

**Point Set Topology**  
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**Week 03**  
**Lecture 15**

So, in the previous lecture we had introduced metric spaces and the last result we had proved was the following lemma. So, we introduced metric spaces, and we saw how given a metric space, we can put a topology on it and we proved the following lemma. We proved that if  $A$  is a subset of  $X$  which is a metric space. Then  $x$  belongs to  $A$  closure if and only if there is a sequence  $x_n$ 's in  $A$  with  $x_n$ 's converging to  $x$ . Using this lemma, we will give a convenient criterion for a subset in a metric space to be closed. Let  $X$  be a metric space, and let  $A$  contained in  $X$  be a subset.

Then  $A$  is closed if and only if it has the following property. So, if  $x_n$ 's is a sequence in  $A$  and  $x_n$ 's converge to  $x$ , then  $x$  is also in  $A$ . So, let us prove this lemma. Recall we had proved that  $A$  is closed in a topological space if and only if  $A$  is equal to  $A$  closure.

So, we will use this criteria. First let us assume that, it is enough to show that  $A$  is equal to  $A$  closure if and only if  $x_n$  belongs to  $A$ , and  $x_n$  converges to  $x$ , then  $x$  belongs to  $A$ . First, assume that  $A = A$  closure right. Then we have to show that the set  $A$  it has this property. So, let us see this.

So, if  $x_n$ 's are in  $A$ , and  $x_n$  converges to  $x$ , then by the lemma we proved in the previous lecture,  $x$  belongs to  $A$  closure, and we are assuming that  $A$  is equal to  $A$  closure. So, this implies that  $x$  belongs to  $A$ , thus  $X$  has this property. Conversely suppose  $A$  has this property that every time we have  $x_n$ 's in  $A$  and  $x_n$  converges to  $x$ , then  $x$  is in  $A$ . So, we assume that  $A$  has this property and we want to show that  $A = A$  closure. Let  $x$  be an element of  $A$  closure.

It is obvious that  $A$  is contained in  $A$  closure. We just have to prove the converse. We take an element  $x$  in  $A$  closure. So, by this lemma that we proved in the previous class, there is a sequence  $x_n$ 's in  $A$  such that  $x_n$ 's converge to  $x$ . By this property that  $A$  has, every time we have such a sequence then  $x$  belongs to  $A$ , because of this property.

So, we have a sequence  $x_n$ 's in  $A$  and  $x_n$  converges to  $x$ . So, we have  $x$  belongs to  $A$ . This implies that  $x$  belongs to  $A$ , is equal to  $A$  closure. Thus  $A$  is closed. To check that a subset is closed, we just need to check that, every time a sequence of points in that subset  $A$ ,  $x_n$ 's converge to  $x$ , then we just have to check that this point  $x$  is in  $A$ .

To check the continuity of a map between metric spaces. So, that is this theorem. So, let  $X$  and  $Y$  be metric spaces, and let  $f$  from  $X$  to  $Y$  be a map of sets. Then  $f$  is continuous if and only if for every sequence  $x_n$ 's converging to  $x$ , we have  $f(x_n)$  converges to  $f(x)$ . This sequence is in  $X$ , and this sequence is in  $Y$ .

So, this gives us a way to check continuity of maps between metric spaces. So, let us prove this. So, let us assume, we first assume that  $f$  is continuous, and let  $x_n$  converging to  $x$  be a sequence. For  $\varepsilon > 0$ , consider the open subset, the  $\varepsilon$ -ball around  $f(x)$ . Here we have  $B_\varepsilon(f(x))$  in  $Y$ , and this is an open subset of  $Y$ .

Here we have  $B_\varepsilon(f(x))$  in  $Y$ , and here we have  $f(x)$ . We just take this open ball around  $f(x)$  of radius  $\varepsilon$ , and we look at its inverse image. Since  $f$  is continuous, its inverse image may be some open subset like this, implies that,  $f^{-1}(B_\varepsilon(f(x)))$  is open in  $X$ . Moreover, since  $x$  is in this open subset. So,  $x$  is somewhere over here and this is an open subset  $U$  which contains  $x$ .

So, therefore, there is a small  $\delta$ -ball of radius  $\delta$  around  $x$ , which is completely contained in this open subset. So, this implies that there is  $\delta > 0$  such that this ball of radius  $\delta$  around  $x$  is completely contained inside  $f^{-1}(B_\varepsilon(f(x)))$ . As  $x_n$ 's converge to  $x$ , by the definition of convergence, there exists  $N$ , such that for all  $n \geq N$ , these  $x_n$ 's are in this ball. We have this sequence of  $x_n$ 's which converge to  $x$ . After finitely many, all these  $x_n$ 's are going to be in this  $\delta$  open ball, This implies that,  $f(x_n)$  is contained in this subset.

So, this implies that for all  $n \geq N$ ,  $f(x_n)$  belongs to  $B_\varepsilon(f(x))$ , and by the definition of convergence, this implies that  $f(x_n)$  converges to  $f(x)$ . This proves one part of the lemma. Now let us prove the converse. So, to prove the converse, it suffices to show that, to prove the converse, we are given that  $f$  satisfies a particular property: it takes convergent sequences to convergent sequences, and we have to show that  $f$  is continuous. To show that  $f$  is continuous, it suffices to show that the inverse image of a closed subset is closed.

When  $Z$  contained in  $Y$  is closed and to show that this is closed, we will use this lemma that we just proved. So, let us see. To show that  $f^{-1}(Z)$  is closed, we will use, we will show that  $f^{-1}(Z)$  is equal to its closure. So, let  $x$  be an element in  $f^{-1}(Z)$  closure. So, this implies that there is a sequence of  $x_n$ 's converging to  $x$ , where  $x_n$ 's is in  $f^{-1}(Z)$  of  $Z$ .

So,  $x_n$  is in  $f^{-1}(Z)$  implies that  $f(x_n)$ 's is in  $Z$ . Moreover, since  $f$  takes convergent sequences to convergent sequences, that is the property that  $f$  has, if  $x_n$

converges to  $x$ , then  $f(x_n)$  converges to  $f(x)$ . So, as  $x_n$  converges to  $x$ , this implies  $f(x_n)$  converges to  $f(x)$ . Now, as  $Z$  is closed, this implies that  $f(x)$  belongs to  $Z$ ,  $Z$  is closed,  $f(x_n)$  is in  $Z$ , and  $f(x_n)$  converges to  $x$ , this implies that  $f(x)$  belongs to  $Z$ . Plus  $x$  belongs to  $f^{-1}(Z)$ .

So, we started with the point  $x$  in  $f^{-1}(Z)$ , and we proved that  $x$  belongs to  $f^{-1}(Z)$ . This implies that  $f^{-1}(Z)$  is equal to  $f^{-1}(Z)$ . This implies that  $f^{-1}(Z)$  is closed, which implies that  $f$  is continuous. So, that kind of brings us to an end of the second part of this course. In the first part, we introduce the notion of topological space, and basic examples.

In the second part, we introduced continuous maps between topological spaces, and we studied their properties. In the second part, we also defined metric spaces, which gives us a large collection of topological spaces, and most important collection of topological spaces. But now we are coming to the third part of this course. So, in the third part, we will introduce topological properties of topological spaces. So, the first property we are going to introduce is that of connectedness.

So recall that two topological spaces are called homeomorphic, if there is a bijective continuous map  $f$  from  $X$  to  $Y$  such that  $f^{-1}$  is also continuous. Two topological spaces are "the same". So, the word same is in quotes "same" as in as far as continuous maps are concerned, they cannot be differentiated if they are homeomorphic. So, it is natural to ask to try and classify two topological spaces upto homeomorphism. For instance we can take the interval  $[0,1]$  and we can take the disjoint interval  $[0,1]$ , I mean this topological space, which is the disjoint union of two intervals, and we can ask if these are homeomorphic.

If we can find some topological property which  $[0,1]$  has and this disjoint union of two intervals does not have, then it would mean that these two cannot be homeomorphic. We will introduce this notion of connectedness now, which will help us distinguish between two topological spaces. In this lecture we shall introduce the notion of connectedness. Definition: let  $X$  be a topological space. Suppose there are nonempty open subsets  $U$  and  $V$ . So, they are both nonempty, that is important.

So,  $U \cap V$  is empty, they are disjoint and  $X$  is the union of both these. Then, we say that  $X$  is disconnected. Otherwise if  $X$  is not disconnected, then we say  $X$  is connected. Immediately, we have this property, we have defined this property of topological spaces. So, we can ask the following questions We can ask is  $\mathbb{R}$  connected? So,  $\mathbb{R}$  is with the standard topology, what about  $\mathbb{R}^n$ ? Again with the standard topology.

What are the connected subspaces of  $\mathbb{R}^n$ ? Is there a nice way to describe these, or for that matter of  $\mathbb{R}^n$ . 3: if  $X$  and  $Y$  are connected, can we say something? One of the topological spaces we constructed, out of  $X$  and  $Y$ , is the product. About the connectedness of  $X \times Y$ . Then we saw various examples, So connectedness of  $GL_n(\mathbb{R})$ ,  $SL_n(\mathbb{R})$ ,  $O_n$ ,  $SO_n$ ,  $GL_n(\mathbb{C})$ ,  $SL_n(\mathbb{C})$  and  $U_n$ . What can we say about the connectedness of these objects.

So, in order to answer these questions, we will need to develop some results and tools which we can use here. So, let us begin that. The first proposition we are going to prove is: let  $U$  contained in  $X$  be dense. If  $U$  is connected, then  $X$  is connected.

Let us prove this. Let us assume that  $X$  is disconnected, and arrive at a contradiction. As  $X$  is disconnected, there are non empty open sets  $U_1$  and  $U_2$ , which are disjoint, such that we can write  $X$  as a disjoint union of  $U_1$  and  $U_2$ . Now, we simply intersect both sides with  $U$ . Intersect both with  $U$  to get  $U = U \cap U_1 \cup U \cap U_2$  disjoint union. As  $U$  is dense in  $X$ , this implies that  $U \cap U_1$  is nonempty, and  $U \cap U_2$  is nonempty, but this shows that  $U$  is disconnected, which is a contradiction. So, this implies that our hypothesis is wrong.

That means  $X$  is connected. As a corollary, let  $A$  contained in  $X$  be a subspace. If  $A$  is connected, then the closure is connected. Proof: When we talk about  $A$  and  $A$  closure, all these are subsets of  $X$ , and we are always in the subspace topology. Recall that we have proved that  $A$  is dense in  $A$  closure. Applying the previous proposition implies that  $A$  closure is connected. So, we will end this lecture here.