

**Point Set Topology**  
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**Week 04**  
**Lecture 18**

Welcome to this lecture. In the previous lecture, we had defined connectedness and today we will continue with that topic and we are going to define path components of  $X$ . I am sorry not the path components, the connected components. Let  $X$  be a topological space and we put an equivalence relation. Let us put an relation on the points of  $X$  as follows: We say that  $x$  is equivalent to  $y$  if there is a connected subset  $T$  contained in  $X$  such that  $x$  and  $y$  are in  $T$ . So, let us check that this defines an equivalence relation on  $X$ . So, there are three things which we need to check.

First we need to check that  $x$  is equivalent to  $x$  for all  $x$  in  $X$ . This is clear because by taking  $T$  equal to just  $\{x\}$ . Any  $\{x\}$  is connected, because for a set to be disconnected, it has to have at least two points. We cannot write the singleton set as a disjoint union of nonempty open subsets.

In fact as a disjoint union of non empty subsets. Second point we need to check is if  $x$  is equivalent to  $y$ , then  $y$  is equivalent to  $x$ . This is also clear as if  $T$  is a connected subset which contains  $x$  and  $y$ , then it obviously contains  $y$  and  $x$ . So,  $y$  is equivalent to  $x$ . Three if  $x$  is equivalent to  $y$  and  $y$  is equivalent to  $z$ , then  $x$  is equivalent to  $z$ .

To show this, recall that in the previous lecture we have proved that if  $T_1$  and  $T_2$  are connected subsets of  $X$  such that  $T_1 \cap T_2$  is non empty, then the union of  $T_1$  and  $T_2$  is connected. Now since  $x$  is equivalent to  $y$  there exists a connected subset  $T_1$  such that  $x, y$  are in  $T_1$ . Similarly, since  $y$  is equivalent to  $z$ , there is connected subset  $T_2$  such that  $y, z$  belongs to  $T_2$ . So, then  $y$  is in the intersection, and so it is nonempty and thus  $T_1 \cup T_2$  is connected, and contains  $x$  and  $z$ . Thus  $x$  is equivalent to  $z$ , this shows that the  $\sim$  is an equivalence relation.

Now every time we have an equivalence relation, on a set  $X$ , we can decompose it into equivalence classes. So, this equivalence relation breaks  $X$  into disjoint union of equivalence classes. So, we can write  $X$  as a disjoint union of  $X_i$ 's in some index  $I$ . So, each  $X_i$  is the equivalence class of some  $x$  and  $x \sim y$  if and only if  $x$  and  $y$  belong to the same  $X_i$ . We have the following proposition, which has, let us say three parts.

Every connected subspace of  $X$  is contained in  $X_i$  for some  $i$ . Given any connected subspace of  $X$ , it is going to be contained in one of the  $X_i$ 's. So, the second part is each

$X_i$  is connected. So, every connected subspace is contained in some  $X_i$ , and each of these  $X_i$ 's is connected. Therefore, these two will imply that  $X_i$ 's are maximally connected subsets of  $X$ , and the third point is each  $X_i$  is a closed subspace of  $X$ .

So, let us prove this proposition. So, let  $T$  be connected subspace of  $X$ . So, assume that  $T \cap X_i$  is not empty and  $T \cap X_j$  is not empty. So, we will prove the first point by contradiction. So, let us assume that a connected subspace meets two  $X_i$ 's, it meets  $X_i$  and  $X_j$  where  $i$  is not equal to  $j$ .

So, let  $t, s$  belong to  $T$ . Then by the definition of the equivalence relation,  $t \sim s$  as  $T$  is connected and it contains both  $t$  and  $s$ . So, I should have written: let  $t$  be in  $T \cap X_i$  and  $s$  be in  $T \cap X_j$ . Then as  $s, t$  are in  $T$  this implies  $s$  is equivalent to  $t$  but then this implies that the equivalence class of  $t$ , which is  $X_i$  is equal to the equivalence class of  $s$ , which is  $X_j$ . This is a contradiction Because  $X_i$  is not equal to  $X_j$ .

Therefore, given any connected subspace, it can be contained only in one of the  $X_i$ 's. Next, let us prove (2). So, let  $X_i$  be an equivalence class and let  $x$  be in  $X_i$ . So, we fix this point ' $x$ '. Then for any  $y$  in  $X_i$ , since  $x$  and  $y$  are both in  $X_i$ , as  $x \sim y$ , there is a connected subspace  $T_y$  contained in  $X$ , such that  $X_i$ , sorry  $X$ , I am just using the definition of the equivalence relation such that  $x, y$  belong to  $T_y$ .

So, clearly from point (1), it follows that  $T_y$  is contained in  $X_i$ . Thus we can write  $X_i$  as union over all these  $y$  in  $X_i$  of  $T_y$ . This inclusion is obvious, because for each  $y$  in  $X_i$  the subset  $T_y$  is contained in  $X_i$  and therefore, when we take a union, it is going to be contained in  $X_i$ . Conversely, for the other inclusion  $T_y$  for every  $y$  in  $X_i$ ,  $T_y$  contains  $y$ . So, therefore, this is actually equal. So, now we have this variant of the earlier lemma we have proved.

Consider the following Let  $T_i$  be a family of connected subsets of  $X$ , and suppose the intersection of all these is nonempty. Then, the union of  $T_i$ 's is connected. The proof of this is very similar to the proof of the lemma that we proved. Let us just try and prove it. Suppose the union is not connected, then there exists nonempty open subsets  $U$  and  $V$ , both in  $X$ , such that union of  $T_i$ 's is written as the disjoint union intersected with  $U$ .

Both these are nonempty and they are of course disjoint open subsets in this union. Now, let we fix this element ' $a$ ', the idea is the same as in the earlier lemma, we fix this element ' $a$ '. Now, we can intersect both sides with  $T_i$ . This will imply that  $T_i$  is equal to  $T_i$  intersected  $U$ . We can fix  $a$  and before we and assume  $a$  is in this union.

So, now, we intersect this equation with  $T_i$  on both sides. So, we get  $T_i$  is equal to  $T_i$

intersected  $U$  disjoint union  $T_i$  intersected  $V$ . Now this contains  $a$ , and so is nonempty, and on the other hand  $T_i$  is connected, which forces that  $T_i$  intersection  $V$  has to be empty. This implies that  $T_i$  is completely contained inside  $U$  for every  $i$ , and this happens for every  $i$ . This implies that this union  $T_i$  intersected with  $V$  is empty, which is a contradiction.

This intermediate lemma tells us that this union is connected when all the  $T_i$ 's are connected and their intersection is nonempty right. Now we will apply this to our situation. Applying this to our situation we get that  $X_i$  can be written as a union of  $y$  in  $X_i$  of  $T_y$ , each  $T_y$  is connected, and the intersection of all these  $T_y$ 's at least contains  $x$ . This implies that the intersection is nonempty because all the  $T_y$ 's contain  $x$ . This implies that  $X_i$  is connected.

This proves (2) and finally, let us prove (3) as  $X$  is connected, as  $X_i$ 's are connected, we had seen this corollary that  $A$  is connected in  $X$  implies  $A$  closure is connected, we have proved this earlier. So, this implies that  $X_i$  closure is connected, but by (1), each  $X_i$  closure is contained in a unique  $X_j$  right, thus, this  $X_j$  has to be  $X_i$  because  $X_i$ 's containing  $X_i$  closure, thus  $X_i$  closure is contained in  $X_i$ , which implies that  $X_i$  is equal to  $X_i$  closure. This implies that  $X_i$  is closed. This completes the proof of the proposition. The proposition says that: given any topological space  $X$ , we can break it up into these equivalence classes, classes and each equivalence class is defined by the relation that we started the lecture with, and each of these equivalence classes are connected.

And they are closed, and given any connected subspace, it is contained in one of these equivalence classes. So, let us just give a definition. So, the equivalence classes are called the connected components of  $X$ . So, as an example let us work with this: So, let  $\mathbb{Q}$  contained in  $\mathbb{R}$  have the subspace topology. So, what are the connected components of  $\mathbb{Q}$ ? We claim that, so before that, note that given any  $a$  in  $\mathbb{Q}$ , the subset  $\{a\}$  is connected.

So, the question is, what is the maximal connected subset which contains this  $\{a\}$ , and that will be the connected component containing  $a$ . Now, we claim that the connected component containing  $a$  is just this set  $\{a\}$ . Basically what we are saying is we have taken  $\mathbb{Q}$ , we can write it as a disjoint union of connected components  $X_i$ 's. So now, each  $X_i$  is nonempty obviously, and just pick any  $a$  in  $X_i$ , and the claim is  $X_i$  is equal to  $\{a\}$ .

Let us see this. If not, suppose  $b$  also belongs to  $X_i$  and  $b$  is not equal to  $a$ . Suppose that  $b > a$ . Then we have this  $a$  is over here,  $b$  is over here, and we can choose an irrational  $c$  here, So, then we can write  $X_i$ . So,  $c$  is not in  $\mathbb{Q}$  and  $X_i$  is in  $\mathbb{Q}$ , therefore  $c$  is not in  $X_i$ . Then  $X_i$  does not contain  $c$  and we can write  $X_i$  as a disjoint union of open

subsets  $(-\infty, c)$  disjoint union  $(c, \infty)$ , but this contradicts these are both nonempty, because this contains  $a$  and this contains  $b$ , and both these are open, and it is a disjoint union, this contradicts the connectedness of  $X_i$ .

Thus  $X_i$  is forced to be  $\{a\}$ . Thus  $X_i$  is forced to be a single point. Therefore, all the connected components are just  $\{a\}$  in  $\mathbb{Q}$ . So, this is the decomposition of  $\mathbb{Q}$  into connected components. Let us make a remark: It is obvious  $X$  is connected iff it has only one connected component. This remark is obvious because if  $X$  is connected, then given any two points  $x$  and  $y$ , then  $T=X$  right.

Therefore, there will be just one equivalence class in the decomposition, and conversely if there is just one equivalence class, then that has to be  $X$ , and therefore  $X$  is going to be connected, because each equivalence class is connected. So, we will end this lecture here, and in the next lecture, we will introduce the topic of path connectedness.