Point Set Topology

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Week 06

Lecture 30

So in the last few lectures we discussed the notion of compactness. We discussed the notion of compactness, however as we saw there are lots of important spaces which are not compact. Note that there are several important spaces which are not compact, for example $\mathbb{R}^n$. These spaces are however locally compact, which is the next property we shall study. Let us define local compactness. Let X be a topological space, we say that X is locally compact, if every point x in X has an open neighborhood U such that U closure is compact, and example it is clear that $\mathbb{R}^n$ is locally compact, because we can take any point, and we can take a disk of let us say radius one around it, so the closure will be this closed disk, and this is obviously closed and bounded in $\mathbb{R}^n$ and therefore it is compact.

And so one of the main propositions, a very useful proposition we are going to prove about locally compact spaces is the following: Iff for any pair (x,W) where W is an open subset containing x, we can find an open subset V such that x is in V, which is contained in V closure, and V closure is contained in W, and V closure is compact. So in other words given any x and given any open subset W which contains x, we can find neighborhood V of x such that V closure is compact. This is V closure. So let us prove this.

It is clear that if this condition holds for any pair (x,W), we can find V such that x belongs to V, V closure is contained in W and V closure is compact, then X is locally compact. We simply apply this condition to the pair (x,X). So when we apply this condition to this pair, we will get that there is an open subset U, which contains x and U closure is obviously going to be contained inside X and U closure is compact. So this proves that. That's precisely the meaning of being locally compact.

Let us prove the converse. Let us assume that X is locally compact. So now recall the following assertion, which we have already seen before. So let Y contained in X be a compact subspace. This is more general.

So consider the following more general assertion. So let Y contained in X be a compact subspace. And let x be a point in X\Y. Then there are open subsets C and D such that y is contained in C, x is contained in D and the intersection C and D are disjoint. Let's see how to prove this.

This is our x, and this is our y and this is our x. So for each point y in Y, we can find a neighborhood, since x is Hausdorff, we can find a neighborhood U_y of x and V_y of y such that, U_y intersected V_y is empty. So we can cover Y as, we can write as a union of all these V_y's. As Y is compact, this implies there is a finite subcover, And then we let U be equal to the intersection of these U_{y_ j}'s. Each U_{y_ j} contains x, so this implies that x belongs to U, U is a finite intersection of open sets and therefore it is open and, right, so it's not we can cover.

So Y is contained in this, sorry, Y is going to be contained in this. And since Y is compact and from what we had seen in one of the earlier lectures, if we can put Y into a collection of open subsets, then we can find a finite subcollection of that collection such that Y is contained in that. So x belongs to U, And let us look at, so let D be equal to U and let C be equal to this union j = 1 to n V_{y_j}'s. So this V_{y_j}'s could be something like this. Then C contains y and D intersected C is equal to U intersection union j = 1 to n V_{y_j}'s, this is equal to union j = 1 to n U intersection V_{y_j}, but this contained in union of j =1 to n U_{y_j} intersection V_{y_j}, which is empty.

So this proves our general assertion, which is something and we had seen the same proof before earlier as well. So we will use this general fact. So suppose now we return to the proof of our assertion. So we are assuming that X is locally compact and we want to show that X has this property. So suppose we are given a pair (x,W), where W is open, and x is contained in W.

As x is locally compact, this implies there exists a neighborhood open subset such that U closure is compact. So this is our x, this is W and let us say U closure is this. So consider the closed subset, Y defined as U closure \W, but this is same as U closure intersected with X\W. W is open, so X\W is closed, and we are intersecting two closed subsets, so that is, U closure is compact and compact subspace of a Hausdorff space is closed and therefore the intersection of these two is going to be closed. So this closed subset, it looks something like this.

So from U closure we are removing this. So this region is Y, and clearly x does not belong to Y. So being a closed subspace of a compact space, since Y is a closed subspace of U closure which is compact, this implies that Y is compact. So thus, using this general result that we proved, yeah, so thus there is, there are open sets C and D such that Y is contained in C, x is contained in D, and C intersection D is empty. C could be some open set like this, so C contains Y, and D could be some open set like this, So we can replace D by an open subset which is smaller than D as long as it contains x.

So we may replace D by D intersection U. So U contains x, therefore D intersection U also contains x, and assume that D is contained inside U. So our set D is going to be contained inside U. So this implies that D closure is also going to be contained inside U closure, and as U closure is compact, this implies D closure is also compact because it is a closed subspace of a compact space. Now, no point of Y can be in the closure of D, because if y is a point in Y, then there is a neighborhood of y, namely C, such that C intersection D is empty.

So this implies that y does not belong to D closure, which implies D closure intersection Y is empty. So as D closure is contained inside U closure, and D closure intersection Y is empty, this implies D closure is contained in U closure\Y. And as Y is equal to U closure minus W, this implies that U closure minus Y is equal to U closure intersection W, which is contained in W. So thus, we have found an open set D such that x is in D and D closure is contained in W and D closure is compact. This completes the proof of proposition.

So this proposition is very useful, and this set theoretic check is very easy, and I will leave it to you as a simple exercise. Next, we are going to talk about compactifying locally compact spaces. So note the space, the open interval (0,1) is locally compact. So that is clear because, I mean, if we take this interval (0,1), and given any point x, we can find a small closed interval around x, and we know that this closed interval is homeomorphic to [0,1], which we proved is compact. Now, notice that it is contained, so this interval (0,1) is contained as a dense open subset in, it is obviously contained in the interval [0,1] and also in S^1.

So to see this, note that (0,1) is homeomorphic to ℝ, and using the stereographic projection, ℝ is homeomorphic to S^1\{NP}. So ℝ is homeomorphic to S^1\{NP}. So this implies that this is homeomorphic to ℝ, and this is embedded inside S^1. So although we can compactify a locally compact topological space in many ways, there is one compactification which is special among all these, and which is now, which is what we are going to explain now. So thus, there can be many ways to compactify a locally compact topological space.

However, one particular compactification is very special, and we shall next describe that. So we will do that in the next lecture. So we will end this lecture here.