Point Set Topology

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Week 05

Lecture 22

So, welcome to this lecture, in the previous lecture we saw that $GL_n(\mathbb{R})^+$ is path connected. So, let us begin this lecture by showing that $GL_n(\mathbb{C})$ is path connected. So, to show that $GL_n(\mathbb{C})$ is path connected, so given a matrix A in $GL_n(\mathbb{C})$, we will find a path joining A to identity. So, we have to find a path in $GL_n(\mathbb{C})$, so let us just look t this straight line path. So, let consider this $\gamma(t)$, this is $tI+(1-t)A$. Now note that if t is a complex number, then $\gamma(t)$ is actually a matrix in, so this I_n is the nxn identity matrix, then $\gamma(t)$ is a matrix in $M_n(\mathbb{C})$, it is an nxn matrix with complex entries.

So, what we need, so we need to find and note that $\gamma(0)$ is equal to A and $\gamma(1)$ is equal to I_n. So, let P(t) be determinant of $\gamma(t)$, so then this is a polynomial, so this is a non zero polynomial with coefficients in complex numbers. And this polynomial is nonzero as P(0) is equal to determinant of $\gamma(t)$, let us look at t=1 or even $\gamma(0)$ is fine, but $\gamma(0)$ we know is A, which is not equal to 0 because A is in $GL_n(\mathbb{C})$. So, thus P(t) has at most finitely many roots, has finitely many roots, sorry has at most n roots, since degree of $P(t)\le n$.

So, if you make the complex plane, then here we have 0 and here we have 1 and let us say the roots of P are λ_1 up to some λ_r, where $r\le n$. So, this is $\lambda_1, \lambda_2, \lambda_3$ upto λ_r and none of the λ_i's can be 0 or 1 because at P(0), is determinant of A, which is nonzero and P(1) is determinant of identity, which is nonzero. So, we can simply take a path joining 0 and 1, which misses all these λ_i's in the complex plane. So, let S from [0,1] to $\mathbb{C}$ be a continuous path, such that S(0)=0, S(1)=1 and the γ_i's do not belong to the image of S. Then $\gamma o S$ is from [0,1] to $M_n(\mathbb{C})$, the image lands in $GL_n(\mathbb{C})$ because if for any t in [0,1] the determinant of $\gamma oS(t)$ is nonzero as determinant of $\gamma oS(t)$ is equal to P(S(t)), which is nonzero.

P of a complex number, $P(\lambda)$ is equal to 0, if and only if λ is in the set $\lambda_1, \ldots \lambda_r$ and the image of S misses all these λ_i's. Therefore the determinant is never zero.

Thus, $\gamma oS([0,1])$ the image lands inside $GL_n(\mathbb{C})$ and clearly $\gamma oS(0)$ is equal to $\gamma(0)$, which is equal to A and $\gamma oS(1)$ is equal to $\gamma(1)$ is equal to identity. So thus, every matrix A in $GL_n(\mathbb{C})$ can be joined to identity by a continuous path in $GL_n(\mathbb{C})$. Thus $GL_n(\mathbb{C})$ is path connected. Therefore using this nice trick we saw that $GL_n(\mathbb{C})$ is path connected and this much easier to prove than the case of $GL_n(\mathbb{R})^+$. So, let us make this simple observation, so this completes the proof.

Lemma: let X be path connected and let f from X to Y be a continuous map. So then f(X) contained in Y with the subspace topology is path connected. So, let us see this, this is easy to prove. So, recall that we had seen and we have used it many times now, if we view f as a map from X to f(X), where f(X) has a subspace topology from Y, then f is continuous. So, now we want to show that f(X) is path connected.

So, if we choose, so let $f(x_1)$ and $f(x_2)$ be any two points in f(X). We have X over here and here we have x_1 and x_2 and here we have let us say f(X), this $f(x_1)$ and this $f(x_2)$. So, since X is path connected there exists a path which joins x_2, x_1 and this is f, since X is path connected there exists γ from [0,1] to X such that $\gamma(0)=x_1$ and $\gamma(1)=x_2$. So, this implies that $f\circ\gamma$ from [0,1] to f(X) is a continuous path joining $f(x_1)$ and $f(x_2)$. This implies that f(X) is path connected, so this completes the proof.

As a corollary of this, we have if X is path connected and f from X to Y is a surjective continuous path, then y is path connected. This corollary is immediate because since f is surjective it follows that Y is equal to f(X) and now we just use the previous lemma. So, as a corollary of this corollary we have these nice applications: $SL_n(\mathbb{R})$ are nxn matrices such that determinant of A is equal to 1, is path connected. In order to show that $SL_n(\mathbb{R})$ is path connected, all that we have to do is we have to construct, using the previous corollary, a surjective map from a path connected space to $SL_n(\mathbb{R})$. So, you will simply take this map to $SL_n(\mathbb{R})$.

So, this map is so if I take a matrix A, A gets mapped to so this is A. So, we just divide all the entries in the first column with determinant of A and the other entries are as it is. So, let us call this map f. So, clearly the image of f, so determinant of f(A) is equal to, since the first column has been scaled by 1 upon determinant of A, this is equal to 1 upon determinant of A into determinant of A, which is just 1. Now, $SL_n(\mathbb{R})$ is contained inside $GL_n(\mathbb{R})$ and here we have f from $GL_n(\mathbb{R})^+$, sorry $SL_n(\mathbb{R})$ and this composition is equal to identity.

This implies that f is surjective. So, it only remains to show that f is continuous, but that is easy to see because so $SL_n(\mathbb{R})$ has the subspace topology from $M_n(\mathbb{R})$, $SL_n(\mathbb{R})$ is a closed subspace of $M_n(\mathbb{R})$ and it has a subspace topology and therefore to show that f is continuous, it is enough to show that continuous if and only if $i\circ f$ is continuous and to say that $i\circ f$ is continuous, it is enough to show that the coordinate functions are continuous. So, to show $i\circ f$ is continuous, enough to show that the coordinate functions are continuous, but what are the coordinate functions? If we start with A then the coordinate functions of $i\circ f(A)$. So, the first, in the first column, the entries are of this type $a_{i1}/\det(A)$, but $a_{i1}/\det(A)$ is once again a continuous function. Since determinant does not vanish on 0, this map is continuous and nonzero, this implies the function A goes to $1/\det(A)$ is continuous.

So if I take A and send it to a_{i1} divided by $\det(A)$, this is continuous and similarly A goes to A_{ij} where j is strictly greater than 1, is also continuous. So, thus the coordinate functions of iof are continuous, this implies iof is continuous, this implies f is continuous and since $GL_n(\mathbb{R})^+$ is path connected and f is surjective and continuous, this implies $SL_n(\mathbb{R})$ is path connected. So, this is one this is nxn matrix with complex entries such that determinant of A is equal to 1, is path connected. Again we use the same map from $GL_n(\mathbb{C})$ to $SL_n(\mathbb{C})$ where we just scale the first column by $1/\det(A)$. So, we will end this lecture here and in the next lecture we will talk about compactness.