Point Set Topology
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Week 05
Lecture 25

In the previous lecture, we had used the following two exercises. One let us show that the real line is homeomorphic to the interval (0,1). So, let us see how to do this. I just want to sketch a quick proof. So, we will give a homeomorphism from the real line to (-1,1) and you can easily check that using a linear isomorphism that the interval (-1,1) is homeomorphic to the interval (0,1). So, let us give a map from ℝ to (-1,1).

So, we simply send x to x/(|x|+1) and easily check this is bijective and continuous. And the inverse of this map is given by (-1,1) to ℝ: y maps to y/(1-|y|) and this is also continuous. There remain to check that these are bijective and continuous. that is left as an exercise.

This is one exercise. The other exercise I had mentioned was, Let X and Y be topological spaces. Fix a point x in X and consider the inclusion from Y to XxY given by y to (x,y). So, clearly this factors like this. So, I and let us call this f_0 is a homeomorphism.

Obviously {x}xY has the subspace of origin from XxY. Let us quickly check this. So, f is continuous as both projections are continuous. Recall we had seen that to give a map to a product it is enough to give continuous maps to each of the factors, and both the factors, the first factor is just the constant map y to x, that is obviously continuous, The second factor is the identity. So, both these are continuous and since the image lands in {x}xY, it follows that f_0 is continuous.

Here we are using the following exercise. Not exercise, the result which we had proved when we talked about subspace topology. Suppose A to B, we have a continuous map f, and let us say the image lands inside a subset C of B and then this f factors as f_0. So, if we give C the subspace topology, then this map f_0 from A to C is continuous. So, f_0 is bijective is clear.

Therefore, to show that f_0 is a homeomorphism, it is enough to show that f_0(V) is open for an open V in Y, but f_0(V) is exactly equal to {x}xV and this is equal to XxV intersection {x}xY. Thus f_0(V) is open, which implies f_0 is a homeomorphism. This completes both the exercises which were mentioned in the previous lecture. Now let us proceed with our discussion on compactness. Last time we proved that close subspace of a compact space is compact.

This time, Let us begin today's lecture with the following proposition: Let X be a topological space, and let Y contained in X be a subspace which is compact. So, here Y is a subspace of X, and therefore it inherits the subspace topology, and this topological space is compact. that is our assumption with the subspace topology. So, our claim is then Y is closed in X. So, let us prove this.

We shall show that the set $X\setminus Y$ is open in X. So, once again let us make a picture, let us say this is our X and say this is our Y. So, let us pick up any point X over here. So, let x be in $X\setminus Y$, it is a point outside Y. So, we need to show that there is a small neighborhood, there is a small open set around x which does not meet Y.

So, we choose any y in Y. So, let us say this is Y, then as X is Hausdorff, as I had mentioned, from now on we will always be assuming that our topological space is a Hausdorff if not mentioned, but as X is Hausdorff there exist open sets U_y containing x and V_y containing y. So, this is our U_y and this is our V_y such that U_y intersection V_y is empty. So, we can do this for all y in Y. Then the sets V_y's form an open cover for Y.

We can write, Y is contained in this union of y in Y of V_y's So, this implies that Y is equal to union y in Y of Y intersection V_y's and since there is an open cover, this has a finite sub cover. So, this is equal to union i=1 to n, Y intersection V_{y_i}. So, this implies that Y is actually contained in the union of finitely many of these V_{y_i}. So, now let U be equal to the intersection if all i=1 to n of U_{y_i}'s. Now each of these U_y's is contained X.

Therefore this implies that, and this is a finite intersection. So, this implies that U is an open subset of X and X is in U. So, let us look at U intersected with the union of V_{y_i}'s (i=1 to n). This is contained in the union i=1 to n U intersection V_{y_i}'s which is contained in the union U is contained in U_{y_i} and so in U_{y_i} intersection V_{y_i}, but each of these is empty, this is equal to ϕ. So, this implies that U intersected this union is empty.

This implies that U intersection Y, which is contained in U intersected union V_{y_i}'s, as we saw that Y is contained in this union, is equal to empty right. Given this point x in $X\setminus Y$, we have found an open set U contained in X, open in X such that X belongs to U and U is contained in $X\setminus Y$, thus $X\setminus Y$ is open. So, this completes the proof of the proposition. So, as a result of this proposition, using this proposition, we will do this very useful theorem: Let Y contained in $\mathbb{R}^n$ be a subspace. Then Y is compact iff Y is closed and bounded.

I have not defined what bounded means, but it means the obvious. So, let Y contained in ℝ^n be a subspace, we say that Y is bounded if there exists some M>0, such that Y is contained in the ball of radius M around the origin. Let us say in the Euclidean metric. Let us prove this theorem. So, let us first assume that Y is compact right.

We can write Y, so first note that in ℝ^n, we can write as the union n>=1, open balls of radius n. So, this is a open cover for ℝ^n because every point is contained in one of these open balls, these keep getting larger and larger. So, this implies that Y is equal, to we simply intersect both sides with Y, Y intersection B(0;n), this is an open cover for Y, and since Y is compact, this has a finite subcover, but these balls are contained (in each other). So, B(0;1) is contained in B(0;2), is contained in B(0;3) and so on. So, this implies that there exists some n such that Y is contained in B(0;n) thus Y is bounded.

Because of this previous proposition, the previous proposition says that ℝ^n is a Hausdorff topological space and we have Y, which is compact. So, Y is also closed right. So, the previous proposition implies that Y is closed. So, thus Y is closed and bounded. So, this proves one direction.

Conversely assume that Y is closed and bounded. So, since Y is bounded, we can find M large such that Y is contained in a product of these intervals. So, Y is bounded means Y is contained in some large open ball, and this open ball we can put inside a square like this, inside a closed square. So, our Y is this thing. Moreover note that, we proved that [0,1] is compact, as [0,1] is homeomorphic to, we can just in fact even write a linear homeomorphism, this implies [-m,m] is compact, this implies [-m,m]^n is compact.

As Y is closed in ℝ^n, this implies Y is also closed in this subset, and by what we proved in the previous lecture, we proved that the closed subspace of a compact space is compact. So, using that, as closed subspaces of a compact space is compact, this implies Y is compact. So, this completes the proof of the theorem. As an application of this theorem let us prove that SO(n) is compact. Recall SO(n) or the same proof for O(n) will work, SO(n) is the set of those matrices A, with real entries such that A^T*A is equal to the nxn identity.

So claim: SO(n) has the subspace topology and we can identify M(n,ℝ) with ℝ^{n^2}. So our claim is that SO(n) is compact. It suffices to show that SO(n) is closed and bounded. Let us look at this map. Let us first show that SO(n) is closed.

Let us consider the map from M(n,ℝ) to M(n,ℝ)xℝ. So, this map is given by A maps to (A^T*A, det(A)). Oh sorry I forgot to put the determinant condition. We claim that this map is continuous. So, to show that it is continuous it is enough to show that, first we look at the map from $M(n,\mathbb{R})$ to $M(n,\mathbb{R})$, A goes to A^T*A.

this map is continuous, and map from $M(n,\mathbb{R})$ to $\mathbb{R}$, A goes to $\det(A)$. This map is continuous is clear because the determinant of A is a polynomial in the entries of A, and we know that polynomial in the entries of A are simply the projection maps, and projection maps are continuous. From this product and polynomials, therefore polynomials in these continuous maps are continuous. Let us look at this map over here. So, once again $M(n,\mathbb{R})$ is identified with $\mathbb{R}^{n^2}$ with the product topology.

So, to say that this map is continuous it is enough to say that each of the coordinate functions is continuous, but once again when we look at A^T*A, the coordinate functions are polynomials in the entries of A. Thus A goes to A^T*A is continuous. So, this shows that ϕ is continuous. Now the point $(Id_{nxn}, 1)$ in $M(n,\mathbb{R})x\mathbb{R}$ is a closed subset. In fact, in any $\mathbb{R}^n$ a point, so we can just take any point $\{(a_1,a_2,...,a_n)\}$, this just the single set, is a closed subspace. So, this implies that ϕ inverse of this $(Id_{nxn}, 1)$ is a closed subspace, but this is precisely $SO(n)$. Therefore $SO(n)$ is closed and next let us see that $SO(n)$ is bounded. This is easy, because when we look at the condition $A^T*A=Id$, so, if A is the matrix $[a_{11}, a_{12}, ...],[a_{21}, ...],[..., a_{nn}]$. $A^T*A=Id$. This will imply that, when we look at the i^{th} entry of A^T*A this is equal to summation j=1 to n a_{ji}^2, and since this A^T*A is Id, this sum has to be equal to 1. When we take sum over all i's, this implies summation of a_{ij}^2 is equal to n, for every a in $SO(n)$. Thus $SO(n)$ is bounded, I mean the topology on $\mathbb{R}^{n^2}$ is given by this Euclidean metric, and what we have just proved is that every element a is contained in the ball around 0 of radius $n^{1/2}+1$. I can add a $n^{1/2}+1$. So, we will end this lecture here.