Point Set Topology
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Week 03
Lecture 13

Okay, so let us begin this lecture. So, in the previous lecture we had studied the notion of a closed subset. We had introduced the notion of a closed subset and closure of subset and we also saw in the exercises that the closure of a subset A, is a smallest closed subset which contains A. So, let us continue our discussion with closed subsets. So, today we will begin with the definition of a dense subset. As always X is a topological space.

A subset T contained in X is said to be dense in X if for every non-empty open set U contained in X, we have T intersection U is non-empty, and the proposition, the main proposition is let A be a subset of X and let A closure denote its closure. So, then let A closure have the subspace topology. Then A contained in A closure is dense. So, let us prove this.

So, we need to show. By the definition of denseness, we need to show that if V contained in A closure is a non-empty open subset then V intersection A is non-empty. By the definition of subspace topology this set V is equal to U intersection A closure, for some open subset U contained in X, U open in X and as V is nonempty this implies there exist some x in V which is equal to U intersection A closure. Therefore U is an open subset of X, U contains x, and x is in A closure. Thus by the definition of closure, this implies that U intersection A is nonempty.

Recall the definition of closure X is in A closure if every open subset containing X meets A. So, this implies that U intersection A which we can write as U intersection A closure intersection A, because recall that A is contained in A closure, that is trivial and this is equal to B, this is non-empty. Thus V intersection A is non-empty, and which shows that A is dense in A closure. Let U be open in X, and V be open in U, where U has been given the subspace topology. Then we want to say that then V is open in X.

So, in simple words open subset of an open subset is open in the larger topological space. So, let us prove this, the proof is easy. By the definition of subspace topology, there is an open subset V tilde in X such that this V is equal to U intersected V tilde. But now U is open in X, V tilde is open in X, and finite intersection of both U and V tilde is open in X, this implies U intersection V tilde is equal to V is open in X. So, this completes the proof here.

and similarly, we have a similar result for closed subsets. So, before we prove that result for closed subsets we need a small lemma. Let A contained in X be a subset. The closed subsets of A, where A is given the subspace topology are precisely of the form Z intersection A, where Z is closed in X. So, let us prove this lemma.

So, the open subsets of A are precisely of the form U intersection A where U is open in X and the closed subsets are the compliments of open subsets here. Thus, closed subsets of A are precisely of the form A and remove an open subset. Now a simple set theoretic check shows that $A\setminus(U\cap A) = (X\setminus U)\cap A$, But X minus U is a closed subset of X. Thus closed subsets are precisely of the form Z intersection A, where Z is a closed subset. This is our Z, and as a corollary of this lemma, let us see the analog of the proposition, which says that closed subset of a closed subset is closed in X.

So, let Z be a closed subset of X and let Z_1 be closed in Z. Then Z_1 is closed in X and the proof of this corollary is very similar. So, by the previous lemma this subset Z_1, we can write as some Z tilde intersection Z, where Z tilde is a closed subset of X and as an intersection of closed subsets is closed, this implies Z_1 is closed in X. Next we want to say that checking whether a subset is closed or not, can be checked by restricting our attention to closed subsets. So, precisely what I mean is the following.

Let Z_1 and Z_2 be closed subspaces such that this X we can write as Z_1 union Z_2. So, a subset Z containing X is closed if and only if Z intersection Z_i is closed in Z_i for i equal to 1,2. So, this proposition will have a useful corollary which we will see next, but let us prove this proposition first. If Z is closed then simply because intersection of closed subsets is closed, this clearly implies Z intersection Z_i is closed by the lemma. Let me just write this implies that Z intersection Z_i is closed in Z_i because of this lemma.

And so let us prove the converse conversely assume Z intersection Z_i is closed in Z_i for i equal to 1, 2. Then by this corollary, Z intersection Z_1 and Z intersection Z_2, both are closed in X. Then by the above corollary, let me just keep the corollary here, Z intersection Z_1 and Z intersection Z_2 are both closed in X. And since the finite union of closed subspaces is closed, this implies that Z which is equal to Z intersection Z_1 union Z intersection Z_2 As X is equal to Z_1 union Z_2. And finite unions of closed subspaces are closed, this implies Z is closed.

So, as a corollary of this proposition let us prove the following useful result. We have the following useful result about continuous maps. Theorem: Let A and B be subsets of X such that be closed subsets of X is a union of A and B. So, let A and B have the subspace topology. Now, let f from X to Y be a map.

Right now it is just a map of sets, such that f restricted to A and f restricted to B are continuous. Then f is continuous. What do we mean by f restricted to A and f restricted to B. Look at this inclusion. So, let us call this i_A, this is f right.

So, f restricted to A is simply f o i_A. If f is continuous, then since both the inclusions are continuous obviously the restriction is continuous. Here this theorem is telling us that if the restriction of f to both A and B is continuous, then f is continuous on X. To check that a map is continuous it suffices if we can write our topological space X as a union of two closed subspaces, two or finitely many, and such that f restricted to each of these is continuous. We will only prove this for the union of two closed subspaces, but it will be immediately clear that the proof works so finitely.

Proof. So, to show f is continuous, it enough to show that f inverse of a closed subspace is closed in X. So, by the previous proposition it is enough to show that f inverse of Y' intersected A and f inverse of Y' intersected B are closed in A and B respectively. But, note that f inverse of Y', (this is a simple set theoretic check) intersected with A is simply f restricted to A inverse image of Y'. And similarly f inverse of Y' intersected B is equal to f restricted to B inverse of Y'. Thus, as f restricted to A and F restricted to B are continuous, this implies that f restricted to A inverse of Y' is closed in A.

And similarly, f restricted to B inverse of Y' is closed in B. This implies that f inverse Y' is closed in X, this implies F is continuous. So, as an application of this proposition. Use this theorem to show that the maps. We have two maps let us say f_1 from ℝ^2 to ℝ.

f_1(x, y) is equal to the maximum of x and y. And f_2(x, y) is equal to minimum of x and y, are continuous. So write ℝ^2 as a union of two closed subspaces and f_1 and f_2, both restricted to each of these subspaces should be continuous. That should be trivial and therefore both these maps are continuous. So, we will end this lecture here.