Point Set Topology

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Week 01

Lecture 01

So, welcome to the first lecture of this course on Point Set Topology. So, lecture 1. So, in this lecture we shall see the definition of a topological space and some very basic examples. So, before we define a topological space. So, recall the definition of a power set. So, let X be any set then the power set of X represented P(X) is the set consisting of all subsets of X.

So, in particular, the empty set, which we denote by the symbol "φ", and the full set X are both elements of P(X). So, having recalled this definition. So, let us define what a topological space. Definition of a topology.

Let X be a set. Let "$\mathcal{T}$" be a subset of P(X) which satisfies the following three conditions. So, the first condition is the empty set φ and the full set X are both in $\mathcal{T}$. So, recall that $\mathcal{T}$ is a collection of subsets of X and what we need is that $\mathcal{T}$ contains φ and X. φ and X are both elements of the power set and what we need is that $\mathcal{T}$ contains both these elements.

The second condition we need is: Suppose U_1, U_2, ..., U_n are finitely many subsets of X and all these are elements of $\mathcal{T}$. Then their intersection from i =1,.

..,n this finite intersection, which is a subset of X, should also be an element of $\mathcal{T}$. So, over here let us emphasize we emphasize that in this condition there are only finitely many U_i's. So, this is the second condition, and the third condition is: let I be any set (possibly infinite). Suppose for each i in I, we are given a subset U_i of X, such that U_i is an element of $\mathcal{T}$. Then the subset of X which is the union over all these i's, $\cup(U_i)$, is a subset of X, should be in $\mathcal{T}$.

And once again let us emphasize that in this condition, the collection of U_i's may be infinite. So, suppose we have a collection $\mathcal{T}$, a subset of P(X) which satisfies the above three conditions is called a topology on X. Often we shall write: Let (X,$\mathcal{T}$) be a topological space. So, by this we mean that X is a set and $\mathcal{T}$ is a topology on X. So, this is the definition of topological space and now we will see some examples of topological spaces.

So, the first example. This is called the trivial topology. In all these examples, let X be any set. So here that $\mathcal{T}$ be the subset of P(X) consisting of just two elements, namely, $\mathcal{T}$ just contains the empty set φ and the entire set X. So, to check that $\mathcal{T}$ defines a topology on X we have to check three conditions.

So let us check these. First condition is that it should contain the empty set φ and X. Clearly $\mathcal{T}$ contains the empty set φ and the set X. So, the first condition is satisfied. So, the second condition for being a topology was that if we take finitely many elements in $\mathcal{T}$, if U_1, U_2, .

.., U_n are in $\mathcal{T}$, then their intersection is in $\mathcal{T}$. In this case, an element of $\mathcal{T}$ is either the empty set φ or the full set X. So, first consider the case where any one U_i = φ then the intersection of U_i's is φ, and so is in $\mathcal{T}$. So, next we consider the case when this does not happen. That means that all U_i = X.

in this case also intersection of U_i's is again equal to X and so is in $\mathcal{T}$. So, therefore the second condition is also satisfied. So, finally let us check the third condition. So, the third condition says that: let I be a set and assume that for each i of I we are given an element U_i in $\mathcal{T}$. So, it is in u i is an element of $\mathcal{T}$ which means it is a subset of X and moreover since $\mathcal{T}$ contains just two elements, then each U_i is either φ or X.

If all the U_i's are empty sets, then this arbitrary union is φ and is in $\mathcal{T}$. Otherwise there exists some j of I such that U_j =X which implies that, when we take the union, this is going to be X, which is in $\mathcal{T}$. So, therefore the third condition is also satisfied. This shows that $\mathcal{T}$ = {φ,X} satisfies Therefore, this shows that $\mathcal{T}$ = {φ, X} all three defining conditions to be a topology on X. Thus $\mathcal{T}$ defines a topology on X, and this is called the trivial topology.

So, this completes the first example. So, in the same way let us see another example this is called the discrete topology. So, here again, X is any set and we take $\mathcal{T}$ = P(X). So, once again to check that $\mathcal{T}$ defines a topology on X, we need to check that $\mathcal{T}$ satisfies the three defining conditions. So, let us check them one by one.

Once again, clearly $\mathcal{T}$ contains φ and X because $\mathcal{T}$ = P(X), and the power set contains φ and X. So, this condition is satisfied. The second condition is Let U_1, U_2, .

.., U_n be finitely many subsets of X which are in $\mathcal{T}$, then their intersection of U_i's should be in $\mathcal{T}$. But this condition is also satisfied because the intersection is a subset of X, it is in P(X) = $\mathcal{T}$. So, therefore the second condition is also satisfied, and similarly the third condition is satisfied. So, let I be any set and suppose we are given for each i of I, a subset U_i of X.

subset U_i of X is in $\mathcal{T}$. So, then we need to show that the union is in $\mathcal{T}$, but again, exactly as in the previous point, this is clear as the union is a subset of X and so is a member of $P(X) = \mathcal{T}$. So, thus the third condition is also satisfied. So, therefore $\mathcal{T}$ defines a topology on X which is called the discrete topology. So, the first two examples were fairly simple. So, let us take a third example.

This is a little bit more interesting. So, this is called the finite complement topology. So, here once again X is any set and let $\mathcal{T}$ be those elements of the power set of X or in other words those subsets of X such that either $U = \varphi$ or $X \setminus U$ is a finite set. So, for this $\mathcal{T}$ let us check that it satisfies the three defining conditions for a topology. So, recall that the first condition we need to check was that φ and X are in $\mathcal{T}$.

Clearly φ is in $\mathcal{T}$ and also since $X \setminus X = \varphi$, is finite (of cardinality 0), this implies that X is in $\mathcal{T}$. Therefore the first condition is satisfied. So, the second condition was the following: So, if U_1, U_2, .

.., U_n are elements in $\mathcal{T}$ then we need to show that the intersection of these U_i's is also in $\mathcal{T}$. So, let us first consider the case where one of these U_i's is empty. So in this case the intersection is empty and so is in $\mathcal{T}$. So, if this does not happen, the other possibility is that U_i is non empty for all $i = 1$, .

.., n. So, in this case what happens is that this will imply that for each $i = 1,...,n$, the set $X \setminus U_i$ is a finite set.

Let us consider $X \setminus (\cap U_i)$. Now, some simple set theory shows that this is equal to the union $i = 1,...,n$ of $X \setminus U_i$'s. Moreover, as each $X \setminus U_i$ is a finite set this implies that their finite union is also a finite set.

So, therefore we have proved that $X \setminus (\cap U_i)$ where $i = 1,...,n$ is a finite set, which implies that the intersection of U_i's is in $\mathcal{T}$. So, this shows that $\mathcal{T}$ satisfies the second condition for being a topology and finally, let us check the third condition.

Here we have to start with I, any set and suppose for each i in I, we are given a subset U_i which is in $\mathcal{T}$. So, what does this mean? We have to check that the union of all these this is in $\mathcal{T}$. So, once again we consider two cases so first consider the case when all the U_i's $= \varphi$. In this case, clearly their union is φ, which is in $\mathcal{T}$. If this does not happen that means that the other possibility is that there is atleast one j for which U_j is non-empty.

So, this will imply that $X \setminus U_j$ is a finite set. Now, as before let us consider the set $X \setminus (\cup U_i)$. Some simple set theory tells us that this is equal to intersection of $X \setminus U_i$'s and this intersection is obviously contained in each of these pieces,each of these subsets. In particular, this is going to be contained in $X \setminus U_j$, which is a finite set. So, this shows that $X \setminus (\cup U_i)$ is a finite set which implies that the union of U_i's is in $\mathcal{T}$.

So, therefore $\mathcal{T}$ satisfies the third condition also. So, the tau we define is here using this condition this collection $\mathcal{T}$ satisfies the three conditions which define a topology. This topology is called the finite complement topology. So, therefore what we have seen so far is that, given a topological space X, any I am sorry, given a set X, any set X, we saw three examples of topologies that we could put on it. So, we will end the first lecture here, and in the next lecture we will see some specific examples which we will often encounter in this course. So, thank you.