Point Set Topology
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Week 07
Lecture 34, Part I

In the previous lecture we ended our lecture with an example from group theory. We saw that given a group, and given a normal subgroup, we can look at the equivalence classes. The normal subgroup defines an equivalence relation on a group and we can look at the equivalence classes, we can give the set of equivalence classes of group structure such that some nice things happen. And today in this lecture we will see that we can do something similar in topology. Let us address the question first, what the question is. Let X be a set and suppose we have a equivalence relation on X, so let X mod equivalence be the set of equivalence classes.

So, then we have this obvious map, the natural map π from X to X mod equivalence. What does it do? It takes x and sends it to its equivalence class. Right now everything is only happening for sets. Now suppose we have a map of sets f from X to Y such that x~y implies f(x)=f(y).

Or we can phrase this property as follows: We have already stated f is constant on equivalence classes. So, then this function this map of sets it factors uniquely as, we have f, X mod equivalence, π, so it factors like f_0. We get a map of sets from the set of equivalence classes to Y and what is f_0? So, f_0 is defined as follows: So, let α be an equivalence class, and let x in X be such that the equivalence class of x is this α right. We define f_0(α) to be equal to f(x). Now this is well defined because f is constant on equivalence classes.

So, clearly this definition makes sense as f is constant on equivalence classes. So, now we are going to bring in topologies and we are going to complicate this problem. So, now further assume that, consider the situation where X is a topological space and we have this equivalence relation on X. So, this is an equivalence relation on the underlying set. So, what we want to do is we want to put, so the question is can we put a topology on X mod equivalence? So, this is just like, can we put a group structure on G/N such that the following two things happen: One the natural map π from X to X mod equivalence is continuous, and two, suppose we are given a continuous map a continuous map f from X to Y.

So, Y is another topological space and we are given a continuous map such that f is constant on equivalence classes. So, then as a map of sets we are going to get a factorization like this f, is this we have π we have X mod equivalence and we get this map of sets f_0, then we get a map of sets f_0 from X mod equivalence to Y. So, the second condition we want is that f_0 must be continuous. So, we get a unique map of sets, in fact even here the f_0 is unique, it is completely determined by f. So, let me make a remark right.

If you only impose condition one, then this problem has a very easy solution, namely we can just give X mod equivalence the trivial topology. So, since there are only two open subsets the empty set and the entire set, the inverse image of both these will be open and therefore π will be continuous here, but if you give this the trivial topology then the second condition the second property may not be satisfied. So, the main theorem we are going to prove in this lecture: There is a unique topology $\mathcal{T}$ on X mod equivalence, which satisfies the above two conditions. Let us prove this theorem. So, let us first show that there is such a topology $\mathcal{T}$ on X mod equivalence which satisfy the two conditions.

So, we are going to define $\mathcal{T}$ first. So, let $\mathcal{T}$ be the collection of subsets U contained in X mod equivalence, such that this $\pi^{-1}(U)$ is open in X. It is easily checked, and I will leave it as an exercise that $\mathcal{T}$ satisfies the conditions to define topology on X mod equivalence. That is a very easy check and I will leave it as an exercise. So, obviously now let us prove that both the these two conditions are satisfied.

So, first we have to show that this map π from X to X mod equivalence is continuous. So, one the map π X to X mod equivalence is continuous. For this we have to show that, given an open subset V in X mod equivalence, $\pi^{-1}(V)$ is open in X, but this is clear from the very definition of the topology on X mod equivalence from the definition of $\mathcal{T}$. V is in $\mathcal{T}$ if and only if $\pi^{-1}(V)$ is an open subset of X. The second condition we want to check is suppose we are given a continuous map f from X to Y, which is constant on equivalence classes.

So, then we get this resulting map f_0, a map of sets determined by f. So, we need to show that f_0 is continuous. For this, let V contained in Y be an open subset. We need to show that $f_0^{-1}(V)$ is open in X mod equivalence. So, for this, applying the definition of $\mathcal{T}$, this happens if and only if $\pi^{-1}(f_0^{-1}(V))$ is open in X, but simple set theory tells us that $\pi^{-1}(f_0^{-1}(V))$ is equal to $f^{-1}(V)$ which is open as f is continuous.

So, this shows that $\mathcal{T}$ satisfies both the conditions. Next let us prove that this topology is unique. So, suppose there is another topology $\mathcal{T}'$ on X mod equivalence such that conditions 1 and 2 hold for $\mathcal{T}'$. So, let us make this diagram. So we have X, we have this map π, this map to (X mod equivalence,$\mathcal{T}'$).

So, the first condition is that the obvious map X goes to the class of X, this should be continuous if X mod equivalence has the topology, should be continuous in our topology. So, we are giving X mod equivalence a topology $\mathcal{T}$ ' and by assumption since $\mathcal{T}$ ' satisfies condition 1 this map is continuous. But we also have this map (X mod equivalence,$\mathcal{T}$). Now as $\mathcal{T}$ satisfies condition 2 we get this dotted arrow like this. So, let us call this π_0.

So, the dotted arrow we will get anyway. Note that the map of sets π_0 is forced to be the identity. So, this map of sets π_0 that we get, that is forced to be the identity right. So, since $\mathcal{T}$ satisfies condition 2 this implies that π_0 is continuous. So, thus if V is in $\mathcal{T}$', V is an open subset in $\mathcal{T}$', then $\pi^{-1}(\pi_0^{-1}(V))$ which is equal to V is open in $\mathcal{T}$.

This implies that $\mathcal{T}$' is contained in $\mathcal{T}$. So, similarly we can switch the rules of $\mathcal{T}$ and $\mathcal{T}$ '. So, we have this. We get this π_0. As $\mathcal{T}$ ' satisfies the second condition this implies that π_0 is continuous.

So, once again this implies that $\mathcal{T}$ is contained in $\mathcal{T}$'. So, thus $\mathcal{T}$ is equal to $\mathcal{T}$'. So, this completes the proof of the theorem. The above topology on X mod equivalence is called the quotient topology. So, we will end this lecture here. Thank you.