An Introduction to Point-Set-Topology (Part II)
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Lecture 55
Definition and Examples of Manifolds

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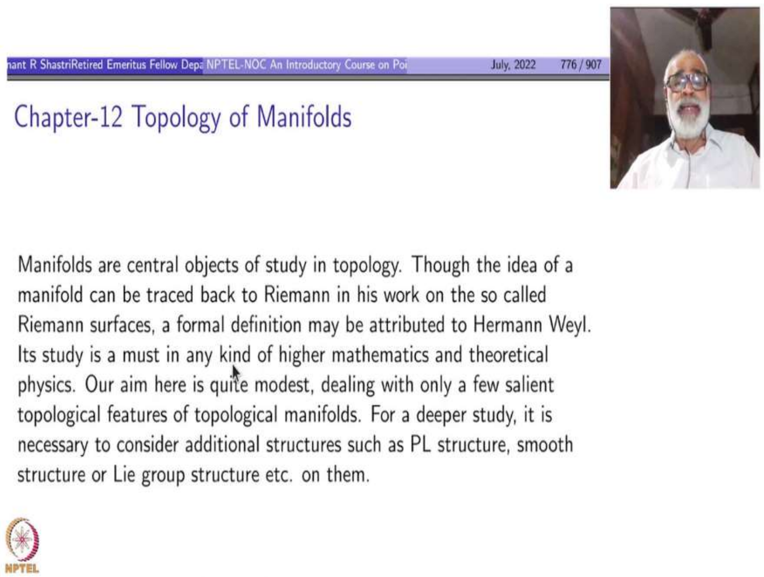

Hello, welcome to NPTEL NOC an introductory course on Point Set Topology Part 2. Today we shall start a new topic, topology of manifolds. Manifolds are central objects in the study of topology. Though the idea of a manifold can be traced back to Riemann in his work on the so called Riemann surfaces, of course, Riemann did not call them Riemann surfaces, obviously), a formal definition of manifold may be attributed to Hermann Weyl. Its study is a must in any kind of higher mathematics and things that use higher mathematics like theoretical physics and so on.

Our aim here is quite modest dealing with only very few salient topological aspects of topological manifolds as compared to deeper studies such as you know, additional structures such as PL manifolds, smooth manifolds, Lie groups and so on so forth, Complex manifolds and so on.

So, we are just taking topological manifolds which in some sense encompass all such special things. So, it is so, general, and therefore, it is going to be somewhat weak. Statements are somewhat weak here. But they will be available for all those objects of studies. So, they are good in that way.

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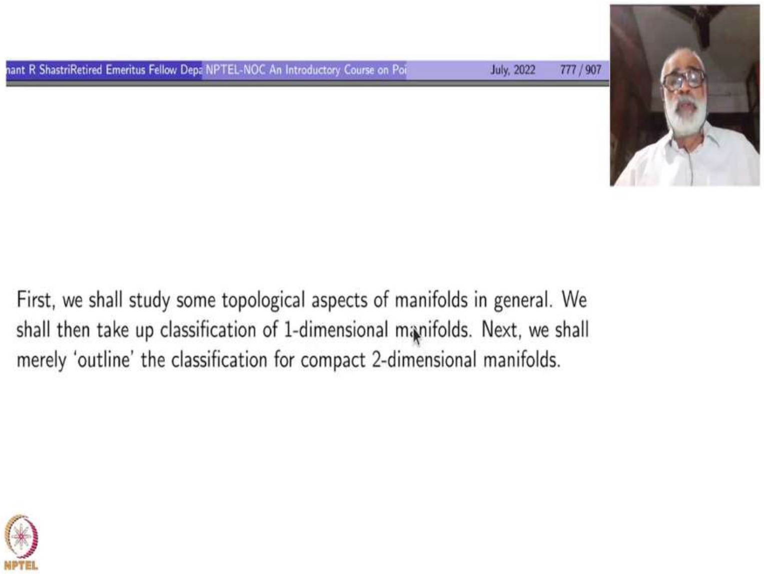

First we shall study some topological aspects of manifolds in general. We shall then take up the classification of 1-dimensional manifolds. Next we shall merely outline the classification of compact 2-dimensional manifolds. This is aimed at motivating you people to study other topological areas like algebraic topology and so on. Especially the NPTEL course on algebraic topology, there are two courses. So, this will be a motivation as well as this entire course will be a very good preparation for those courses finally.

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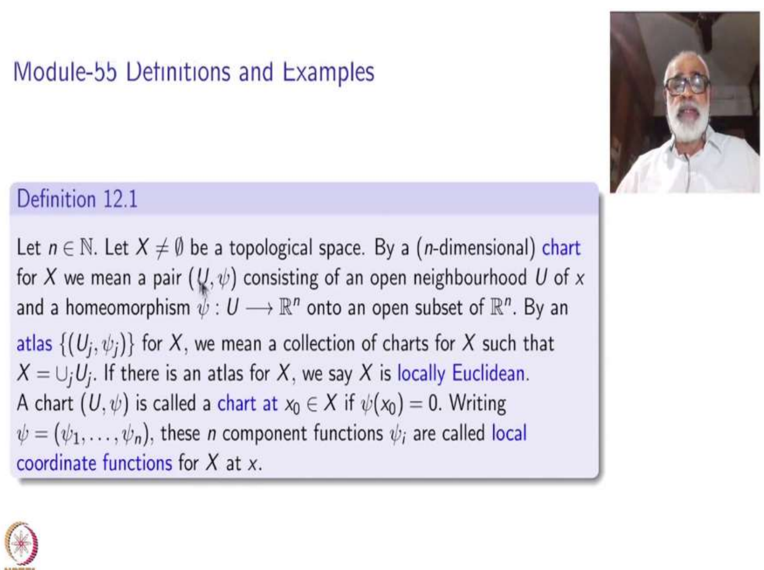

So, today module 55, definitions and examples of manifolds. Fix an integer n take a non empty topological space X. By an n dimensional chart for X, we mean a pair (U, ψ) consisting of an open neighbourhood U of some point x inside X and a homeomorphism ψ from U to $\mathbb{R}^n$. U is an open subset of X, ψ is from U to $\mathbb{R}^n$ onto an open subset of $\mathbb{R}^n$. Often when I say homeomorphism, it is not necessarily onto here but onto some open subset. you take that subspace topology there, it is a homeomorphism.

Here I specifically mentioned that the image of ψ is open $\mathbb{R}^n$ that is very important here. U is open inside X does not mean that $\psi(U)$ is open inside $\mathbb{R}^n$. So that has to be told specifically. That is the condition here. Now, once you have defined a chart, I am going to define an atlas. These terms are borrowed from geography. I think all these things go back to Gauss. So, by an atlas we mean a collection $\{(U_j, \psi_j)\}$ of charts, such that what is the condition? The union of all these domains of the charts must be the whole of X.

If there is an atlas for X, then we say X is locally Euclidean. In other words, the chart whose domain contains a point x of X, gives you local Euclideanesss of X at the point x. If it happens at all the points of X, then X is called locally Euclidean.

A chart (U, ψ) is called a chart at x_0 if ψ is that $\psi(x_0) = 0$. This is a very special definition, I have cooked up. So, that I do not have to keep on saying that psi of x_0 is 0. (U, ψ) is a chart at x_0 already means that x_0 is in U and the function is such that the image of x_0 is chosen to be the origin.

We also write $\psi_1, \psi_2, \ldots, \psi_n$, for the n coordinate functions of ψ. It makes sense because ψ is taking values in $\mathbb{R}^n$. So, these n coordinate components functions of ψ are also called local coordinates for X at the point x_0. The local coordinates, remember, may not be defined on the whole of X, they are defined in a neighbourhood of this x_0 and depend upon the choice of the chart ψ.

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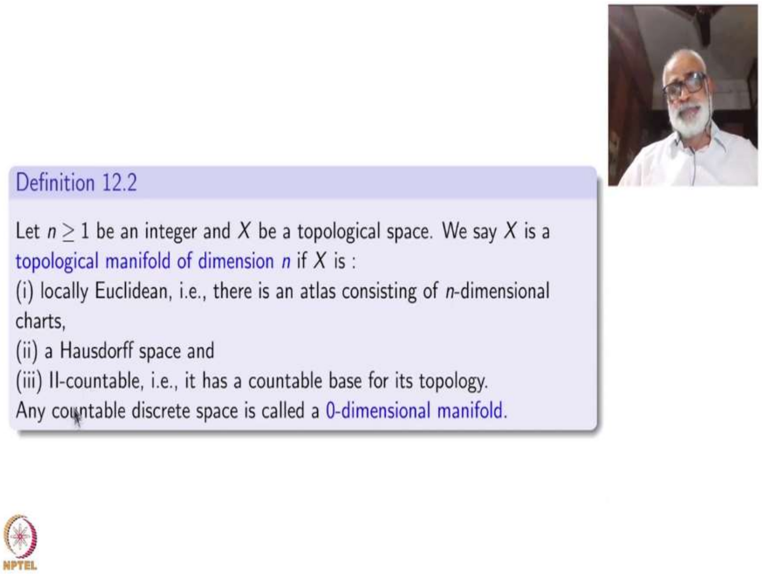

Let $n \geq 1$. X be a topological space. We say X is topological manifold of dimension n.

(i) if it is locally Euclidean in the above sense that there is an atlas consisting of n dimensional charts, viz., (taking values inside $\mathbb{R}^n$), that is the first condition, and

(ii) and (iii) conditions are just ordinary topological conditions viz., X is Hausdorff space and II-countable.

So, these two extra conditions have been put, because we do not want to deal with non Hausdorff space to begin with and we do not want to deal with non II-countable spaces either. Because then we cannot do any fruitful analysis.

Any countable discrete space is by definition, a 0-dimensional manifold.

Do not confuse this one with the definition of 0-dimensionality for a separable metric space that we have studies earlier. Of course, 0-dimensional manifolds form a subclass of the class of all 0-dimensional spaces. But we insist that they are countable discrete spaces. We do not want to take all arbitrary 0-dimensional spaces, viz., second countable separable metric spaces which satisfying SII. So that is the difference here. The converse is not true because true for example you may take the space of rational numbers inside $\mathbb{R}$, which is a 0-dimensional spaces but not a manifold.

This takes care of definition of manifolds for $n = 0$. If you want to cover the case when $n = -1$, the you should take an empty set as a manifold to be consistent with the topological dimension theory.

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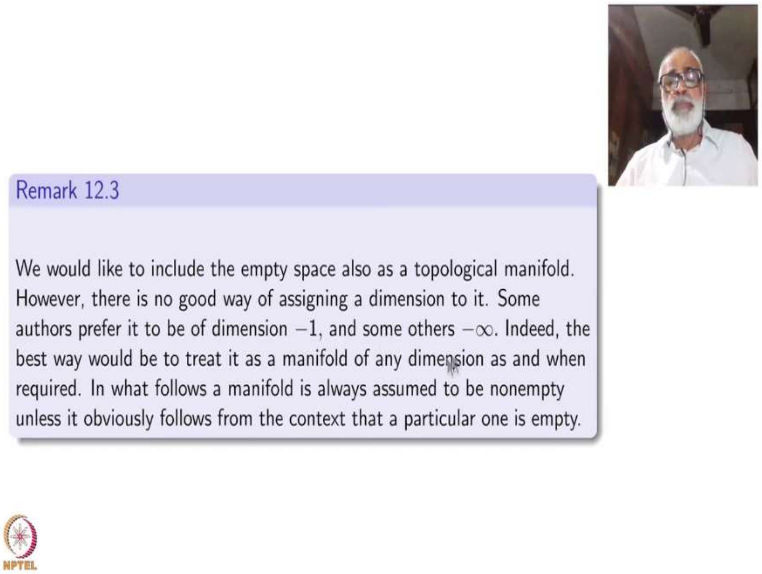

However, there is no good way of assigning a manifold dimension to empty set. Some authors prefer it to be dimension -1 just like in our dimension theory, and some others put it as $-\infty$. Indeed, there are theories in the topology of manifolds itself where it is better to treat the empty set to have any dimension whatsoever depending on the context. (It is like assigning a degree to the 0 polynomial. For the 0 polynomial usually people do not define the degree. But it is better to keep 0 polynomial having any degree it wants, depending upon the context.)

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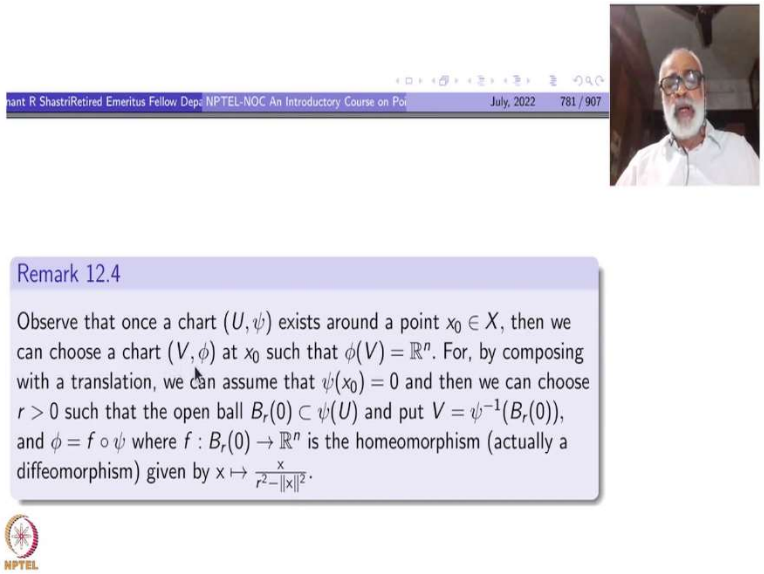

Observe that once the chart (U, ψ) exists at a point x_0 belonging to X (means what now, $\psi(x_0)$ is 0). Then we can choose another chart, (V, ϕ) at x_0 such that, this $\phi(V)$ is the whole of $\mathbb{R}^n$. In fact, this ψ is going to take U to some open subset of $\mathbb{R}^n$, may not be the whole of $\mathbb{R}^n$. You can first take a smaller neighbourhood V of x_0 such that $\psi(V)$ is an open ball around 0. Then you can change ψ to ψ by composing i with a homeomorphism of the open ball onto the whole of $\mathbb{R}^n$. Therefore, you could have assumed right in the beginning, the definition of chart itself, that all the charts have this property viz., their image is the whole of $\mathbb{R}^n$. But that is not always convenient. It is better to have as liberal definition as possible. Strongest definitions allow less chance for free work. That is all.

Given any chart (U, ψ), by composing with a translation we can assume first of that $\psi(x_0) = 0$ and then we can choose r positive such that an open ball of radius r around 0 is contained inside this open subset around 0 and then take V to be $\psi^{-1}(B_r(0))$. Then what happens ψ itself is a homeomorphism from V to $B_r(0)$. But now $B_r(0)$ to the whole of $\mathbb{R}^n$, you have lots of homomorphisms, you compose ψ with them. For example, you can take x mapsto $x/(r^2 - \|x\|^2)$. This is a very nice homeomorphism from $B_r(0)$ the open ball of radius r centred at 0 to the whole of $\mathbb{R}^n$.

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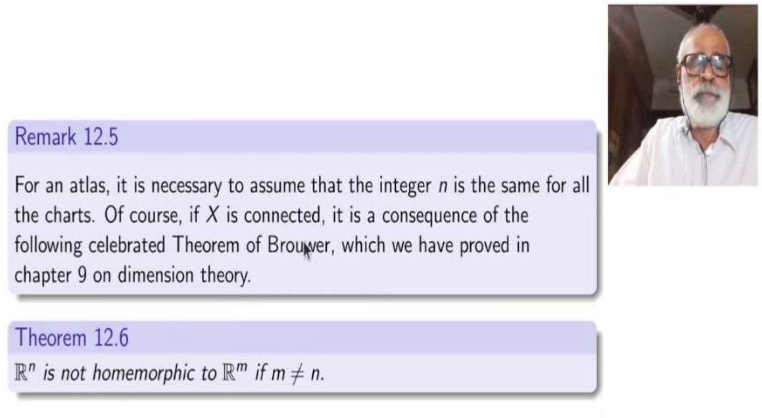

For an atlas, it is necessary to assume that the integer n is the same for all the charts, do not change the n within an altas. Of course, if X is connected you are free to change also but you have to use a certain theorem which we have seen namely Brouwers invariance of domain. Namely $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$ if $n \neq m$. So, you will not have much chance there. Because once different n's are used, the dimension is anyway a locally constant function on X. But X but is connected, therefore, a locally constant function has to be a constant.

So different integers can not occur for a connected locally Euclidean space.

So, that is a minor point. It is better to assume that you do not have spaces like a line and a disjoint union with a point as a manifold. Such things we do not call a manifold. Because line can be parametrize by open subsets inside $\mathbb{R}$, itself is a single parameterization is there. Single point you have take it as a discrete space. Single discrete therefore it is 0 dimension. So do not allow different dimensions with a single manifold. That is all.

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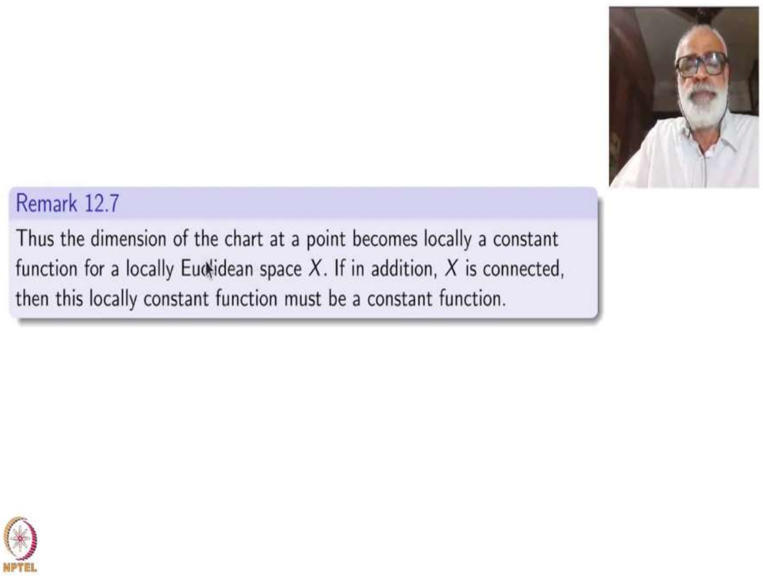

Thus the dimension of the chart at a point becomes locally constant and a locally constant function on a connected space is a a constant function.

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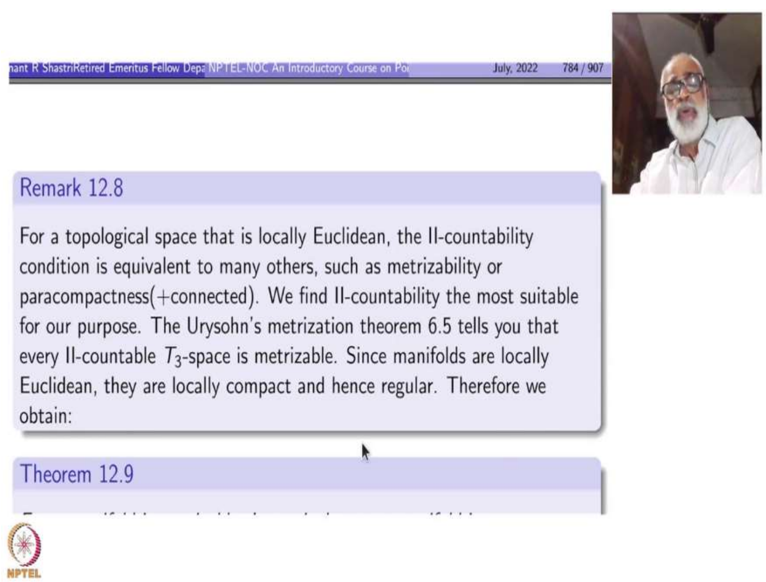

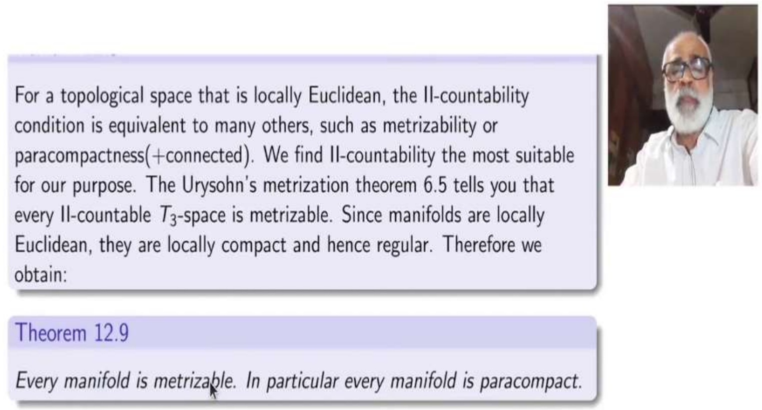

For topological space that is locally Euclidean Hausdorff, II-countabilty condition is equivalent to many other conditions. A space is locally Euclidean means a lot of things such as locally compact, locally connected, locally convex, etc. In addition, II-countability condition is equivalent to many others such as metrizability plus separability, paracompactness plus connectedness etc. All these things are equivalent. So we find second countability the most suitable for our purpose and easy to understand.

The Urysohn's metrization theorem that we have done tells you that every II-countable T_3-spaces is metrizable. Locally compact Hausdorff spaces are T_3-spaces. Therefore, under II-countability they are metrizable. So, I could have put metrizability and separability also. Then I can conclude that it is second countable.

Therefore, we obtain every manifold is metrizable. In particular every manifold is paracompact. Because we have proved metrizable space are paracompact. So, to sum up, suppose you started with a locally Euclidean plus Hausdorff space. Then you add the condition that it is metrizable and separable. Then it will II-countable. Similarly, you can add the extra condition that it is paracompactness and connected. That is also good enough to conclude that it is II-countable. So, many such properties are interchangeable.

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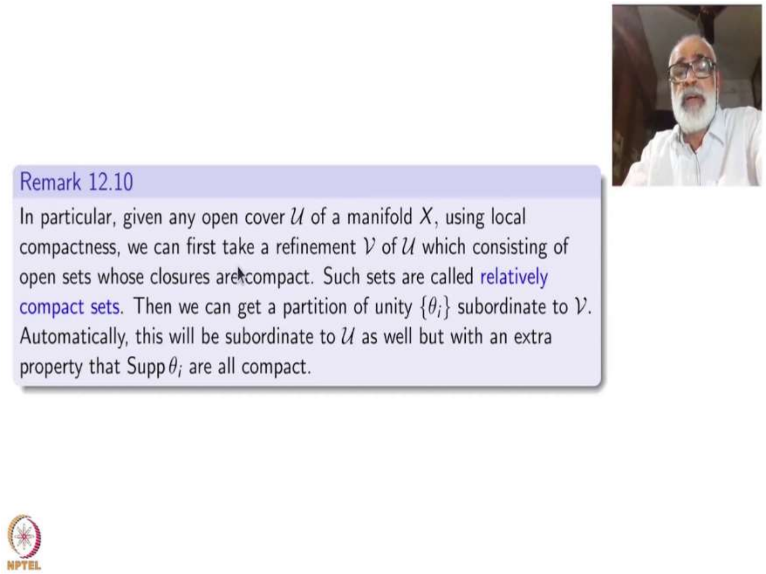

In particular given any open cover $\mathcal{U}$ of a manifold X, using local compactness, we can first take a refinement $\mathcal{V}$ of $\mathcal{U}$ consisting of open sets whose closures are compact. See I am now talking about paracompactness. once you have paracompact you have partition of unity also. I want to see something more here in the case of manifolds. So, what is that?

Start with a manifold and an open cover. Take a refinement which consists of open sets whose closures are compact. So, here I am using local compactness of X. Such sets are called relatively compact sets. (See we have never used this term so far. Many times we come across open sets U such that $\overline{U}$ are compact. This terminology, relatively compact means that and that is quite convenient. You could have used that terminology.) Then we can get a partition of unity $\{\theta_i\}$ subordinate to this $\mathcal{V}$.

Subordinate means, what? The family of support of θ_i form a refinement of $\mathcal{V}$. Support of a real values function θ is by definition, the closure of the set of all points at which θ is not zero. But members of $\mathcal{V}$ are such that their closures are compact. Therefore this θ_i's have compact support.

So, that is what automatically this will imply that will be subordinate to $\mathcal{U}$ also and with the extra property support of the θ_i's are all compact. So, this is not true in general an arbitrary paracompact space. You have a partition of unity, which is subordinate to any given cover. So, this extra property will be quite useful now. I am going to use this one.

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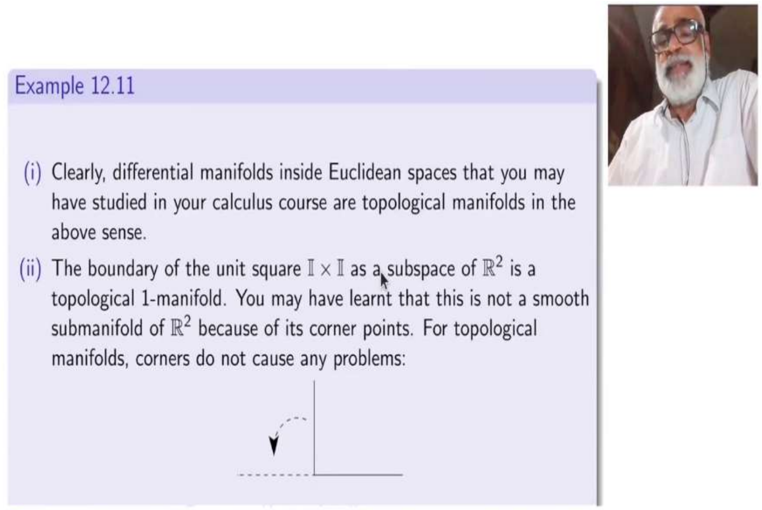

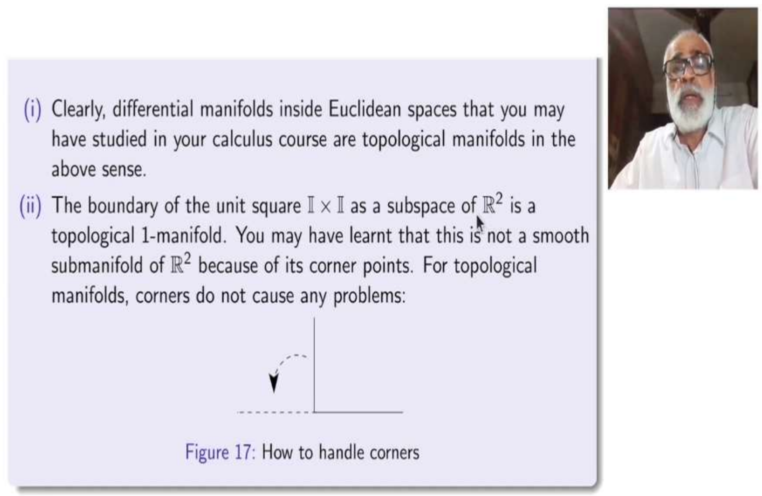

So, let us see some examples now. If you have studied some differentiable manifolds inside $\mathbb{R}^n$, in you calculus course, like a circle, a sphere, parabolas, ellipses, ellipsoids and so on, they are all manifolds. Take some time to see that. But I am not going to do that. I am going to do only simpler examples. So that is your calculus course, full of examples of manifolds there.

The boundary of the unit square as a subspace of $\mathbb{R}^2$ is a topological 1-manifolds. Even simpler one is if you take a triangle, any triangle, the boundary of that triangle consisting of three sides that will be a topological 1-manifold. You may have learned that this is not a smooth manifold inside $\mathbb{R}^2$. It is not a smooth manifold because you know triangles have `corners', the squares have corners and so on.

The corner point cause problems. If you do not know that, do not worry, they are all topological manifolds. Why? Because suppose you have a corner like this. What you can do? You can take the two line segments on either side, the union is homoeomorphic to a line segment in $\mathbb{R}$. All that you have to do is turn this vertical segment and make it horizontal like this on the other side.

You can write down your own formula for this using $\cos\theta, \sin\theta$ etc. and write down the homeomorphism. In fact, you can try to get a homeomorphism from this first quadrant to the entire upper half space also. So, this is a nice illustration. This picture tells you how to handle corners. See here at his point, locally the space looks like an open line segment. There is no problem. But here, what is the argument? You have to just straighten it out to get a homeomorphism into an open interval, that is all.

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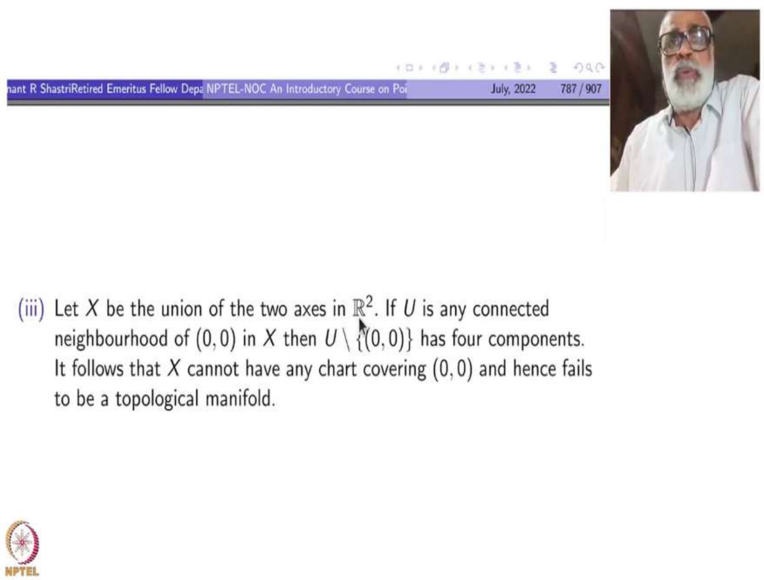

Now, let X be the union of two axes in $\mathbb{R}^2$, x-axis and y-axis. (You can draw similar picture anywhere you do not have to take the axes itself). If U is any connected neighbourhood of $(0,0)$, how does it look like? If you throw away the point $(0,0)$ from U, it will have four connected components. Therefore, you can see that such an open neighbourhood of U of this point $(0,0)$ inside X, where X is the union of the two axes, cannot be homoeomorphic to an open interval. So, X cannot have any chart covering $(0,0)$ and hence fails to be a topological manifold. Note that if at all you should have a chart at $(0,0)$ it should be a homeomorphism into $\mathbb{R}$, because everywhere else except this point $(0,0)$, you have neighbourhood which is homoeomorphic to an open interval. But at that point $(0,0)$, you do not have.

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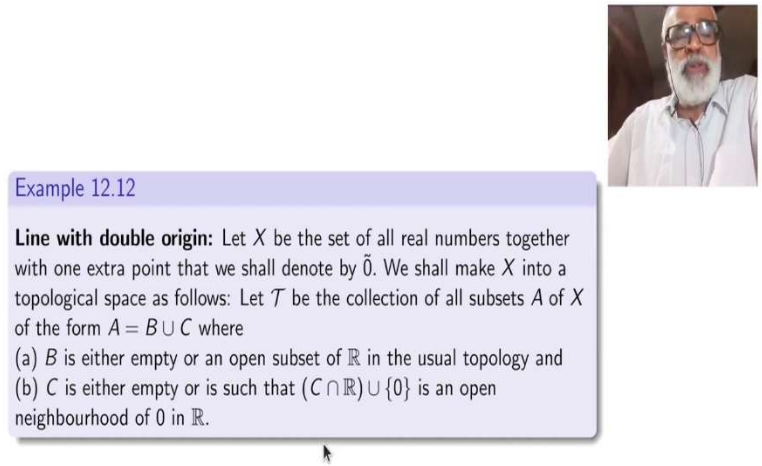

Now, I will give you a very specific example here which violates the Hausdorffness. We have put Hausdorffness forcibly. Suppose Hausdorffness is a consequence of Local Euclideannss, then you do not have to put that extra condition. No, it is not a consequence. That means we must give an example. What I am going to construct is a locally Euclidean, locally 1-dimensional Euclidean space, which is II countable also, or even compact also, no problem. But it will not be Hausdorff. Let us see how.

Let X be the set of all real numbers, (that is the real line) together with one extra point that we shall denote by $\tilde{0}$. We shall make X into a topological space as follows. Let $\mathcal{T}$ be the collection of all subsets A of X of the form $A = B \cup C$, where B is allowed to be empty or an open subset of $\mathbb{R}$ in the usual topology (for empty set is also open subset in any space, but I want to emphasize this fact that it could be just empty, why are you writing A as $B \cup C$) and what about C? C is either empty, or C has the property. So, first part I said, B is a subset of $\mathbb{R}$ itself. That means $\tilde{0}$ is not there in B. Now, the second part C may be empty or $\tilde{0}$ belongs to C. there, and if you intersect C with $\mathbb{R}$ that means you are throwing away $\tilde{0}$ from C, and then take union with 0, throw a $\tilde{0}$ and put back just 0) then that must be a neighbourhood of 0 inside $\mathbb{R}$. Is that clear?

What is the relation between 0 and $\tilde{0}$? How does X look like. If you just ignore the other one, then the rest of them looks like real line. If you ignore one of them, the corresponding subspaces, both of them homeomorphic to $\mathbb{R}$. The two points 0 and X are very closed to each other in the sense that you cannot separate them with disjoint neighbourhoods. The moment you take neighbourhoods those neighbourhood will intersect.

You see any neighbourhood of 0 in X contains an open interval in $\mathbb{R}$. Throw away 0 from it and put back $\tilde{0}$ you a neighbourhood of $\tilde{0}$. And the other way round, vice versa. Therefore 0 and $\tilde{0}$ cannot be separated by open sets. But they are different points in X. So, this is why X is called the line with a double origin.

You can have such examples wherein several points will `overlap' instead of just two points, or you can start with $\mathbb{R}$ and choose a discrete subset and double all the points in it in a similar manner. They all will be locally $\mathbb{R}$ but not Hausdorff.

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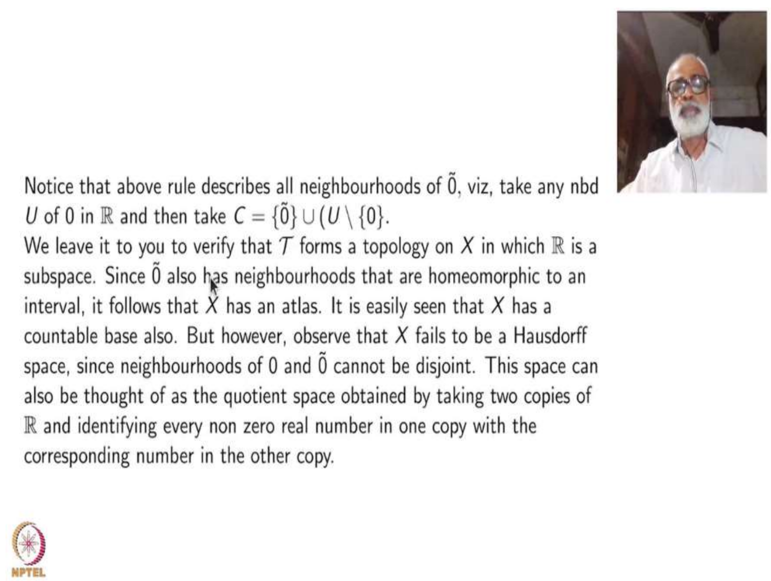

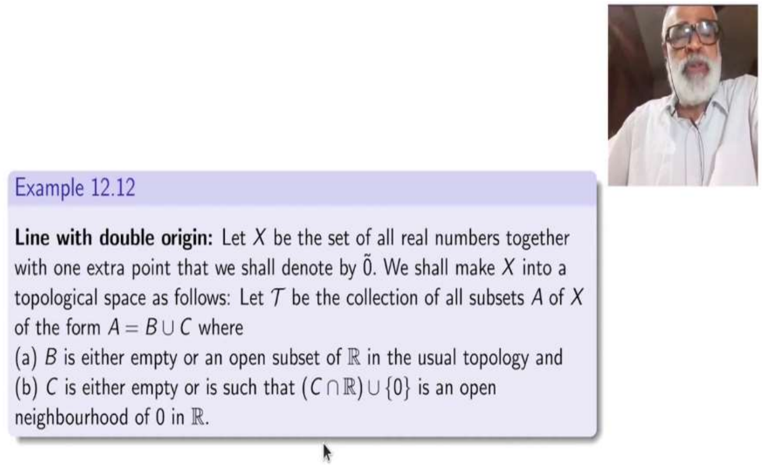

You will get some strange spaces, which we do not want to deal with. They do not occur naturally, it is only a consequence of our definition, you know, the deficiency of a definition and therefore, we have to impose Hausdorffness.

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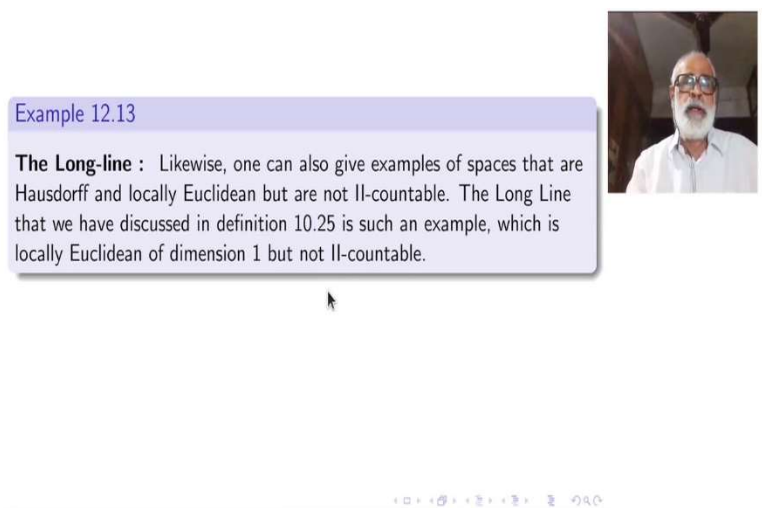

Let us, go ahead now. So, this is one which we have studied a couple of days back. The long line, using our well ordering order topology and so on. We know that the long line is locally Euclidean of dimension 1. It is Hausdorff also, what is it not? It is not II-countable. (It is not even first countable if you allow the point Omega, but Omega is not allowed inside the long line). So, it is I-countable, but it is not II-countable. So, the long line is not treated as a manifold. So, that is an example, we do not want to consider long lines as manifolds because which you cannot do much analysis on them.

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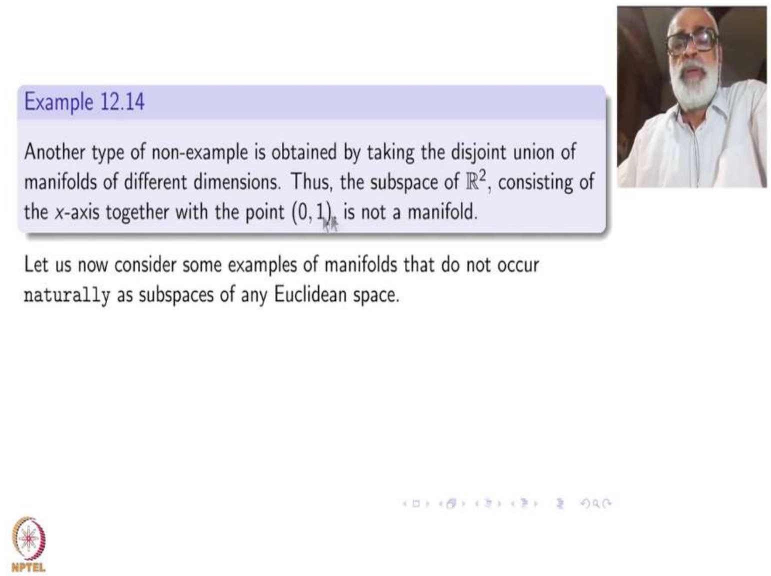

Another type of non example is obtained by taking disjoint union of manifolds of different dimensions. This I already dealt with. Like taking the real line and a singleton point outside or a line and a disjoint plane. Such things are not allowed, we do not want to consider. So, for example, such spaces are there if you go to algebraic geometry, Wherein, even larger class of objects are studied.

Let us, consider some examples of manifolds that do not occur naturally as subspaces of $\mathbb{R}^n$. So far our examples where all subspaces of $\mathbb{R}^n$ and only some counter examples were not subspace of $\mathbb{R}^n$.

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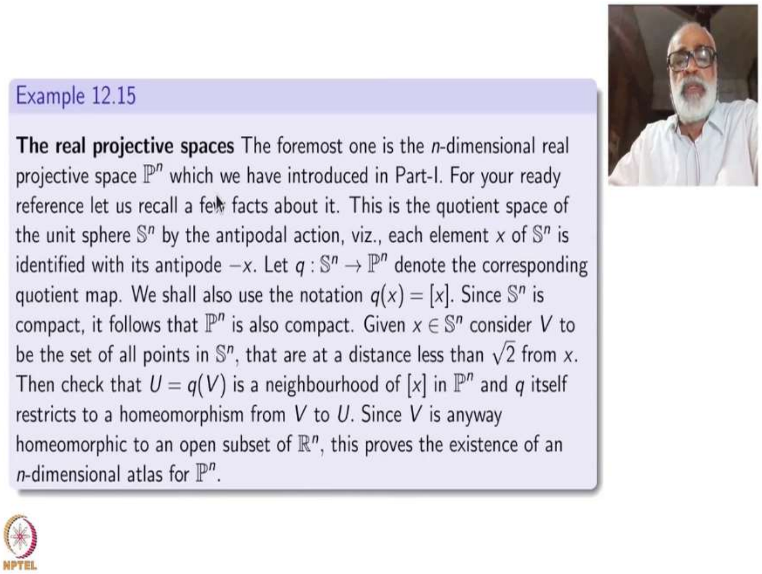

Now, let us see something which does not occur naturally as a subspace but they are manifolds. So, this is a very important example, the projective space. The foremost one is the n-dimensional real projective space which we have introduced in part I. But let me recall at least a few aspects of this. I hope you know it, if you do not know you can read it from Part I notes or from elsewhere.

So, what is the definition of $\mathbb{P}^n$? $\mathbb{P}^n$ is the quotient of the unit sphere $\mathbb{S}^n$ inside $\mathbb{R}^{n+1}$ by the antipodal action. What does that mean, x and $-x$ are identified for every $x \in \mathbb{S}^n$. Antipodal action means x goes to $-x$.

So, look at the equivalence classes, I can denote them by $[x]$ and take the quotient map q from $\mathbb{S}^n$ to $\mathbb{P}^n$ to be $q(x) = [x]$. So, put the quotient topology on $\mathbb{P}^n$. So, the topological space is defined. Automatically, since q is a surjective map, and $\mathbb{S}^n$ is compact, therefore $\mathbb{P}^n$ will be compact.

So, we have to see why it is locally Euclidean and Hausdorff etc. We are happy because it is compact space already and so it will be II-countable also. Now, given any x belonging to $\mathbb{S}^n$, consider V to be the set of all points y in $\mathbb{S}^n$ such that the distance of that point y from $x < \sqrt{2}$.

So, take the open ball of radius $\sqrt{2}$ around $x \in \mathbb{R}^{n+1}$ and intersect it with $\mathbb{S}^n$, that will be open subset of $\mathbb{S}^n$. Now, if you take y going to $-y$, under that, this open subset will go to a disjoint open subset. Therefore, if you restrict the map q to V that will be an injective map and is actually a homoeomorphism and its image in $\mathbb{P}^n$ will be open. It is not difficult to see that V itself is homeomorphic to an open subset of $\mathbb{R}^n$ (by taking the linear projection parallel to the vector x).

So, that is the nice neighbourhood. So, this will give you local description that every point inside $\mathbb{P}^n$ has a neighbourhood which is homoeomorphic to an open subset of $\mathbb{R}^n$.

Indeed this also tells you that the space is Hausdorff now. Because what you can do is, let me see I will tell you something more.

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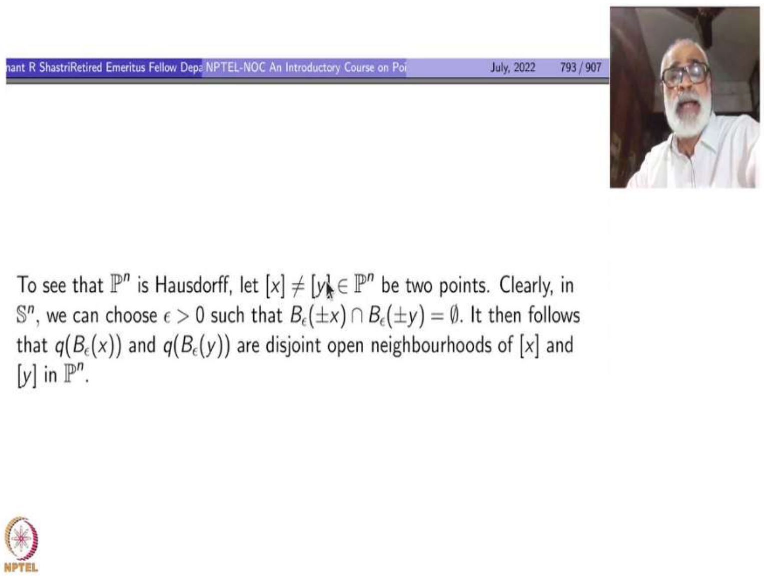

Take $[x]$ and $[y]$ two distinct points in $\mathbb{P}^n$, what does it mean? $\pm x$ is not equal to $\pm y$. If $[x]$ is not equal to $[y]$, these classes are different means $\pm x$ is not equal to $\pm y$. So you have four different points in $\mathbb{S}^n$. You take the least distance between any two of them. Call that epsilon and take open epsilon balls $B_\epsilon(z)$ around each of these four points $z \in \mathbb{R}^{n+1}$ and intersect with $\mathbb{S}^n$. You get four mutually disjoint neighbourhoods of these four point in $\mathbb{S}^n$.

First of all, observe that the antipodal map takes $B_\epsilon(z)$ to $B_\epsilon(-z)$. It follows that under q these four balls define two disjoint neighbourhoods of $[x]$ and $[y]$ respectively.

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Exercise 12.16

Show that a manifold is connected iff it is path connected.

Exercise 12.17

(This exercise is intended as some kind of an explanation for our obsession with Hausdorffness and locally Euclidean spaces.) Consider $\mathbb{R}$ with the cofinite topology. Certainly it is not Hausdorff. But is it locally Euclidean? More generally, take the affine space $\mathbb{C}^n$ with the Zariski topology. Is it locally Euclidean? Is it contractible?

So I included some exercises here. Show that a manifold is connected if and only if it is path connected.

So, this is a very easy exercise. But there is another exercise here this is not very central even if you do not understand this, it is okay for some time. So, this any actually explain our obsession with Hausdorffness and local Euclideaness.

So, you look at the co-finite topology on $\mathbb{R}$. Clearly it is not a Hausdorff space. But is it locally Euclidean?

(You can ask other questions too. We have given examples of spaces which are locally Euclidean and not Hausdorff. Does Hausdorffness imply local Euclideanness? There are many such questions. So that is why I am discussing this example.)

You can take a look at it and if you have difficulties you can come back to us. So, today let us stop here. Next time, we shall introduce a larger class of manifolds, viz., manifolds with boundary. Thank you.