Point Set Topology

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Week 06

Lecture 27

This is lecture 27 and before we start a discussion on compact metric spaces, we will see a nice application of what we have seen so far. So, today we will prove a very interesting theorem. So, SO(n) is connected. In fact, SO(n) is also path connected but that takes a little bit more work and so in this course we will prove that SO(n) is connected. Before we start the proof, it is perhaps worth appreciating that the definition of SO(n) is somewhat complicated. So, recall that SO(n) is all those matrices A in M(n,ℝ) such that A^T*A is equal to identity and determinant of A is 1.

So, to show that if you were to try and show directly that SO(n) is path connected then it would probably be quite hard. Let us begin the proof. So, first consider this map from SO(n) to the sphere S^{n-1}. Before that so let E1 in ℝ^n denote the column vector (1,0,0,...). Okay, and consider the map from SO(n) to S^{n-1} given by, we take a matrix A, let us say n>=2. So, we take a matrix A and it gets mapped to A*E1. So, we are taking this matrix A in SO(n) and this is getting sent to the first column.

So, a-priori it is not clear that the image lands in S^{n-1}, but let us just check that. So, A*E1, so we need to check like a-priori, this is a map from SO(n) to ℝ^n, and we need to check the image lands in S^{n-1}. So, SO(n) has the subspace topology from M(n,ℝ) and this is just a projection on some coordinates. So, therefore this map is continuous. So, this map is continuous.

Let us compute the norm of this A*E1. The <A*E1,A*E1> this is equal to, so this is the standard inner product on ℝ^n given by <V,W> is defined to be W^T*V. When we use this, so we get this inner product is equal to (A*E1)^T*(A*E1), which is equal to E1^T*A^T*A*E1, but now A is in SO(n), which means A^T*A is identity. So, this is equal to E1^T*E1, which is just equal to 1. So, this implies that the image lands in S^{n-1}.

The norm of this vector A*E1 is 1. So, we have this SO(n), we have this continuous map to ℝ^n and the image actually lands inside S^{n-1} and since S^{n-1} has a subspace topology from ℝ^n, this implies that this map we have defined the map from SO(n) to S^{n-1} is continuous. Next, let us consider the subgroup H equal to SO(n-1) sitting inside SO(n) as follows: 1, 0's here, 0's here and here we have this matrix SO(n-1). We claim that A*E1 is equal to B*E1. So, if you call this map ϕ, so what we are saying is ϕ(A) is equal to ϕ(B) if and only if B^{-1}*A is in H.

So, let us check this. So, if B^{-1}*A is in H, so this implies that B^{-1}*A is equal to h for some h in H. This implies that A is equal to B*h simply by multiplying on the left with B. So, now we apply both these on E1. This implies that A*E1 is equal to B*h*E1, but now note that h*E1 is simply equal to E1, because when we apply this matrix of this type on E1, we get back E1.

So, this implies A*E1 is equal to B*E1. So, now let us prove the converse. Conversely, suppose A*E1 is equal to B*E1, then multiplying with B^{-1} on both sides, then this implies that B^{-1}*A*E1 is equal to E1. So, we let C be the matrix B^{-1}*A. So, then C*E1 is equal to E1.

This implies that C is a matrix which looks like this. So, C*E1 is equal to E1, which means the first column looks like this and the others can be anything. Now, as A and B are in SO(n), this implies B^{-1}*A is equal to C, is also in SO(n). And from the condition C^T*C is equal to identity. So, if we write C as [C1,C2,...,Cn], So, this C^T*C is equal to identity. This becomes [C1^T,C2^T,...,Cn^T] into this column [C1,C2,...,Cn] is equal to identity. So, this implies Ci^T*Cj is equal to 0 for i not equal to j. So, if we now, but this Ci^T*Cj that is just the inner product Ci and Cj. This implies that <Cj,C1> is equal to 0 for j not equal to 1. But C1 is this column vector (1,0,0,...). So, C1 is this column and let us take C2, C2 will be this column. So, when we take <C1,C2>, we get this first entry over here. So, since the inner product is 0, this implies that this entry will be 0, and similarly this entry will be 0 and similarly all these entries will be 0. So, as C1 is this, from this we conclude that C, this matrix looks like this and here we have whatever else. This implies that C belongs to H.

So, thus B^{-1}*A belongs to H which is what we want to prove. Next we claim that the map from SO(n) to S^{n-1} is surjective. Recall that given a vector v with ||v||=1. So, v is in ℝ^n, we may extend it to an orthonormal basis V ={v1,v2,...,vn} of ℝ^n. And so now let A be equal to matrix v1, we write these vectors as column vectors and we let this matrix A is an nxn matrix. Then clearly this matrix A^T*A is going to be equal to this matrix obtained by taking the inner products of vi and vj. This is an easy check. And this is clearly equal to identity because the vi's form an orthonormal basis. If determinant of A is equal to -1, then we simply replace the last column by negative of that, then that A' be the matrix [v1,v2,.

.., -vn] So, then we easily check that A'^T*A' is equal to identity and determinant of A' is equal to 1. So, moreover, the first column of A' is v1 is equal to v. So, this proves that this ϕ is surjective. And we need one more ingredient, so let us look at that. So, if A is in SO(n), then consider the map from SO(n) to SO(n), this is given by left translation by elements of A.

So, precisely this map sends a matrix B in SO(n) to A*B. It is easily checked that if A and B are in SO(n), then obviously A*B is in SO(n) if SO(n) is subgroup of GL(n,ℝ). So, let us check that this map is continuous. To show this map is continuous, note that SO(n) has a subspace topology from M(n,ℝ). So, it is enough to check that this L_A, after we compose with this inclusion, is continuous.

This composite is continuous, but this composite map, this dotted arrow, it factors like this M(n,ℝ) to L_A. So, both these triangles commute, we can take this matrix A, this matrix A is fixed, and we can define left multiplication by A on M(n,ℝ) itself. Here also it is A goes to A, but clearly this map is continuous because all the coordinates, they are just some linear combinations of the coordinates of B, are simply linear combinations of the coordinates of B. So, therefore this horizontal map is continuous. As a result when we look at this map, that is continuous because we have just restricted this continuous map to SO(n), which means the dotted arrow is continuous which means the map L_A which we started with, that is continuous.

Similarly, L_{A^{-1}}, let me just write. So, let us call these maps a,b,c,L_A, let us call this L_(A~). So, to show L_A is continuous, it is enough to show boL_A is continuous, but it is enough to show boL_A is continuous because SO(n) has a subspace topology from M(n,ℝ), but boL_A is equal to this dotted arrow a, which is equal to L_{A~}oc, and L_(A~) is continuous because L_(A~), it is enough to check what it the coordinate functions of L_(A~) are continuous and c is just a restriction of L_(A~) to SO(n), this implies L_(A~)oc is continuous. So, this implies L_A is continuous. Similarly L_{{A^{-1}} is continuous, but clearly we have L_AoL_{A^{-1}} is equal to identity, which is also equal to L_{A^{-1}}oL_A.

Thus we have L_A is continuous and L_A is a bijective continuous map and it is invertible, it is clear L_{A^{-1}} and that is also continuous here so thus L_A is a homeomorphism ok. So, now with these ingredients we are ready to prove that SO(n) is connected. We will prove the theorem by induction on m. So, the base case is n=1. In this case SO(1) is simply, this is one element which is obviously connected.

Assume we have proved that n>=2, and we have proved that SO(k) is connected for,..., sorry. So, let us assume that n>=1 for 1<=k<=n.

So, now we will show that SO(n+1) is connected So, consider the map SO(n+1) to S^n. Note that as n>=1, we have n+1>=2. So, therefore we can consider this map which we analyzed before. So, if SO(n+1) is disconnected then we can write it as the disjoint union of two nonempty disjoint open sets, and are also closed. So, let us pick any element of SO(n+1), and let H be the subgroup SO(n) contained inside SO(n+1).

So, A is in SO(n+1) So, A is either in U or it is in V. Assume that A belongs to U. Then we can take this subset AH, and we can write it as AH intersection U disjoint union AH intersected V right. As L_A from SO(n+1) to SO(n+1) is a homeomorphism, and we have H sitting over here, with L_A is sent to AH sitting over here.

So, L_A is a homeomorphism. So, therefore it is a homeomorphism, this implies that L_A restricted to H, this is map from AH to AH is also a homeomorphism. So and as H is connected, H is homeomorphic to SO(n), and we are assuming by induction that SO(n) is connected. As H is connected and AH is homeomorphic to H, this implies that AH is also connected. So, thus one of these two open sets has to be empty, but A belongs to this set. This implies that this set over here has to be empty, or in other words AH is completely contained inside U.

So, we have proved that if A belongs to U then AH is completely contained inside U. If B belongs to V then BH is completely contained inside it. Now let us consider this map from SO(n+1) to S^n. So, U is closed and SO(n+1) is compact implies U is compact, which implies ϕ(U) is compact in S^n, which implies that ϕ(U) is closed.

So, similarly ϕ(V) is also closed. So, as ϕ is surjective, this implies S^n is equal to ϕ(U) union ϕ(V). So, we claim that this union is disjoint. If not there exists some vector v in ϕ(U) intersection ϕ(V). This implies that there exists A in U and B in V such that ϕ(A) is equal to A*E1, is equal to V, is equal to B*E1 is equal to ϕ(B). But now we know that if this happens, this will imply that B^{-1}*A belongs to H, which implies that A belongs to BH and since B belongs to V*BH is contained in U, contained in V, which is a contradiction, because we assumed that A is in U and U and B are disjoint.

Thus we have written if SO(n+1) is disconnected then S^n is going to be ϕ(U) disjoint union ϕ(V). We can write it as a disjoint union of nonempty closed subsets, which means it will be also be the disjoint union of nonempty open subsets which means S^n will be disconnected. But this contradicts, what we have seen before that S^n is connected with the fact that S^n is connected. So, thus SO(n+1) is connected.

So, this proves, this completes the theorem. So, we will end here.