Point Set Topology
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Week 03
Lecture 11

So, welcome to this lecture, this is lecture 11. So far we have defined topological spaces and continuous maps between them. Next we will define continuous maps in terms of limits. We want to define continuity in terms of limits in a metric space. So, we will have to define metric spaces, and then limits and finally, give a criterion of when a map is continuous in metric spaces in terms of limits. And that is the theme of the next few lectures.

So, we begin with discussion on closed subspaces. We have not reached metric spaces, but we will reach there soon. So, definition, this is a general discussion. The definition of a closed subset is very easy.

So, let X be a topological space. A closed set $Z \subseteq X$ is said to be closed if the complement $X \setminus Z$ is an open subset of X. Recall what we mean by this, so, X is a topological space. So, it has a topology $\mathcal{T}$ and Z is said to be closed if when we take the complement that is in $\mathcal{T}$. Let us make a remark.

One easily checks the following properties of closed subsets. These follow from the definition of a topology. So, the first property is that the empty set ϕ and the full set X are both closed. The second property is that let Z_1 .

.. Z_r be finitely many closed sets. Then their union Z_1 ∪ Z_2 ∪ ... ∪ Z_r is closed.

The third property is that: let I be a set and suppose we are given a selection of closed subsets, then the intersection of these closed subsets, which is a subset of X, is closed. All these three properties follow from the analogous properties for open subsets, from the respective properties of open subsets. That is one remark and the second remark is that suppose we are given a collection of subsets, suppose we are given a subset S contained in P(X) which means we are given a collection of subsets of X, which satisfy the above three properties. So, S satisfies these properties 1, 2 and 3. Then, let $\mathcal{T}$ be the collection of those U in X such that X \ U is contained in S.

Then $\mathcal{T}$ defines a topology on X. In other words, what this remark tells us is that to define a topology on X, it is enough to specify the collection of open subsets which should satisfy the three conditions, which we saw in the first lecture, either that or we can specify a collection of subsets which satisfy these three conditions, these three conditions here and that would specify a topology. So, this remark is left as an exercise, the proof of this remark. So, this is left as an exercise. And another simple statement which is again left as an exercise, which just follows from the definition.

Let X and Y be topological spaces and let f : X -> Y be a map of sets. Then f is continuous if and only if, for every closed subset in Y, $Z \subseteq Y$, the subset $f^{-1}(Z)$ is, this is the subset of X, is a closed subset. So, the main point behind this exercise is this easy check that $f^{Y\setminus Z} = X \setminus f^{-1}(Z)$. We have to use this well known property of the inverse image of a map. In one of the lectures we saw after we introduced various ways of specifying topologies, we saw several examples here.

So, let us just revisit those examples and see which of those subsets are closed and which are open. The first example, or one of the examples we had seen, was the set S^1, this is those (x, y) in $\mathbb{R}^2$ such that $x^2+y^2 =1$. So, can we say something about this set, is this set open or closed? So, the answer to this is quite easy. First, we have to make the following observations. So, recall that we proved that the set of continuous maps is closed under multiplication and addition.

And also that we also proved that the projection maps, the projections from a product are continuous. So therefore, under this map (x, y) maps to x, is continuous. So, just for fun let us just make a simple map. We have both these maps, (x, y) goes to x, and (x, y) goes to y is continuous. We can make the graph of this map (x, y) goes to y.

So, let us say this is going to the y axis. If we take a small open subset here, the inverse image of that is going to be this strip over here. If this is A and this is B. Then this is going to be (0, a) and this is going to be (0, b), this point here is (0, a) this point here is (0, b). And we have to take this open, the inverse image of this interval is this tube, is this strip if I may say so.

And clearly this is open because given any point over here, we can always find a square like this. This is a picture of the projection on to the y coordinate. And similarly, we can make a picture of the projection on to the x coordinate. So, here the inverse image of (a, b) will be this open strip. So, this point is (a, 0), this point here and this point is (b, 0) And clearly this is also open because given any point here, we can always find a, So, now both these projections are continuous, so therefore (x, y) goes to x^2 is also continuous, because this is simply the map, let me call this map p_1 and let me call this map p_2.

So, this is simply the map p_1^2. So, if you want, so (x, y) maps to $p_1(x, y)$ and the square of this. So, both these are continuous. Similarly, (x, y) goes to y^2 is continuous, and addition is also continuous. So, therefore this implies that, so the results we had proved in the previous class, (x, y) goes to x^2+y^2 is a continuous map.

And now inside $\mathbb{R}$ the subset $\{1\}$ is a closed subset because what is the complement of $\{1\}$? so, this is 0 and this is the point 1. So, we are just deleting this point, we are deleting this. So, if I take any x over here, then I can always find a small ε such that the neighborhood, the ε neighborhood around x is going to be contained in $\mathbb{R}\setminus\{1\}$. So, in other words, for every x in $\mathbb{R}\setminus\{1\}$, we can find $\varepsilon > 0$ such that this ε neighborhood around x is contained in $\mathbb{R}\setminus\{1\}$. Therefore $\mathbb{R}\setminus\{1\}$ is open which implies the subset which contains only 1 is closed.

This in turn implies, so let us call this function, give this function name f. So, since f is continuous, this implies that f^{-1} of a closed subset is closed. Let me just write that of a closed subset is closed, which implies that $f^{-1}(\{1\})$ is closed. But what is $f^{-1}(\{1\})$? It is exactly those (x, y) by definition in $\mathbb{R}^2$ such that $f(x, y) = 1$. This is same as saying that $x^2+y^2=1$, but this is precisely equal to the set S^1.

So, therefore, we see that S^1 is a closed subset of $\mathbb{R}^2$. On the other hand, we could have proved this fact directly from the definitions because we can look at S^1 and S^1 is this. So, if we take any point in $\mathbb{R}^2 \setminus S^1$, we could have directly shown, and it is not very hard that we can find a small ball or a square of radius ε which is completely contained inside the complement. So, we could have proved that directly as well. So, let us take the second example, another example which we had seen was the spheres S^n.

This is those $(x_0, x_1, ..., x_n)$ in $\mathbb{R}^{n+1}$ such that $x_0^2 + x_1^2 + ... + x_n^2 = 1$. Is S^n open? Or can we say, let me just phrase it like: is S^n open or closed in $\mathbb{R}^{n+1}$ By the same reasoning as above, so since the function f from $\mathbb{R}^{n+1}$ to $\mathbb{R}$ given by $f(x_0, x_1,$.

.., $x_n) = x_0^2 + x_1^2 + ... + x_n^2$ is continuous. Once again because projections are continuous, and therefore the squares of each of the projections is continuous, and when we add them, that is continuous here. This implies that, so once again $f^{-1}(\{1\})$ is a closed subset of $\mathbb{R}^{n+1}$, which implies that S^n is a closed subset of R^{n+1} and this is exactly equal to S^n by the same reason as the previous example. So, here are two examples, another example we had seen was the subset $GL(n, \mathbb{R})$, this is equal to those A in $n \times n$ matrices (in $M(n, \mathbb{R})$) such that determinant of A is not equal to 0. So, again can we say if this is open or closed? Let us look at the determinant map from $M(n, \mathbb{R})$ to $\mathbb{R}$.

So, as an example let us look at the determinant of a 2×2 matrix, so let us say this is x_{11} x_{12} x_{13} x_{14}. So, we know that the determinant is $x_{11}x_{22} - x_{21}x_{12}$. Each of the projections are continuous, so each x_{ij} is continuous and the product of continuous maps is continuous, so this implies that this is continuous, and similarly this is continuous because the projections are continuous, and this function is the product of two projections. So, for instance this function is this first function is X maps to $x_{11}x_{22}$ so this function is continuous. Therefore, in this case this is continuous and in general we know that, since the determinant is a polynomial in the coefficients of a matrix.

So, this determinant function is actually a polynomial function in the entries here. Therefore, and each coefficient is the projection map, and the projection is continuous, therefore polynomials in projections are continuous. Therefore, the determinant map is continuous and now note that $GL(n, \mathbb{R})$ is simply $\det^{-1}(\mathbb{R}\setminus\{0\})$. So, determinant is continuous and as $\mathbb{R}\setminus\{0\}$ is open, this implies $GL(n, \mathbb{R})$ is an open subset of $M(n, \mathbb{R})$. So, another example we can take is the set $SL(n, \mathbb{R})$, this is those A in $M(n, \mathbb{R})$, so determinant of A is equal to 1.

This is clearly, so this is closed as $SL(n, \mathbb{R})$ is equal to $\det^{-1}(\{1\})$. And yet another example we saw was the set of orthogonal matrices, this is those A in $M(n, \mathbb{R})$ such that A^tA is equal to identity. We claim that this is a closed subset. Now, in each of the above examples, showing that these subsets are closed directly could be somewhat complicated, and we see that once we use, once we obtain these subsets, once we can realize these subsets as inverse images under continuous maps of certain subsets, then it becomes a lot more easier. So, here also we are going to do the same thing.

So, first we have to look at this map from $M(n, \mathbb{R})$ to $M(n, \mathbb{R})$, so let us call this map F. So, the map F is A goes to A^tA. So, let us see what this does for 2×2 matrices. So we have x_{11}, x_{12}, x_{21}, x_{22}. This is going to map to the transpose of this times the matrix, but this is equal to $x_{11}^2 + x_{21}^2$.

This is $x_{11} x_{12} + x_{21} x_{22}$. This is $x_{12} x_{11} + x_{22} x_{21}$ and this is $x_{12}^2 + x_{22}^2$. So, notice that on the right hand side each entry is a polynomial in the entries of A, right and $M(n, \mathbb{R})$ here has the product topology, which is the same as the standard topology. So, if we view $M(n, \mathbb{R})$ as $\mathbb{R}^{n^2}$, right, so F is continuous if and only if each F_{ij} is continuous, right. So, F_{ij}, so in our example, let us say what is F_{11}. So, F_{11} is precisely this coordinate, so F_{11} of this matrix, Let us denote this matrix by X for simplicity of notation, right.

Similarly, $F_{12}(X)$ is equal to this coordinate, that is $x_{11} x_{12} + x_{21} x_{22}$. And similarly, we have $F_{21}(X) = x_{12} x_{11} + x_{22} x_{21}$ and finally, we have $F_{22}(X)$, right. So, all these each of the coordinate functions, so each of the coordinate functions, coordinates is a polynomial in the entries of X. In other words each of these coordinate functions, we can write it as a polynomial obtained using the projection maps. Each of the projections is continuous, and since we take product and add etc, that is also going to be continuous.

Therefore each of these coordinate functions is a continuous function which implies that the map the F is a continuous function. Because we can view F as a map from $M(n, \mathbb{R})$ to $M(n, \mathbb{R})$, well standard topology or the product topology it does not matter because as we saw the product topology on $\mathbb{R}^n$ is same as the standard topology on $\mathbb{R}^n$, right. So, therefore so this shows that F is continuous, right and now so note that. So, this is for the 2×2 example, so in the $n \times n$ situation, so in the $n \times n$ situation, or let me now write this. So, you can write down situation so in the n cross n situation or let me now write this.

So, you can write down explicit coordinates for general n, right. So, what are F_{ij} for general n in terms of x_{ij}? So, this is an easy exercise so you can just write this down in any case, this shows that F is continuous in general. So, F from $M(n, \mathbb{R})$ to $M(n, \mathbb{R})$ is continuous. So, clearly this $O(n, \mathbb{R})$ is equal to $F^{-1}(I_n)$, right. So, $M(n, \mathbb{R})$ contains the identity matrix and therefore to show that $O(n, \mathbb{R})$ is closed in $M(n, \mathbb{R})$, and suffices to show that this $\{I_n\}$ in $M(n, \mathbb{R})$ is a closed subset, right.

But this is a very general result so we will write this as lemma, right.

Let $a_1 \ldots a_m$ in $\mathbb{R}^n$ be an element, right. So, let us denote this by A. Then $\mathbb{R}^n \setminus \{A\}$ is a closed subset. Proof, this is left as an exercise. So, we will end here.