Fourier Analysis and its Applications

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IX - Odds and Ends

We now take up a discussion of a few points that have hitherto been left out.

(i) Uniform convergence of Fourier series

(ii) Dirichlet's theorem on pointwise convergence at points the function has one sided derivatives.

(iii) Riemann's localization principle.

(iv) Case of monotone functions. Use of Bonnet's mean value theorem.

The first of these is quite easy and we dispose off this first.

Uniform convergence of Fourier series We prove the following theorem

Theorem: Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a 2π-periodic continuous, piecewise smooth function then the Fourier series of $f(x)$ converges absolutely and uniformly to $f(x)$.

Proof: This is a nice application of Bessel inequality which we established in chapter 2. Denote by a_n, b_n the Fourier coefficients of $f(x)$ and by a'_n, b'_n the Fourier coefficients of $f'(x)$. Note that f' is continuous on $[-\pi, \pi]$ except for (finite) jumps at finitely many points. Now

$$a'_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos nx dx.$$

Integrating by parts we see that the boundary terms cancel out and we get at once

$$a'_0 = 0, \quad a'_n = nb_n, \quad b'_n = -na_n.$$

Since $f'(x) \in L^2[-\pi, \pi]$ we can apply Bessel inequality and

$$\sum_{n=1}^{\infty} n^2(|a_n|^2 + |b_n|^2) \leq \frac{1}{\pi}\int_{-\pi}^{\pi} |f'(x)|^2 dx.$$

We are now ready to establish the absolute and uniform convergence of the Fourier series for $f(x)$. We begin with the simple observation that

$$|a_n \cos nx| + |b_n \sin nx| \leq \sqrt{|a_n|^2 + |b_n|^2}$$

If we show the convergence of the series

$$\sum_{n=1}^{\infty} \sqrt{|a_n|^2 + |b_n|^2} \tag{9.1}$$

then the stated result would follow from the Weierstrass's M-test. To see that (9.1) converges,

$$\begin{aligned}
\sum_{n=1}^{N}\sqrt{|a_n|^2+|b_n|^2} &= \sum_{n=1}^{N}\Big(\frac{1}{n}\cdot n\sqrt{|a_n|^2+|b_n|^2}\Big) \\
&\leq (\sum_{n=1}^{N}\frac{1}{n^2})\sum_{n=1}^{N}n^2(|a_n|^2+|b_n|^2) \\
&\leq (\sum_{n=1}^{\infty}\frac{1}{n^2})\sum_{n=1}^{\infty}n^2(|a_n|^2+|b_n|^2) \\
&\leq \frac{C}{\pi}\int_{-\pi}^{\pi}|f'(x)|^2dx.
\end{aligned}$$

Letting $N\to\infty$ the result follows.

Convergence of the Fourier series at points where the one-sided derivatives exist A large number of examples fall under this scheme. The main point is that the with regard to convergence of the Fourier series at a specific point the behaviour of f at points far away from x_0 do not matter as long as $f\in L^1[-\pi,\pi]$ and f *is sufficiently nice in a vicinity of* x_0.

First we take up the case of convergence at points of continuity.

Theorem: Suppose that f is a piecewise smooth 2π-periodic function on $\mathbb{R}$ which is *continuous at* x_0 and whose one-sided derivatives exist at x_0. Then the Fourier series of f converges to $f(x_0)$ at the point x_0.

Proof: We employ the notation of chapter 1 namely $D_n(t)$ denotes the Dirichlet kernel and $S_n(f,x_0)$ is the sum of $2n+1$ terms of the Fourier series evaluated at x_0. We recall that

$$S_n(f,x_0)-f(x_0)=\int_{-\pi}^{\pi}(f(x_0-t)-f(x_0))D_n(t)dt.$$

We split the integral into a sum of two integrals and we show that

$$\int_0^{\pi}(f(x_0-t)-f(x_0))D_n(t)dt\longrightarrow 0,\quad n\to\infty$$

The integral over the other half is handled similarly. We now need to recall the expression for the Dirichlet kernel:

$$2\pi D_n(t)=\cos nt+\sin nt\cot(t/2)$$

We write the integral as

$$\int_0^{\pi}g(t)(t\cos nt+t\cot(t/2)\sin nt)dt \tag{9.2}$$

where the function $g(t)$ is given by

$$g(t)=\frac{f(x_0-t)-f(x_0)}{t} \tag{9.3}$$

The function $g(t)$ has a limit at $t=0$ and so is bounded on $[0,\delta)$ and so $g\in L^1[0,\pi]$. since $t\cot(t/2)$ is also continuous on $[0,\pi]$, the Riemann Lebesgue lemma applies and we see that the integral tends to zero as $n\to\infty$. The proof of the theorem is thereby completed.

Remark: The conditions can be considerably relaxed. One doesn't require piecewise smoothness of the function. See for example *A. Zygmund, Trigonometric functions, page 52.* However a large class of the functions that are of interest in Engineering will be piecewise smooth.

The same argument establishes the following *localization principle:*

Theorem (Localization principle: Suppose $f(x)$ is 2π-periodic function which is integrable on $[-\pi, \pi]$ and $f(x) = 0$ at all points on an open interval $I \subset (-\pi, \pi)$ then the Fourier series for $f(x)$ converges to zero at *all points of* I. In particular if two functions agree at all points of I then their Fourier series behave alike *throughout* I.

Proof: Select a point $x_0 \in I$ and let $(x_0 - \delta, x_0 + \delta) \subset I$. Look at

$$S_n(f, x_0) - f(x_0) = \int_{-\pi}^{\pi} (f(x_0 - t) - f(x_0))D_n(t)dt.$$

Split the integral into two pieces over $|t| < \delta$ and over $\delta < |t| < \pi$. The integral over $(-\delta, \delta)$ vanishes identically. Over the other interval we have

$$2\pi D_n(t) = \cos nt + \sin nt(\cot(t/2))$$

We have $\cot(t/2)$ is continuous on $\delta < |t| < \pi$ and the factor $(f(x_0 - t) - f(x_0))$ is in L^1 so the result follows by Riemann Lebesgue lemma.

Behaviour at a discontinuity: A model case is that of the signum function given by

$$\sigma(x) = \text{sgn}(x), \quad x \in [-\pi, \pi]$$

The Fourier series of $\sigma(x)$ is given by

$$\frac{4}{\pi}\Big(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\Big)$$

By theorem 99, The series converges to $\sigma(x)$ at all the points in $(-\pi, \pi) - \{0\}$.

At the origin the series converges to zero which agrees with

$$\frac{1}{2}(\sigma(0+) + \sigma(0-))$$

We now show that this situation holds in far greater generality. But we shall use this special case in the proof of the general result that we now state.

Theorem: Suppose f is piecwise smooth on $[-\pi, \pi]$ and has one sided derivatives at x_0 and a jump discontinuity there. The Fourier series of f converges at x_0 to the value

$$\frac{1}{2}(f(x_0+) + f(x_0-)).$$

Proof: We reduce it to the continuous case. Let α be the jump namely $\alpha = f(x_0+) - f(x_0-)$ and using the signum function above we define:

$$F(x) = f(x) - \frac{\alpha}{2}\sigma(x - x_0).$$

Now

$$F(x_0+) = f(x_0+) - \frac{\alpha}{2}\sigma(0+), \quad F(x_0-) = f(x_0-) - \frac{\alpha}{2}\sigma(0-).$$

Subtracting we see that

$$F(x_0+) - F(x_0-) = f(x_0+) - f(x_0-) - \frac{\alpha}{2}(\sigma(0+) - \sigma(0-)) = 0.$$

whereby F is continuous at x_0. Obviously F has one-sided derivatives at x_0. Our theorem holds for F namely the Fourier series for F converges at x_0 to the value $F(x_0+) = F(x_0-)$ (each being $(f(x_0+) + f(x_0-))/2$. Check this). Now,

$$S_n(f, x_0) = S_n(F, x_0) + \frac{\alpha}{2} S_n(\sigma(x - x_0), x_0)$$

The second term tends to zero (by the result proved for the special case of signum function) and the first tends to $(f(x_0+) + f(x_0-))/2$ and the result is established.