Point Set Topology

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Week 02

Lecture 10

So, let us begin the next lecture. In the previous lecture, we repeatedly used this result or remark, which I had left as an exercise, but let us just see how to do this. So, let us write it as a proposition: the standard topology and product topology on $\mathbb{R}^n$ are equal. What do I mean by product topology on $\mathbb{R}^n$? $\mathbb{R}^n$ is equal to the product $\mathbb{R}\times\mathbb{R}\times\ldots$

$\times\mathbb{R}$ (n times), and we give each factor the standard topology, and then we take the product topology. Proof: Recall that we had proved that if X is a topological space, it is a set with two topologies, $\mathcal{T}_1$ and $\mathcal{T}_2$, and if $\mathcal{B}_i$ is a basis for $\mathcal{T}_i$ ($i = 1,2$). Then if $\mathcal{B}_1$ is contained in $\mathcal{T}_2$ implies $\mathcal{T}_1$ is contained in $\mathcal{T}_2$. We had used this result to prove that topologies are equal, so we will use this once again.

So, let $\mathcal{T}_1$ denote the standard topology on $\mathbb{R}^n$ and let $\mathcal{T}_2$ denote the product topology. So, a basis for $\mathcal{T}_1$ is given by sets of the form $S_\varepsilon(\underline{x})$, so here $\underline{x}$ is a point in $\mathbb{R}^n$, so $\underline{x} = (x_1, x_2, ..., x_n)$, and this set $S_\varepsilon(\underline{x})$, recall this was those points $\underline{y}$ in $\mathbb{R}^n$ such that $|x_i - y_i| < \varepsilon$, but this set is precisely equal to the product of $B_\varepsilon(x_i)$, where $B_\varepsilon(x_i) = (x_i - \varepsilon, x_i + \varepsilon)$.

And each $B_\varepsilon(x_i)$ is open in $\mathbb{R}$ and therefore, this is an element of the basis for the product topology. Since (The product of $B_\mathbb{R}(x_i)$'s)belongs to $\mathcal{B}_2$, which is equal to the basis for the product topology, this implies that this $\mathcal{B}_1$ which consists of these sets $S_\varepsilon(\underline{x})$ is contained in $\mathcal{B}_2$, which is contained in $\mathcal{T}_2$. So, this shows that $\mathcal{T}_1$ is contained in $\mathcal{T}_2$. Now similarly let us prove that $\mathcal{T}_2$ is contained in $\mathcal{T}_1$, so for that we will show thatv So, next we will show that $\mathcal{B}_2$ is contained in $\mathcal{T}_1$, so what is a typical element of $\mathcal{B}_2$ look like, an element of $\mathcal{B}_2$ looks like product of U_i, $i =1, 2, . .., n$, where U_i's are open in $\mathbb{R}$. So, let $\underline{x}$ be a point in this product of U_i's, so then, for each i, there exists $\varepsilon_i>0$ such that this $B_{\varepsilon_i}(x_i)$ is completely contained in U_i. So, we choose $\varepsilon = \min\{\varepsilon_i, i=1, 2, . .., n\}$, then this implies that $B_\varepsilon(x_i)$ is contained in U_i for all i, which implies that the product of $B_\varepsilon(x_i)$'s is contained in this product of U_i's. But that implies that $S_\varepsilon(\underline{x})$ is contained in the product of U_i's, because this set is equal to this $S_\varepsilon(\underline{x})$. Therefore what this shows is that, if $n = 2$, so this is my U_1 and my U_2, and we have taken any point $\underline{x}$ here and we have found $S_\varepsilon(\underline{x})$ around it. For each $\underline{x}$ in $U_1\times U_2$ there is an $\varepsilon>0$ such that this open square of 2ε is completely contained inside $U_1\times U_2$. So, this implies that, so taking union over all x_, so we get that this product of U_i's is equal to this union of S_ε(x_) over all x_.

Of course, ε depends on X. So, this implies that, so since we have this set product of U_i's and in the standard topology, we can write it as a union of basic open sets in the standard topology and arbitrary unions of open sets are open. So, this implies that this product of U_i's is open in the standard topology, that is $\mathcal{T}_1$, because we have written it as a union of open subsets in $\mathcal{T}_1$. Thus, this shows that $\mathcal{B}_2$ the basis is contained in $\mathcal{T}_1$ which implies that $\mathcal{T}_2$ is contained in $\mathcal{T}_1$. This shows that the standard topology and the product topology on $\mathbb{R}^n$ agree.

So, this is a result which we had repeatedly used in the previous lecture. Now let us continue, so next we are going to see an example. So, in this example we will give a set theoretic description of a map and the exercise is to show that, the map is continuous. So, in order to show that that map is continuous we will use what we have learnt earlier. Let us first describe the map set theoretically.

So, let H contained in $\mathbb{R}^n$ be the hyperplane $H = \{(x_1, x_2, ..., x_{n-1}, 0) : x_i \in \mathbb{R}\}$ (The last coordinate is zero). So, points of this point this type inside $\mathbb{R}^n$, and similarly let H' be the hyper plane $H' = \{(x_1, x_2, ..., x_{n-1}, 1) : x_i \in \mathbb{R}\}$, x_i's could be anything but the last one we want is 1. We will denote H' in red maybe, so this is H' and this is H. So, we are interested in using this, we are interested in considering $\mathbb{R}^n\setminus H'$. So, we will define a map, so we will define a map which we denote by $\phi : \mathbb{R}^n\setminus H' \to H$. So, what is the description of this map? So, we take any point x_ in $\mathbb{R}^n\setminus H'$ and we take this point p_ = (0, 0, ..., 0, 1). Now, we join x_ and p_ by a straight line and we extend this straight line till it meets H. So, this is x_, and so the description in words is let x be in $\mathbb{R}^n\setminus H'$, join the points x_ and this fixed point p_ =(0, 0, ..., 0, 1) and let q_ in H be the point where this line meets H.

So, there will be a unique point. We define the map as $\phi(x)$ = q_. So, the claim we want to make is: let $\mathbb{R}^n\setminus H'$ and H have the subspace topology from $\mathbb{R}^n$. Then ϕ is continuous. Before we prove the claim, let us make a remark over here.

The subspace topology can be easily checked. Note that H is isomorphic to $\mathbb{R}^{n-1}$, it is the inclusion image of the inclusion $i : \mathbb{R}^{n-1} \to \mathbb{R}^n$. So, what is the map $i(x_1, x_2, .$

.., x_{n-1})? It gets mapped to (x_1, x_2, ..., x_{n-1}, 0). And this map is an inclusion and the standard topology on ℝ^{n-1} agrees with the subspace topology from ℝ^n and a similar remark holds for H'.

So, having made this remark let us prove our claim. The idea is to first describe ϕ in terms of coordinates, and then see that each of the coordinate function is continuous. So this is the brief idea or the main point of the idea. So, let us see how to prove this claim. So, the image of ϕ lands in H and H has a subspace topology.

And so, so recall what we have proved. So, we have ℝ^n\H'. So, we can view ϕ as a map to ℝ^n, but the image lands inside H, and we have this inclusion, and since H is given the subspace topology. To show that, this is actually, well this is ϕ and let us call this ioϕ right. So, to show that ϕ is continuous, it is enough to view ϕ as a map from ℝ^n\H' to ℝ^n and show that this map is continuous.

So, that is exactly what we are going to do now. So, we will now compute a formula for this map in terms of coordinates. Let us do that. So, we are going to make this map precise.

So, let x_ = (x_1, x_2, ..., x_n) be in ℝ^n\H'. Every point in the line joining x_ and p_ is of the form x_ + t(p_ - x_) where t is in ℝ. So, that is clear because we have our x_ and we have this direction p_ - x_ or we can take x_ - p_, we have this direction, this is the direction of p_ - x_. So, all points on this line are given by moving along this direction for different values of t. So, in general, a point on this line has this has these coordinates ((1-t)x_1, (1-t)x_2, .

.., (1-t)x_{n-1}, x_n + t(1 - x_n)). So, a general point on this line has this expression. We are looking for the point which lies on H, so, we have to set the last coordinate to be 0. This point is on H if and only if x_n + t(1 - x_n) = 0, that is if and only if t = x_n/(x_n - 1), and note that this is well defined. So, this t, which we will denote by t_0 is well defined since x_n is not equal to 1, as x_ does not belong to H'.

We have this t_0 = x_n/(x_n - 1). So, let us compute what is q_ going to be. So, q_ is going to be. This implies that ϕ(x_) = q_. So, maybe I can write q_ = ϕ(x_1, x_2, .

.., x_n). So, let us compute what is 1-t_0 = 1/(1 - x_n) ϕ(x_1, x_2, ..., x_n) = (x_1/(1 - x_n), x_2/(1 - x_n), .

.., x_{n-1}/(1 - x_n), 0). So, this means that therefore, to show that ϕ is continuous, it is enough to show that each of the coordinate functions are continuous, because ℝ^n now has the product topology, which is the same as the standard topology. So, it is enough to show, suffices to show that $\phi_i: \mathbb{R}^n \setminus H' \to \mathbb{R}^n \to \mathbb{R}$ (then projection onto the i^{th} coordinate to $\mathbb{R}$).

But what is this? $(x_1, x_2, ..., x_n) \mapsto x_i/(1 - x_n)$ if $i < n$, $(x_1, x_2, ..., x_n) \mapsto 0$ if $i = n$.

So, we want to show that ϕ is continuous. So, $\phi : \mathbb{R}^n \setminus H' \to \mathbb{R}^n$. So, here the standard topology is same as a product topology and therefore, it is enough to show that this continuous when $\mathbb{R}^n$ has the product topology and to check that ϕ is continuous in the product topology, we just have to check that each of the coordinate functions is continuous, and that is exactly what we are going to do. So, each of the coordinate functions is given by $x_i/(1 - x_n)$. So, that is when $i < n$, but now note that $-x_i$ So, $\mathbb{R}^n \setminus H' \to \mathbb{R}$, we have these two functions $-x_i$, this is just the projection $\mathbb{R}^n \to \mathbb{R}$, we have that this is just the i^{th} projection, this is negative of the i^{th} projection. So, this $x_i/(1 - x_n)$ this is equal to the constant function $x \mapsto -1$, times the projection map p_i, times the map $1/(x_n-1)$.

So, x_n is a continuous function, x_{n-1} is a continuous function on $\mathbb{R}^n$ and $x_n - 1$ never vanishes on $\mathbb{R}^n \setminus H'$. Therefore, $1/(x_n - 1)$ is a continuous function on $\mathbb{R}^n \setminus H'$. Therefore, all the product of all these is continuous, all these three are continuous on $\mathbb{R}^n \setminus H'$. So, the last one is continuous because $x_n - 1$ never vanishes.

So, their product is continuous. And of course, the last coordinate function is just the constant function 0, and it is an easy check that the constant functions are continuous. This implies that $\phi : \mathbb{R}^n \setminus H' \to \mathbb{R}^n$ is continuous and also $\phi : \mathbb{R}^n \setminus H' \to H$ is continuous. So, this completes the proof of the claim. So, let us make some remarks. So, we could have written ϕ directly as $\phi : \mathbb{R}^n \setminus H' \to \mathbb{R}^{n-1}$ $(x_1, x_2, ..., x_n) \mapsto (x_1/(1 - x_n), x_2/(1 - x_n), ..., x_{n-1}/(1 - x_n))$ and this is clearly a continuous map, due to the same reasons as above, and what we can do is, we can restrict this map to the sphere minus point $p\_$. So, let us just see what is happening. So, let us just copy this diagram over here and we paste it here. So, when we make this sphere, the unit sphere. So, the unit sphere meets H' exactly at this point $p\_$.

So, $S^{n-1} \cap (\mathbb{R}^n \setminus H')$ is exactly $S^{n-1} \setminus \{p\_\}$. This is an easy check, which I will leave to you. So, we have $S^{n-1} \setminus \{p\_\}$ is subset of $\mathbb{R}^n \setminus H'$ and here we have this map $\phi: S^{n-1} \to \mathbb{R}^{n-1}$. So, if we give S^{n-1} the subspace topology, then this ϕ restricted to $S^{n-1} \setminus \{p\_\}$, a function $S^{n-1} \setminus \{p\_\} \to \mathbb{R}^{n-1}$ is continuous because restriction of a continuous map to a subspace is continuous. Here note that, for that to happen, it is important that the subset is given the subspace topology.

So, here are some exercises. So, the first exercises show that ϕ restricted to $S^{n-1}\setminus\{p\_\} \to \mathbb{R}^{n-1}$ is a bijective map of sets. So, exercise 2 is. Let us say $\Psi : \mathbb{R}^{n-1} \to S^{n-1}\setminus\{p\_\}$ denote the inverse. So, a priori, this is just a set theoretic map, a set theoretic inverse of this map ϕ.

So, then show that Ψ is continuous. So, here is a hint to do this exercise. So, we have $\Psi : \mathbb{R}^{n-1} \to S^{n-1}\setminus\{p\_\}$ this is included inside $\mathbb{R}^n\setminus H'$, and $S^{n-1}\setminus\{p\_\}$ has a subspace topology from $\mathbb{R}^n\setminus H'$ which in turn has a subspace topology from $\mathbb{R}^n$. Therefore to show that Ψ is continuous, it is enough to show that this composite is continuous. Therefore, if you can compute the coordinates of this composite function, and show that each coordinate is continuous. Thus, enough to compute coordinate functions of this composite, and show those are continuous.

So, that is exercise 2 and that is also hint to this exercise and that leads us to a definition. Let $f:X \to Y$ be a bijective continuous map. Let g denote its set theoretic inverse which exists since f is bijective. So, if g is also continuous then f is called a homeomorphism. I would like to give one more exercise which is very easy.

Let $f : X \to Y$ be a bijective continuous map. Show that f is a homeomorphism if and only if for every U open in X, f(U) is open in Y. that is part (a) of the exercise, and part (b) of the exercise is, if f is a homeomorphism, then show that its inverse g is also a homeomorphism. So, we will end here.