}$. This implies that V∩Y, which is V, belongs to $\mathcal{T}$_Y.

As we have proved that $\mathcal{T}$_y it contains all subsets of Y which implies that $\mathcal{T}$_Y is the discrete topology. Let us take a third example. Let us take ℝ with the standard topology and let Y be the subspace ℤ. So, the question is, what is the subspace topology on Y. We claim that the subspace topology on Y is the discrete topology.

So, to show this, it is enough to show that if you take a point x in Y, a singleton, then this subset {x} is open in the subspace topology, because if you show this, then this would imply that given any subset W in Y, we can write W as the union over all n in W of the subset singleton {n} and since arbitrary unions of open sets are open this implies that W is open. We are assuming that, suppose we can show that each singleton {n} is open, so each {n} is open and we take arbitrary unions, that is also open. So this would imply that W is open. So, every open subset will be, I am sorry every subset of Y will be open in Y. So, therefore it is enough to show that these singleton {n}'s are open but to see this, for each n, we can write as the open interval (n – ½, n + ½) ∩Y because, here we have the real line, and here is point n.

So let us say this is n – 1 and this is n + 1. So this interval, this is open in the standard topology in ℝ, so U = (n – ½, n + ½) is open in the standard topology in ℝ and U∩Y = {n}. So, therefore this singleton {n} is in $\mathcal{T}$_Y or equivalently it is open in the subspace topology and therefore, from what we have seen above, all subsets of Y are open. So, we will end this discussion here and in the next lecture we will see a slightly more complicated example.