An Introduction to Point Set Topology
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Lecture 36
Global Separation of Sets

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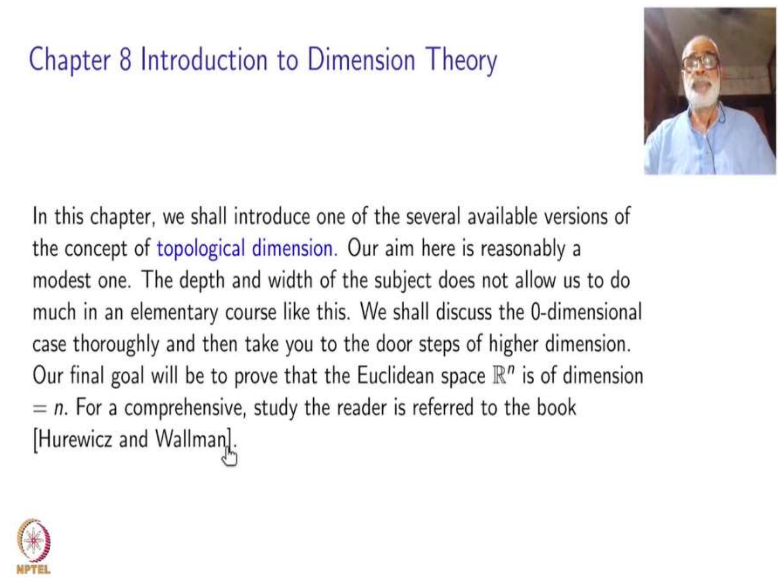

Welcome to Module 36 of NPTEL-NOC, An Introductory Course on Point-Set-Topology part-II. So, we begin a new chapter today, an introduction to dimension theory. So, dimension theory has several approaches. Among all these available versions, we will choose one such suitable for our back ground. Of course, they are all topological dimension theories. Our aim here is reasonably modest one. The depth and width of the subject does not allow us to do much in an elementary course like this.

We should discuss the zero-dimensional case thoroughly and then take you to the doorsteps of higher dimensions. Our final goal will be to prove that the Euclidean space $\mathbb{R}^n$ is exactly of dimension n. If you are interested in more details for a comprehensive study, you are welcome to read this book of Hurewicz and Wallman for this particular dimension theory.

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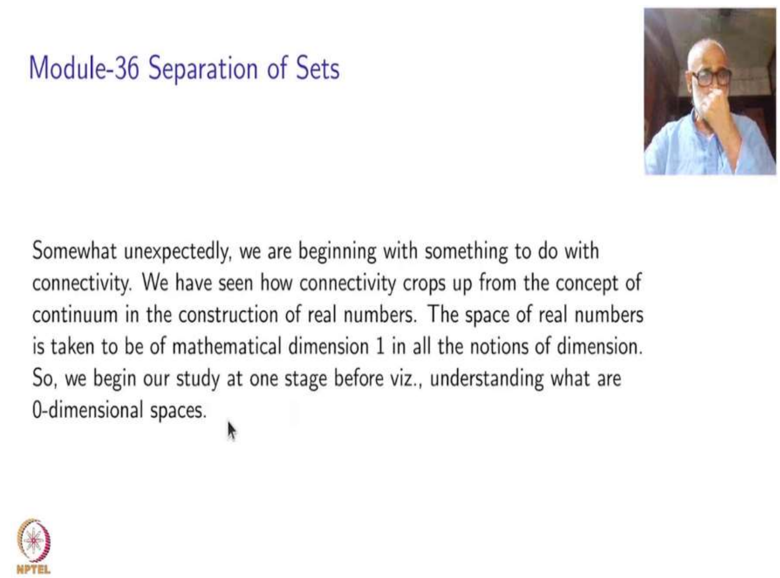

So, Module-36, the title is separation of sets. It has something to do with the separation axioms that we studied so thoroughly and it has something to do with connectivity, but this has nothing to do with the separability of a metric space, another kind of separability of a space as such. So, there are too many different ways the word separation and/or its modifications are used. So, you have to be a bit careful here.

Somewhat unexpectedly, we are beginning with something to do with the connectivity. We have seen how connectivity crops up from the concept of continuum in the construction of real numbers. The space of real numbers is taken to be of mathematical dimension one. Why I am saying this one is whether it is topological dimension or a vector space dimension and so on, various dimension, all of them, you can call out them mathematical dimension, in each of them dimension $\mathbb{R}$ is one.

Note that the word dimension in physics has slightly different connotation. So, in all mathematical dimension theories, the space of real numbers must be having dimension one. So, we begin our study at one stage before that, namely understanding 0-dimensional spaces. For example, if you are studying linear algebra, vector spaces, the 0-dimensional space is just a zero vector space and nothing more than that.

So, that simplicity makes the life very easy with linear algebra. But that is not the case in topology. So, we are trying, we are going to spend considerable amount of time in studying 0-dimensional case itself.

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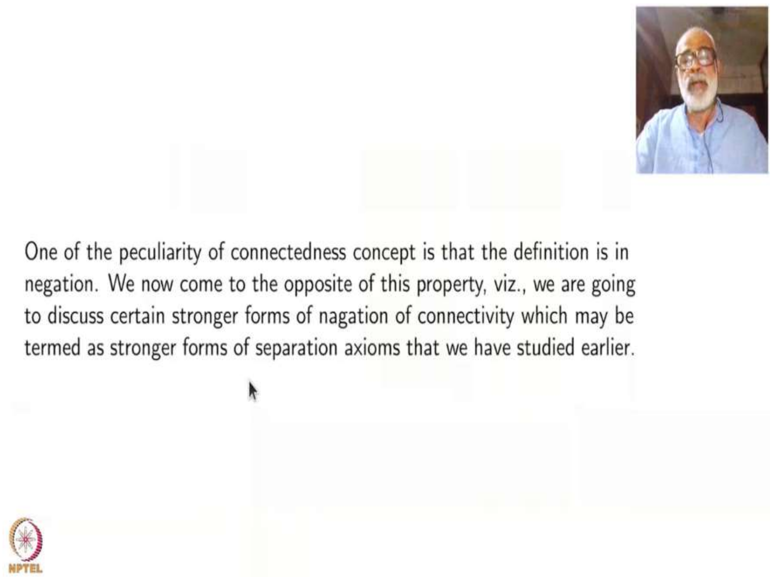

One of the peculiarity of connectedness concept is that the definition is in the negation. In fact, we define disconnectedness first. And then if the space is not disconnected, then we say it is connected. So, we would like to come to the opposite of this property namely, disconnectivity.

So, we are going to discuss certain stronger forms of negation of connectivity, namely, disconnectivity, which may be termed as stronger forms of separation axioms, such as Frechet-space (T_1 space), T_2 space, T_3 space, regularity, normality and so on. So, all those things will come into picture now indirectly.

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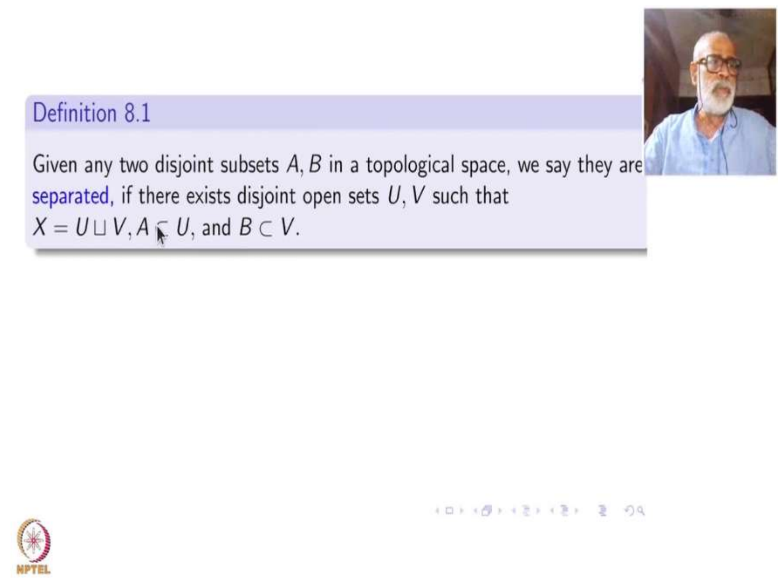

So, let us begin with a definition here. Given any two disjoint subsets A and B in a topological space X, we say they are separated if there exists disjoint open sets U and V such that X is the union of U and V, (U and V are disjoint and that is why I wrote the symbol $\sqcup$ here) A is contained inside U and B is contained inside V, and on emore condition after that, viz., you can assume both of U and V are open or both U and V are closed. It is the same thing.

So, the term here I am going to use is that A and B are separated subsets of X. Of course, to be separated first of all they have to be disjoint. People also use the word disconnected for this one, but that is somewhat confusing for me, so I do not want to use that terminology here. Expresssing X as a disjoint union as above is called a disconnection of X. We will call this a separation of X. So, we had such a notation already introduced also.

We are just writing it as X equal to $U|V$, you remember that. So, that notation also I may use sometimes. In any case, starting with two disjoint subsets, we have enclosed them in U and V. The only thing is the totality of U and V is the whole space X is the extra thing. Otherwise, this is just like normality provided A and B were closed subsets. So, you are reminded of normality.

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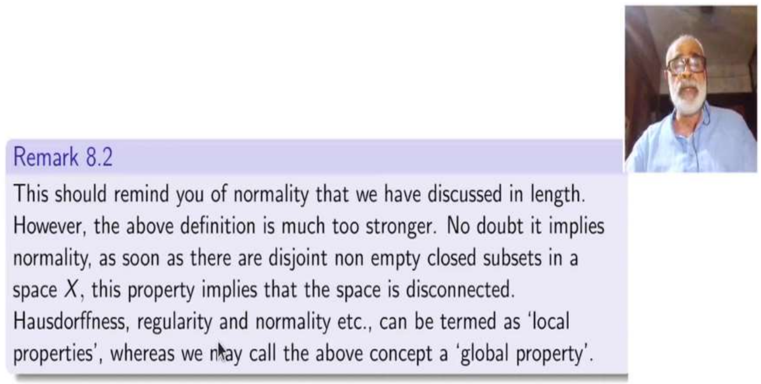

For disjoint closed subsets in a normal space, the two open sets you get may not cover the whole space. They are disjoint fine, but here it is the whole space that makes both of them closed also. So, this is going to be quite a strong condition. No doubt it implies normality as soon as there are disjoint non-empty closed sets in X.

This property implies that the space is disconnected also provided it has more than one point, whereas there are plenty of normal spaces which are connected. If you have this property of separation for every pair disjoint closed sets, then the space will have normal as well as disconnected, so this is very strong.

Hausdorffness, regularity, normality etc. can be termed as local properties, whereas the one which we have introduced now is a global property.

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Let X be a topological space with at least two points. Consider the following four conditions:

Axioms of Disconnectedness (27)

S0: Connected components of X are all singletons.
SI: Any two distinct singletons in X are separated.
SII: Any closed subset $A \subset X$ is separated from any point in $X \setminus A$.
SIII: Any two disjoint closed subsets in X are separated.

So, let us consider the following conditions on a topological space. To make sense out of these things, you should have the topological space with at least two points. The singleton space is connected and it will satisfy this separation property also vacuously. So, do not get confused with that. So, best way is to assume that the space has at least two points.

Then consider these four different axioms of disconnectedness, or axioms of separation. Whatever word you want to use, there is going to be some confusion. So, I can call it the axioms of separation, and that is why I have included notation S0, SI, SII and SIII. Just to include people who are using this terminology I have put that in the title itself.

So, S0 just says connected components of X are all singletons.

Any two distinct singletons in X are separated is SI.

Any closed subset A is separated from any point outside A. That is SII.

Any two disjoint closed subsets in X are separated. That is SIII.

So, S0, SI, SII, SIII you can see that these three things are one stronger than the other. But this S0 seems to be not in this list. Why should it be? This is apparently the odd man out. But this is the one which connects the connectivity with these properties SI, SI, SIII etc.

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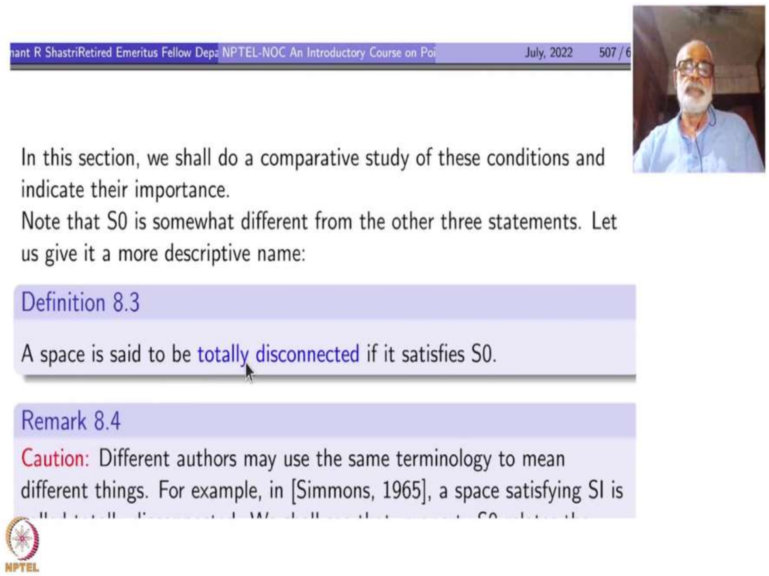

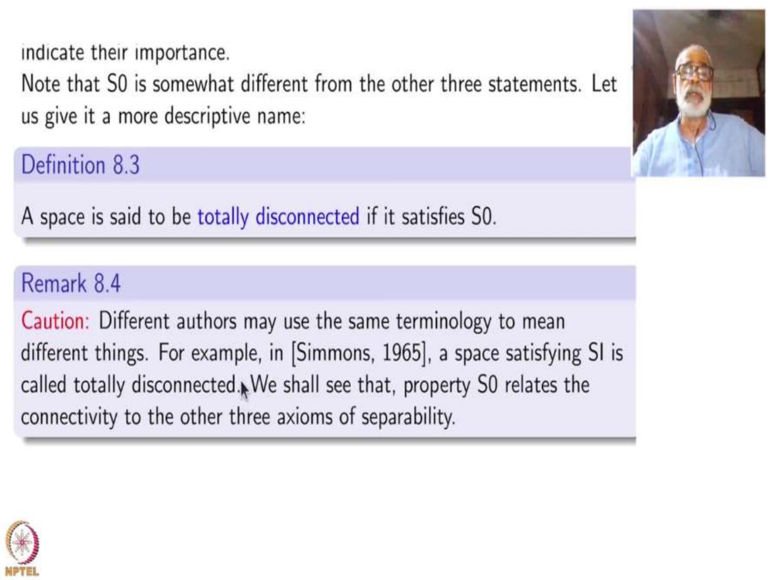

In this section, we shall do a comparative study of these four conditions above and indicate their importance. Note that, S0 is somewhat different from other three statements. Let us give it a more descriptive name. Sometimes I will use that. But for safety I will keep refering to it as S0 only.

So, any space which satisfies S naught will be called totally disconnected.

I want to caution you that different authors may use this same terminology to mean different things. So their total disconnectedness maybe different. For example, one of the books which I am very much using and respect, I have a lot of respect for this one, is Simmon's book: Topology and Modern Analysis.

In this book, a space satisfying our condition SI is called totally disconnected, not S0

We shall see that property S0 relates the connectivity to the other three axioms of separability here. So, let us go ahead.

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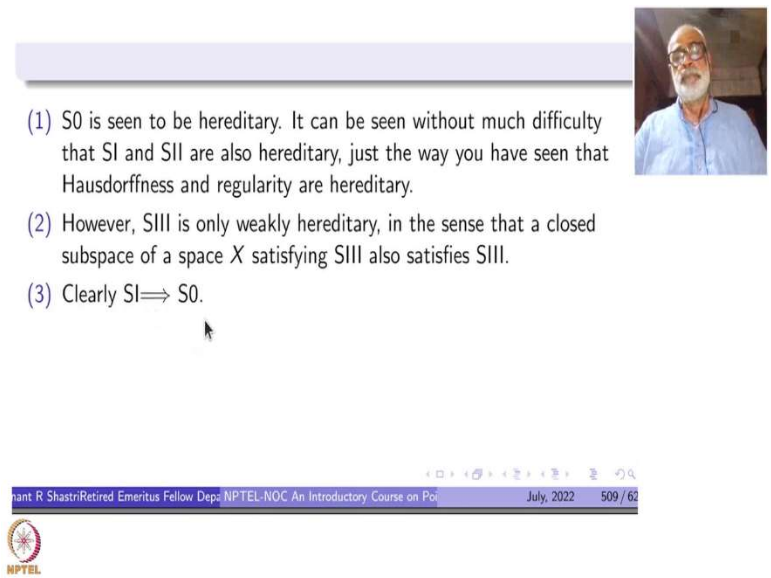

S0 is seen to be hereditary. If something is totally disconnected, namely, all the singletons are the components then for any subspace also singletons will be component. So, it is hereditary. It can be seen without much difficulty that SI and SII are also hereditary. SI is hereditary is just like Hausdorffness being hereditary. SII is hereditary is just like regularity is hereditary.

However, when you come to SIII, it is similar to normality. Just the way normality fails to be hereditary, this will also fail to be hereditary, exactly for same reason. But it is weakly hereditary in the sense that closed subspace of a SIII space will be SIII. Starting with a closed subspace and then taking closed subsets inside that they will be closed in the original space also. Then if you take a separation in the original space, you can restrict it to the subspace. Then you are done. So, that is the proof that it is weakly hereditary.

Now, SI implies S0. What is SI? Any two points, for any two distinct points there is a separation. If there is a separation, they cannot be in the same component. So, if any two points cannot be in the same component, all the connected components are singletons. That is over.

Quite often people confuse S0 with SI. So, caution S0 does not imply SI. We will see an example soon. Unfortunately, the word totally disconnected is used for this one also by some authors. So, you have to be careful with that.

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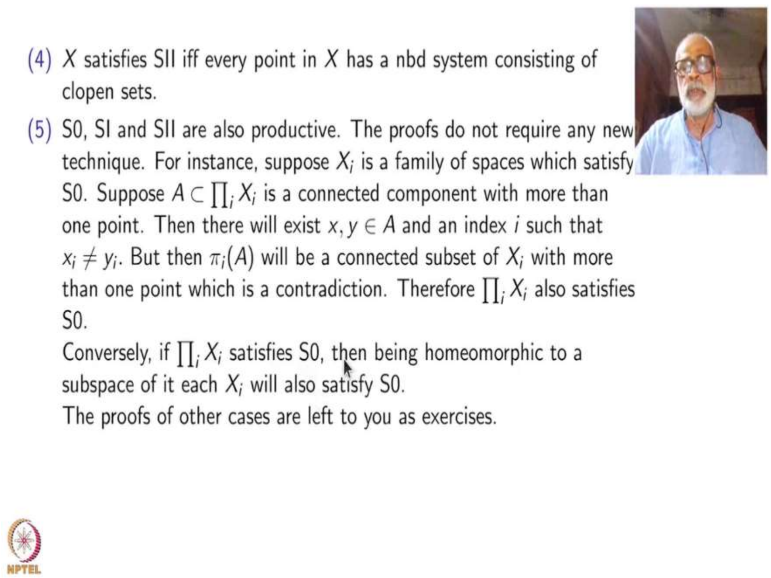

Let us go ahead. If X satisfies SII, then every point in X has a neighborhood system consisting of open as well as closed sets (those things are called clopen sets), and conversely. This could have been taken as a definition of SII. Why? Take a point, take a closed set away from that. Or take a closed set and take a point outside that, which is same thing as taking an open set and taking a point inside of that.

Then what do you have, you have a clopen set along with its complement, that will form a separation on X. So, both ways, I mean this argument can be seen both ways. So, SII, if and only if, at every point of X there is a local base consisting of clopen sets.

S0, SI and SII are all productive also.

Once again the proof is similar to proving that Hausdorffness is productuve. This case of S0 is just like proving product of connected spaces connected. It is much simpler than this one. Product of totally disconnected space is totally disconnected. Suppose each X_i satisfies S0. Now, take a subset A. Suppose A is connected and has more than one point then I should get a contradiction. How? There exists x, y in A and an index i such that $x_i \neq y_i$, because they are different, at least one coordinate must be different. But then if you look at the projection of A on i-th coordinate space space X_i, viz., $\pi_i(A)$ has to be a connected subset because it is image of a connected set under a continuous function.

So, it is a connected subset of X_i with more than one point because $x_i \neq y_i$. That is a contradiction, because we assumed that connected components of X_i are singletons. Therefore, the product satisfies S0.

The converse. Why the converse is true? Suppose the product satisfies S0, then each coordinate space can be thought of as a subspace of the product space, via coordinate inclusion.

(So, this argument you have used several times. X will be homeomorphic to of $X \times \{y\}$ inside $X \times Y$) So, as a subspace because S0 hereditary, it will be S0. So, each coordinate space is S0, which same thing is each X_i is S0.

So, this argument you have used several times. $X_1 \times X_2$ if you look at, X_1 cross one point will be a subspace of $X \times Y$, $X_1 \times X_2$, but it is homeomorphic to X_1. So, as a subspace because it is hereditary, it will be S0. So, each coordinate space is S0, which same thing is each X_i is S0.

So, I will leave the other things here namely, productivity of SI and SII as an entertaining exercise to you. Go through at least that much so that you will be familiar with this concept. The earlier you do them the better. So, before reading the next, before coming to the next module, try to do this exercise so that you will be completely familiar, completely thorough with the definitions at least.

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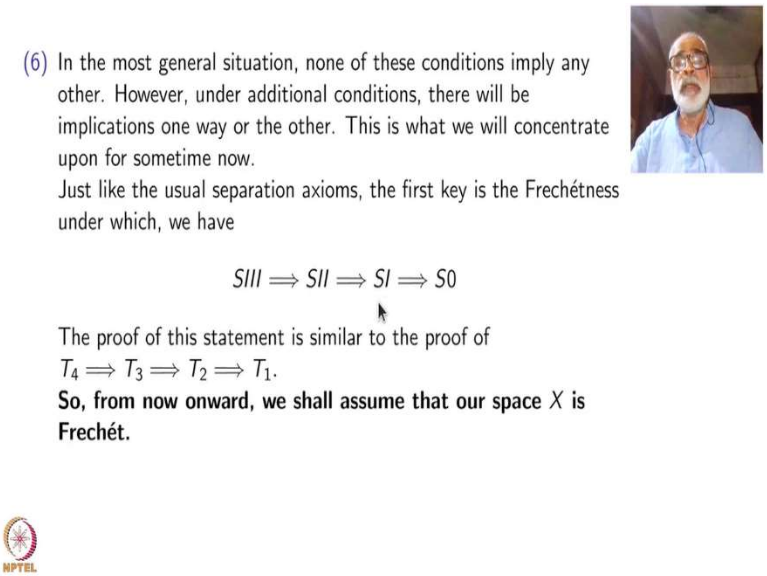

In the most general situation, none of these conditions imply any other. Though I said they looks like stronger, stronger, stronger. Let us see why they are stronger. What is the, in what sense they are stronger. Under some additional conditions there will be implications one way or the other. This is what we concentrate upon for some time. viz., when S0 implies SII implies SIII etc. other way around and so on this is what we are going to do. Just like the usual separation axioms, Hausdorffness, regularity, normality etc.

The first key is the Frechetness. If you admit that all the spaces are T_1 spaces, then automatically SIII will implies SII will implies SI implies S0.

Exactly same reason as T_4 implies T_3 implies T_2 implies T_1. Exactly same reason. First you have to assume that the space is T_1. Otherwise, just normality does not imply regularity right? Same way SIII may not imply SII. As soon as points are closed, SIII will imply SII, because in SII, I start taking a closed set and a point outside. They will automatically give you disjoint closed subsets so you can apply SIII and conlcude to SII.

Similarly, in SI, I start with taking two distinct elements, but they can be treated as a closed set a pint outside and so SII implies SI. And I have already seen that SI implies S0. Why because any two points are disconnected here, there is a separation means they are not in the same connected component. So, this part is easier without the T_1 axiom also. This part we have already seen. So, under T_1 axiom, one is stronger than the other in that order.

So, from now onwards we shall always assume that our space is T_1 space. Then we will try to see whether we can come back. And that is where maybe we have to put more and more conditions.

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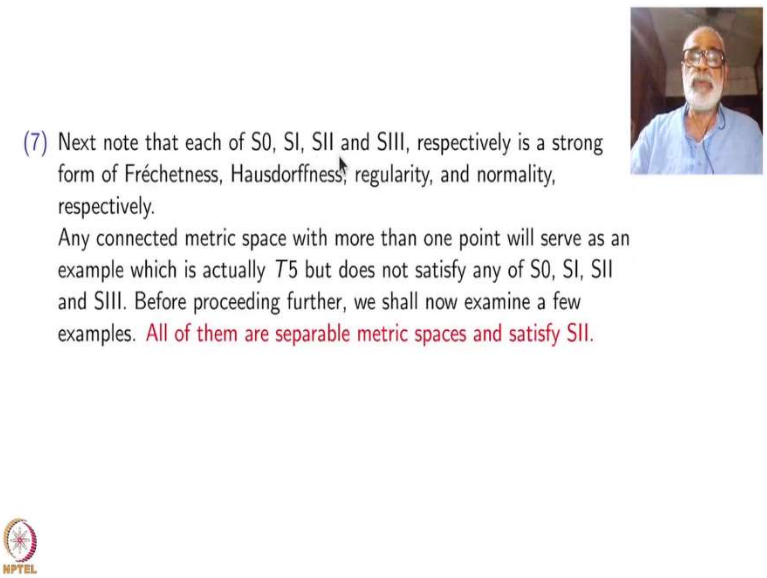

Next, note that each of S0, SI, SII and SIII, respectively, is a strong form of Frechetness, Hausdorffness, regularity and normality, respectively.

By the way, why S0 implies Frechetness? Tell me. Singletons are connected components and connected components are always closed. Therefore singletons are closed.

Any metric space with more than one point will serve as an example which is actually a T_5 space but does not satisfy any of the S0, SI, SII and SIII, only thing you have to assume is that it is connected. Each of these S0, SI, SII, SIII imply disconnectedness (provided there are more than one point in X). That is the meaning of why these axioms bring a connection between connectivity and separation.

So, before proceeding further, we shall examine some examples now. It turns out that all these examples are separable metric spaces and satisfy SII. Why I am doing this one, because finally we are interested in dimension theory developed by Wallman and Hurewicz, in which there is a blanket assumption that they are all metric spaces with a countable dense subset. Separable metric space is a blanket assumption.

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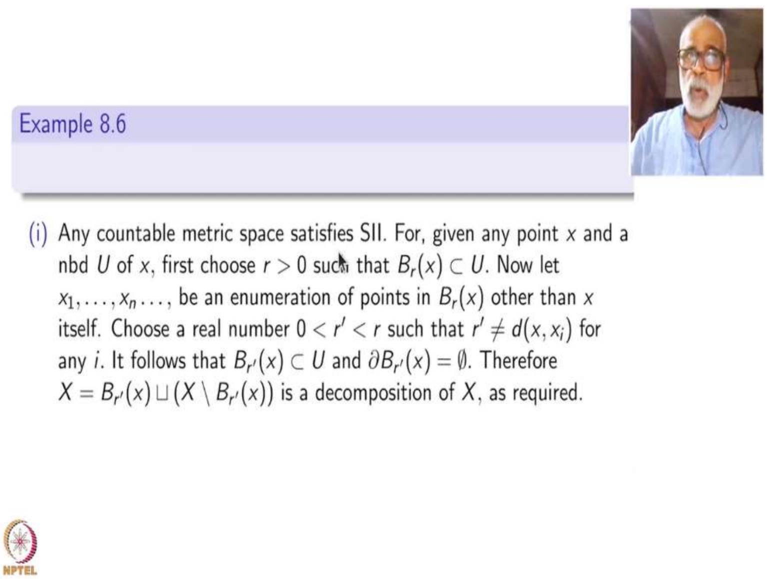

So, let us examine these examples. They are somewhat out of your way so far.

Any countable metric space satisfies SII. For given any point x and a neighborhood U of x, choose r positive such that $B_r(x)$ is inside U. Now, let $x_1, x_2, \ldots, x_n, \ldots$ be an enumeration of points of $B_r(x)$ other than x itself, possible because $B_r(x)$ is countable (the whole metric space X is countable). So, I am just labeling all the other elements.

They maybe finite, they maybe infinite, does not matter dot, dot, dot. I am not saying that it is infinite. Now choose a real number r' so that $0 < r' < r$ and such that $r' \neq d(x, x_i)$ for any i. Note $d(x, x_i)$ as i ranges over $1, 2, 3$ and so on, will give you an only a countable number of real numbers, between 0 and r, whereas there are uncountably many real numbers in $(0, r)$. So, I choose r' which is not equal to any of these countably many numbers.

It follows that the open ball of radius r' around x is contained inside U, because r' is less than r, but its boundary, given by equality, that is empty, because equality occurs and r' is distinct from all $d(x, x_i)$. Whenever you have an open set with boundary is empty, it is closed set also.

Now $B_{r'}(x)$ is contained inside U. So, what we have shown that every point has a neighborhood system consisting of clopen subsets. So, that is the SII that we have seen earlier.

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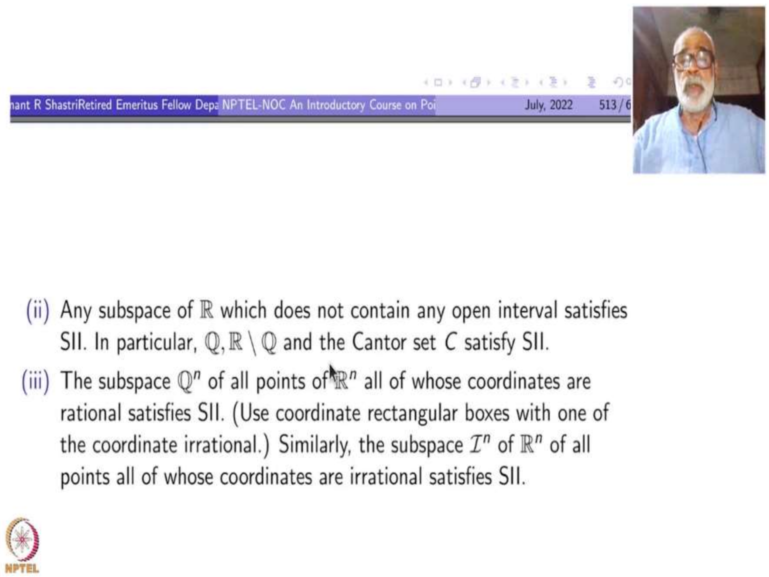

Any subspace of $\mathbb{R}$ which does not contain any open interval satisfies SII. Like we have been studying all these examples for quite some time, viz., $\mathbb{Q}, \mathbb{R} \setminus \mathbb{Q}$, the Cantor set etc. These are all examples of SII. Namely, for each point you can find a neighborhood system such that the boundary of each neighborhood is empty that is the nice way of remembering this SII.

The subspace $\mathbb{Q}^n$ of $\mathbb{R}^n$ of all points whose coordinates are rational also satisfies SII. $\mathbb{Q}$ is contained inside $\mathbb{R}$, you take you $\mathbb{Q}^n$ contained in $\mathbb{R}^n$. That also satisfies SII. All that I have to do is given a point use coordinate rectangular boxes with one of the coordinate of each corner point is irrational. Then the entire boundary of the box will not be inside $\mathbb{Q}^n$.

So, the boundary of the open box will be empty as far as $\mathbb{Q}^n$ is concerned. Similarly, the subspace $\mathbb{I}^n$ of $\mathbb{R}^n$ of all points whose coordinates are irrational, the other way around, (not the complement of the previous example by the way, there all coordinates are rational, here all coordinaties are irrational) that also sastisfies SII. Argument is same thing. Sir, so from II, $\mathbb{Q}$ and $\mathbb{R}$. We can use that here because SII is productive.

Yes, you can use that one also. No problem.

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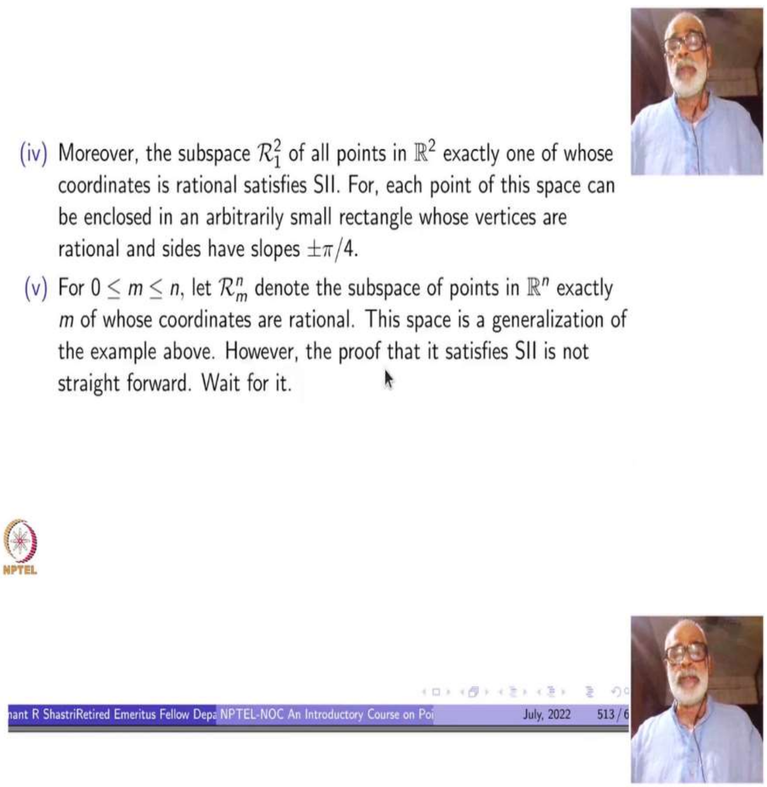

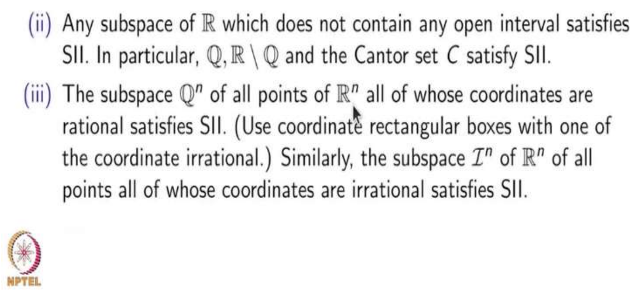

Moreover, now comes the next step, where I am mixing up. For this you will have to use argument as given below, not the productive property.

The subspace $\mathbb{R}_1^2$ of $\mathbb{R}^2$, of all points (I have used this symbol where the upper subscript corresponds to the dimension of the ambient Euclidean space, but the lower subscripts corresponds to the number of coordinates being rational) (x, y) in which either x is rational or y is rational and not both. If both are irrational or both are rational, I am not taking that point. So, this space is also SII.

So, to see this one, you cannot use products and so on. This is not a product space now. So, you have to directly say that for each point, you can get arbitrary small neighborhoods such that the boundary of the neighborhood is empty in $\mathbb{R}^2_1$. That is what you have to do.

So, how do you do that? Each point of this space can be enclosed in an arbitrary small rectangle whose vertices are rational. If the vertices are rational, they are not inside $\mathbb{R}^2_1$. Remember that only one of the coordinates must be rational for a point to be in side this subspace. The vertices are not inside but what are the sides, the sides have slopes plus minus $\pi/4$, and not lines parallel to coordinates lines. So slopes $\pm\pi/4$ which just means that these are points (x, y) such that $x \pm y$ is equal to a constant which is rational.

So, if $x \pm y$ is rational, then either both x, y are rational or both are irratioanl. So, the boundary of this rectangle does not intersect $\mathbb{R}^2_1$. So, this space satisfies SII.

You try to do that inside $\mathbb{R}^3$. Then you will have a lot of problems, this argument would not work.

Using lines with slope $\pm\pi/4$ seems to work only for $\mathbb{R}^2$. But we have other ways of dealing with this problem in higher dimensions.

Start with two indices m and n, $m \leq n$ that is all. Again, similar notation, $\mathbb{R}^n_m$ denotes the subspace of points inside $\mathbb{R}^n$ with exactly m of the coordinates being rational. The same construction.

This 2 is replaced by n and 1 is replaced by m. This space is a generalization of the above example. However, the proof that it satisfies SII is not straightforward. At least the method that we did in (iv) above does not work. Try it.

Of course, you have to wait and then after some stage, I will give you the proof. So, that is a much involved proof here. Just take $\mathbb{R}^3$ here instead $\mathbb{R}^2$. Then you can take any m between 1 and 2. Then the problem will be there.

The cases $m = 0$ and n have beeen taken care of already. You see that they correspond to $\mathbb{I}^n$ and $\mathbb{D}^n$ respectively. So, the problem is only when m is between 1 and $n - 1$. Also $n = 2$ is already done. So, I think today this is enough. So, next time we will continue with more examples so that we are thorough with these concepts S0, SI, SII. So, thank you. That is all for today.