

**Introduction to Point Set Topology, (Part I)**  
**Prof. Anant R. Shastri**  
**Department of Mathematics**  
**Indian Institute of Technology, Bombay**

**Module - 19**  
**Lecture - 19**  
**Baire's Category Theorems**

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Module 18: Three Important Theorems on Complete Metric Spaces  
Module 20: An Application in Analysis  
Module 20: Completion

### Module-19 Baire's Category Theorems

It is convenient (not a logical necessity) to make a couple of definitions, before stating Baire's Category Theorem.

**Definition 1.124**

Let  $X$  be a topological space and  $A$  be a subset.

(a)  $A$  is called an  $F_\sigma$  set in  $X$  if it is the union of countably many closed subsets of  $X$ .

(b)  $A$  is called a  $G_\delta$  set if it is the intersection of countably many open sets in  $X$ .

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Welcome to module 19 of Point Set Topology, part I. Last time, we had announced that we will do three important theorems in metric space theory. The two of them we have already done last time. The third one is Baire's Category Theorem. Before we even state this theorem, it is convenient to make a few definitions. Of course, one can make the statement without these definitions and so on that is not a logical necessity.

The definitions only help in reducing the number of words we use ultimately. So, let  $X$  be a topological space and  $A$  be a subset. So, these notions I am use introducing inside any topological space, not necessarily metric space; remember that.  $A$  is called an  $F_\sigma$  set in  $X$ , if it is the union of countably many closed subsets of  $X$ . This definition is cooked up just to take care of these kind of sets because they are not closed you know, in general, an infinite union of closed sets may not be closed.

But, however, in this theory what happens is countable union of closed sets becomes very important, but you cannot call them closed sets. So, we have to put a name for them usually  $F$  is used for closed sets at least in German (or is it French?) topology in those days,  $F$  was used for closed set. So, the sigma is for countability, you know like countable sum. So,  $F_\sigma$  stands for countable union of closed subsets. Any set which can be written as countable union of closed subsets will be called  $F_\sigma$ .

Similar to this one and dual to that  $A$  is called a  $G_\delta$  set, if it is the intersection of countably many open sets. So, that is like De Morgan law, ok? So, the explanation for the name  $G_\delta$  is same thing  $G$  was used for an open subset and  $\delta$  for intersection; so,  $G_\delta$  is for countably many open subsets and then take the intersection.

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The screenshot shows a presentation slide with a blue header and a white content area. The header contains the text: "Anant B. Shastri/Retired Emeritus Fellow Department of Mathem. NPTEL-NOC An Introductory Course on Point-Set Topology, P". The content area is divided into two columns. The left column has a dark blue background with white text: "Introduction" and "Creating New Spaces". The right column has a light blue background with white text listing modules: "Module 6: Topological Spaces", "Module 7: Examples", "Module 8: Functions", "Module 13: Definitions and examples", "Module 16: Interior, derived set, etc.", "Module 18: Three Important Theorems on Complete Metric Spaces", "Module 20: An Introduction to Analysis", and "Module 20: Completion". Below the content area, the text "Definitions-continued" is displayed. To the right of the slide is a small video feed showing a man with glasses and a beard, identified as "Anant Shastri". Below the slide, there is a white box containing three numbered items: (c) "If  $A$  is a countable union of nowhere dense subsets of  $X$  then  $A$  is said to be of **I-category**. Such a set is also called a **meager set**.", (d) "If  $A$  is not of I-category then it is called a **II-category set**. Such a set is also called a **non meager set**.", and (e) "A topological space which is II-category in itself is called a **Baire Space**". At the bottom of the slide, there is a small logo on the left and navigation icons on the right.

Let us continue this definitions - if  $A$  is a countable union of nowhere dense subsets. Now, that is a stronger word here not just closed subsets, nowhere dense subsets, then,  $A$  is called a 1st category set or belonging to the 1st category or just say 1st category. A set is said to be 1st category, if it is a countable union of nowhere dense subsets of  $X$ .

So, everything is happening in the ambient space  $X$ , ok? If we change  $X$ , the nature may change. Such a set is also called a meagre set. If  $A$  is not of 1st-category, then it is called 2nd category. Just to make distinction between these two things that is all, ok? Such a set is also called non-meagre because it is not meagre, that is all. Now, there is one more terminology here that people are using. If a topological space itself is of 2nd category, you call it a Baire space ok? After the mathematician Baire ok, who introduced these ideas.

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The screenshot shows a presentation slide with a dark blue header and footer. The header contains a table of contents with items like 'Creating New Spaces', 'Module 19: Interior, derived set, etc.', 'Module 18: Three Important Theorems on Complete Metric Spaces', 'Module 20: An Application in Analysis', and 'Module 20: Completion'. The main content area is white with a blue title bar for 'Theorem 1.125' and a grey text box containing the theorem statement: '(Baire's Category Theorem (BCT-1)) Every complete metric space is of II-category.' A small video inset in the top right shows a man with glasses and a beard, identified as 'Anant Shastri'. The footer contains the name 'Anant R. Shastri', his title 'Emeritus Fellow Department of Mathematics', and the course title 'NPTEL-NOC An Introductory Course on Point-Set Topology, P'. A navigation bar with icons is located above the footer.

Now, let us state Baire's category theorem. It becomes very simple because of this terminology. Every complete metric space is of 2nd category. What is the meaning of this? It is not 1st category. What is the meaning of that? It is not the countable union of nowhere dense subsets of  $X$ . The only condition is that  $X$  is a complete metric space now ok?

So, statement becomes very easy that is the whole idea. Now, I have put this as BCT - Baire Category Theorem 1, because there are several versions of this one, ok? So, I have picked up one of them, simplest, very simplest in terms of these definitions. So, this is: every complete metric space is 2nd category.

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Before going into the proof, let us examine a couple of examples.

**Example 1.126**

(1) Of course, every open set is  $G_\delta$ . There are many  $G_\delta$ -sets, which are not open. In some sense, they are the next best things!

(2) In a metric space, every singleton set is a  $G_\delta$ -set. For you can write

$$\{x\} = \bigcap_{n \in \mathbb{N}} B_{1/n}(x).$$

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Before going into the proof, let us examine a couple of examples. Of course, every open set is  $G_\delta$ . There are many  $G_\delta$  sets which are not open. In some sense, they are the next best things to open sets. See, in topology we always keep studying open sets right, but in a metric space  $G_\delta$  sets also become important, ok?

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In particular, every singleton set in  $\mathbb{R}^n$  is a  $G_\delta$ -set. Similarly, every closed interval is also a  $G_\delta$ -set. Many interesting results on continuous real-valued functions follow from this observation. For example, let  $X$  be any topological space and  $f : X \rightarrow \mathbb{R}$  be a continuous function. Then for every  $r \in \mathbb{R}$ , the set

$$A_r = \{x \in X : f(x) = r\}$$

is a  $G_\delta$ -set

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In a metric space especially every singleton set is a  $G_\delta$  set for you can write singleton  $x$  as intersection of balls of radius  $1/n$  with centre  $x$  as  $n$  varies over all positive integers, right. In particular, every singleton set in  $\mathbb{R}^n$  is also a  $G_\delta$ , that is precisely what we have done. In every metric space this is true alright. Similarly, every closed interval is also a  $G_\delta$  set because you can write it as say,  $[a, b]$  is the closed interval; you can take  $(a - 1/n, b + 1/n)$  and then take the intersection ok.

So, many interesting results on continuous real valued functions follow from this observation namely, let  $X$  be any topological space and  $f$  from  $X$  to  $\mathbb{R}$  be a continuous function. Then, for every  $r \in \mathbb{R}$ , the set of  $A_r$  consisting of all points  $x$  belonging to  $X$  such that  $f(x)$  is equal to  $r$ . This is a  $G_\delta$  set why? Because singleton  $r$  is a  $G_\delta$  set.

You can take the inverse image of all those countably many open sets right. They will be open, when you take the intersection it will be the intersection singleton  $r$ . Do you understand what is going on here? Since  $\{r\}$  can be written as intersection of  $(r - 1/n, r + 1/n)$  right? You take the in inverse image of that, they are say  $G_{n,r}$ , intersection of that will be precisely  $A_r$ .

So, take any continuous function into  $\mathbb{R}$ , ok? So, inverse image of open set is open inverse image of  $G_\delta$  is  $G_\delta$ . That is all what I am trying to say it here, more generally, alright.

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+ Go back to application

**Example 1.127**

(a) Any countable set in a metric space is  $F_\sigma$ . Each interval in  $\mathbb{R}$  is both  $F_\sigma$  as well as  $G_\delta$ .

(b) The set of irrational numbers is not  $F_\sigma$  in  $\mathbb{R}$ . For if  $\mathbb{R} \setminus \mathbb{Q} = \cup_n F_n$ , where each  $F_n$  is closed in  $\mathbb{R}$ , then being subsets of irrational numbers, each  $F_n$  will be nowhere dense as observed in Example 1.89. But then

$$\mathbb{R} = (\cup_n F_n) \cup (\cup_{r \in \mathbb{Q}} \{r\})$$

is a countable union of nowhere dense subsets, contradicting Baire's theorem.

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Module 6: Topological Spaces

So, there are some other more interesting examples. Any countable set in a metric space is  $F_\sigma$ . You know by taking complements, because each singleton a closed set this is a countable union. Each interval in  $\mathbb{R}$  is both  $F_\sigma$  as well as  $G_\delta$ . So, just like  $G_\delta$ , similarly we can take  $F_\sigma$  also.

But now comes an interesting one. The set of irrational numbers is not  $F_\sigma$ , ok? So, I have given you examples, but there are not everything is  $F_\sigma$  or  $G_\delta$ . The set of irrational numbers is not  $F_\sigma$  inside  $\mathbb{R}$ . For suppose, it is like this namely  $\mathbb{R} \setminus \mathbb{Q}$  is the irrational numbers; suppose you write it as union of countable union of closed sets, ok.

So, that is the meaning of this is  $F_\sigma$ , right? Then being subsets of irrational numbers, we know that each  $F_n$  will be nowhere dense ok, as observed in example 1.89. Let me just show you this example which you have done earlier, the fourth one here,  $F$  is a closed subset of  $\mathbb{R}$ , and is contained in either  $\mathbb{Q}$  or  $\mathbb{R} \setminus \mathbb{Q}$ . Then, it is nowhere dense right?

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The image shows a presentation slide on the left and a video feed on the right. The slide contains the following text:

Consider the set  $\mathbb{Q}$  of all rationals inside  $\mathbb{R}$  with its usual topology. Check the following facts:

- (i)  $\mathbb{Q}$  is neither an open set, nor a closed set.
- (ii)  $\bar{\mathbb{Q}} = \mathbb{R}$ . That means  $\mathbb{Q}$  is dense in  $\mathbb{R}$ .
- (iii)  $\ell(\mathbb{Q}) = \mathbb{R}$ .
- (iv) If  $F$  is a closed subset of  $\mathbb{R}$  and is contained in  $\mathbb{Q}$  or  $\mathbb{R} \setminus \mathbb{Q}$ , then it is nowhere dense in  $\mathbb{R}$ . For then  $\bar{F} = F$  and Therefore,  $\overset{\circ}{F} = \overset{\circ}{F} = \emptyset$ , since  $F$  contains no intervals.

Below the slide is a navigation bar with the text: "Anant R Shastri Retired Emeritus Fellow Department of Mathemat... NPTEL-NOC An Introductory Course on Point-Set Topology, P...". Below the navigation bar is a table of contents with the following items:

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The video feed on the right shows a man with a white beard and glasses, wearing a blue shirt, speaking.

We have seen this one. For  $F$  is a closed subset, then  $\bar{F} = F$ , interior of this one is same thing as interior of  $\bar{F}$ , So that means that  $F$  contains no intervals. Any subset of irrationals contains no intervals. So, this is what we had seen earlier. So, I am just recalling it. This is an important example here ok? So, if you write  $\mathbb{R} \setminus \mathbb{Q}$  as union of  $F_n$ 's, first of all each  $F_n$  is closed is the assumption, then each  $F_n$  becomes nowhere dense.

But then you can put more you know another countable family of sets namely all singletons ok, singletons are anyway nowhere dense right? They also do not contain any intervals, these singleton are coming from  $\mathbb{Q}$ , But now whole of  $\mathbb{R}$  is a countable union of closed sets and nowhere dense sets. So, that shows that this  $\mathbb{R}$  is written as union of nowhere dense sets. It means  $\mathbb{R}$  is 1st category in our definition right, but  $\mathbb{R}$  is a complete metric space.

So, Baire's theorem just says that, every complete metric space is 2nd category. So, I have given you an application you know very simple mind application of Baire's theorem to show that that the set of irrational numbers cannot be written as a countable union of closed sets. Of course, with rational numbers are  $F_\sigma$  no contradiction, there is no harm ok?

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**Introduction**  
Creating New Spaces

**Example 1.128**

Let  $p \in \mathbb{K}[x_1, \dots, x_n]$  be any non constant polynomial in  $n$  variables. First of all, we know that the zero set  $Z(p) = \{x = (x_1, \dots, x_n) \in \mathbb{K} : p(x) = 0\}$  is a closed set. Next, we claim that  $Z(p)$  contains no non-empty open set. For if  $U \subset Z(p)$  is one such open set, then it follows that the function  $p : \mathbb{K}^n \rightarrow \mathbb{K}$  restricted to  $U$  is the zero function. By taking successive partial derivatives, we can then show that all the coefficients of  $p$  are zero. Therefore,  $p$  itself is the zero polynomial, which is a contradiction.

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I will give you one more example these are the easy consequences of Baire's theorem, but that is not the end of Baire's theorem we will see. So, let us take another example here take a polynomial in  $n$  variables. Of course, if you take zero polynomial that is not interesting. So, non constant polynomials, let us take one ok?

Look at all the zeros of that, the zero set  $Z(p)$  of polynomial  $p$ ; all  $x = (x_1, x_2, \dots, x_n)$ , when you evaluate  $p$  on that:  $p(x) = 0$ ; that is a closed set because  $p$  is continuous alright. We claim that  $Z(p)$  contains no non empty open set, it is nowhere dense ok. So, this is elementary calculus. As soon as there is an open set of  $\mathbb{R}^n$  (or  $\mathbb{C}^n$  does not matter,) contained inside a set, you can study the polynomial on that set ok? Which is identically 0 by definition, that is it is contained in  $Z(p)$ .

But a zero function on an open set has all its partial derivatives 0. If you compute partial derivatives cleverly you can compute all the coefficients of this polynomial; not only in one variable in any variable, any number of variables. You have to do all the partial derivative various partial derivatives, That means what?

All the coefficients are 0; that means,  $p$  itself is a zero polynomial, but we started with a non-constant polynomial, ok. So, the zero set of any polynomial is nowhere dense. What is the



consequence? The entire  $\mathbb{R}^n$  or  $\mathbb{C}^n$  cannot be the union of countably many zero sets of polynomials ok. So, that is the consequence by Baire's theorem.

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The image shows a presentation slide. At the top, there is a header with the text "Anant R Shastri Retired Emeritus Fellow Department of Mathematics" and "NPTEL-NOC An Introductory Course on Point-Set Topology, P". Below the header is a table of contents with the following items:

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Below the table of contents, there is a text box with the following text:

Thus it follows that the zero set of any non-zero polynomial is a nowhere dense set. It would follow from Baire's theorem that  $\mathbb{K}^n$  cannot be written as a union of countably many zero-sets of non-zero polynomials.

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The zero set of any nonzero polynomial is nowhere dense. It follows that  $\mathbb{K}^n$ ,  $\mathbb{K}$  could be  $\mathbb{R}$  or  $\mathbb{C}$  does not matter, cannot be written as a union of countable many zero sets of non-zero polynomials ok?

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Coming to the proof of BCT, we shall actually prove a stronger version of BCT.

**Theorem 1.129**

**(Baire's Category theorem, Version-1 (BCT0))** *Let  $X$  be a complete metric space. Suppose  $\{A_n\}$  is a countable family of nowhere dense subsets of  $X$ . Then  $X \setminus \cup_n A_n$  is dense in  $X$ ; in particular, it is non-empty.*

The proof will make use of theorem 1.107 repeatedly. This theorem says the following:

If  $A$  is a nowhere dense set in  $X$ , then every open set in  $X$  contains the closure of an open ball disjoint from  $A$ .

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So, let us prove this theorem now. We shall actually prove a stronger version of this theorem. The stronger version I have called BCT0 because it sits over all other versions ok? Take a metric space which is complete. Take a countable family of nowhere dense subsets. The complement is actually dense that is a statement.  $X \setminus \cup_n A_n$  is dense. In particular it is non-empty. An empty set cannot be dense because closure of an empty set is empty ok? That it is non-empty same thing as BCT1. That we have seen because if it were empty then  $X$  would have been union of  $A_n$ 's that would mean that  $X$  is 1st category right?

But the statement is  $X$  is 2nd category. So, BCT0 implies BCT1 very easily. So, we are going to prove the stronger statement alright. Once again, we have done the groundwork already.

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Theorem 1.107

Let  $(X, d)$  be any metric space. A subset  $A$  of  $X$  is nowhere dense in  $X$  iff each non-empty open set in  $X$  contains the closure of an open disc disjoint from  $A$ .

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So, this is theorem 1.107. Let us take a look at this one ok?  $(X, d)$  is a metric space. A subset  $A$  of  $X$  is nowhere dense if and only if, each non empty open set in  $X$  contains the closure of an open disc disjoint from  $A$ . So, there will be some  $B_r(x_0)$ , closure of that intersection with  $A$  will be empty ok? And this will be contained inside any given non-empty open set. So, this theorem I am going to use again and again ok. So, I have to go back now. Yeah, we have had this statement here, but I wanted to show that what actually we have done, we might have forgotten it.

If  $A$  is a nowhere dense subset of  $X$ , then every open set in  $X$  contains the closure of an open ball disjoint from  $A$ . So, this is a statement I am going to use again and again ok?

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Figure 6: A nowhere-dense set compared with an open set

So, this is precisely the kind of thing that we are going to be. This is dot, dot, dot, dot, dot is the nowhere dense set. This single line, thin line indicates an open set. Inside that, I can find a ball of some positive radius such that the closure is disjoint from all these points dot, dot, dots ok? A nowhere dense set compared with an open set.

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**Proof:** Let  $B_1$  be any open ball of radius  $r > 0$ . We need to show that  $B_1 \not\subset \bigcup_n A_n$ .  
 $B_1$  contains an open ball  $C_1$  such that  $\bar{C}_1 \cap A_1 = \emptyset$ . Let  $B_2$  be an open ball inside  $C_1$  of radius less than  $r/2$ . It contains an open ball  $C_2$  so that  $\bar{C}_2 \cap A_2 = \emptyset$ . Inductively, we choose open balls  $C_n$  of radius less than  $r/n$  such that  $C_n \subset C_{n-1}$  and  $\bar{C}_n \cap A_n = \emptyset$ . By Cantor's theorem, it follows that

$$C = \bigcap_n \bar{C}_n \neq \emptyset \subset B_1.$$

Since  $C \cap A_n = \emptyset$  for all  $n$ , we see that  $C \not\subset (\bigcup_n A_n)$ .

Let  $B_1$  be any open ball of positive radius you start with. Instead of any open set, if I prove this one for  $B_1$  which is an open ball then it will follow for every open set, I will produce

something inside  $B_1$ . So, starting with an open set, I can start with  $B_1$  instead right? That is why that  $B_1$ , you know, an open ball of radius  $r$ .

We need to show that  $B_1$  is not contained in the union of  $A_n$ 's. That is enough, ok? This  $B_1$  is not contained inside union of  $A_n$  that is all I want to show. First of all  $B_1$  contains an open ball  $C_1$  such that the  $\bar{C}_1 \cap A_1 = \emptyset$ , because  $A_1$  is nowhere dense.

So, this is the first time I am applying this theorem 1.107, ok. Now, I have got  $C_1$ . Let  $B_2$  be an open ball inside  $C_1$  of radius less than  $r/2$ , ok? I am making sure that the radii are going down, down, down to 0 by putting  $1/2$  here ok.  $B_2$  is contained inside  $C_1$ ; This  $C_1$  is a ball, but I want a ball of radius smaller than  $r/2, r/3$  whatever ok? It contains again, now apply the theorem again, it contains an open ball  $C_2$ , such that  $\bar{C}_2 \cap A_2 = \emptyset$ .

Now, you know the game. Inductively, suppose you have chosen  $C_n$  of radius less than  $r/n$  such that  $C_n$  is contained  $C_{n-1}$  and  $\bar{C}_n \cap A_n = \emptyset$ . Once you have that, inside the  $C_n$ , you will get another one and so on. So, you keep taking them.

Now, we apply Cantor's intersection theorem. To what? To these  $C_n$ 's,  $C = \bigcap \bar{C}_n$ . So, these are all closed subsets, their diameters are less than  $2r/n$ , right? So, as  $n$  tends to infinity, they go to 0.

So, therefore, this intersection is actually a single point. It is non empty is all that I require. ok? Since, I want to apply Cantor's theorem I have put  $r, r/2, r/3, \dots, r/n$  so on. So, this will go to 0. Therefore,  $C$  the intersection is a singleton. But I want only non-empty that is ok. They are all contained in  $B_1$ , ok. So, this singleton is inside  $B_1$ , but what is this point? This point is in  $C$  and  $C \cap A_n$  is empty for all  $n$ , ok?

Why?  $C_n$  it is contained in  $\bar{C}_n, \bar{C}_n \cap A_n = \emptyset$ . So,  $C \cap A_n = \emptyset$  for all  $n$  right. So, this point is in none of the  $A_n$ 's. So, this means  $B_1$  is not contained in the union of  $A_n$ 's.

See if you wanted to prove that something is dense, what you have to do? Take any non empty open set, it should intersect that set. Here, we wanted to show that  $(\cup A_n)^c$  is dense. So the open set should not be contained inside the  $\cup A_n$ . So, that is what I have proved.

Take any open set  $B_1$ , ok, it is not contained inside union means what? The compliment intersects  $B_1$ . Therefore, what we have proved is that the compliment of  $\cup A_n$  is is dense. Be sure that we have non-emptiness is not just what we have proved. We have actually proved that  $X \setminus \cup_n A_n$  is actually dense. So, that is the proof of this ok?

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Anant R Shastri Retired Emeritus Fellow Department of Mathem. NPTEL NOC An Introductory Course on Point-Set Topology, I

Introduction Creating New Spaces	Module 6: Topological Spaces Module 7: Examples Module 8: Functions Module 13: Definitions and examples Module 16: Interior, derived set, etc. Module 18: Three Important Theorems on Complete Metric Spaces Module 20: An Application in Analysis Module 20: Completion
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Anant Shastri

**Remark 1.130**

This theorem is very useful in function theory when one has to prove the existence of various types of functions. Even the weaker conclusion that  $X \setminus \cup_n A_n \neq \emptyset$  is quite useful. Indeed, proofs of several fundamental results in Functional Analysis, such as the Closed Graph Theorem, the Open Mapping Theorem, and the Boundedness Principle, etc., use BCT.

So, let me make a few remarks here. This theorem is very useful in function theory when one has to prove the existence of various types of functions. So, this theorem says something is non-empty; that is how it is used ok? Some how you get a complete metric space, you cook up a complete metric space then you cook up a sequence of closed subsets which are nowhere dense in it. They will not cover the whole space means there is something left out.

So, that is an existence theorem. So, that is the way it is used in many existence theorems ok? Indeed proofs of several fundamental results in functional analysis use this theorem. I will quote some of them which are very very fundamental namely, closed graph theorem, open mapping theorem and boundedness principle and so on and so forth.

So, all these things come in elementary functional analysis itself. The first course in function analysis seems you will have all these theorems ok. They are all using Baire's category theorem to prove, alright.

So, all these things come in elementary functional analysis itself. In the first course in function analysis you will have all these theorems ok. They are all using Baire's category theorem in the proof alright.

(Refer Slide Time: 24:08)

The screenshot shows a presentation slide with a table of contents on the left, a video feed of Anant Shastri on the right, and a text box containing Remark 1.131. The table of contents includes: Introduction: Creating New Spaces; Module 6: Topological Spaces; Module 7: Examples; Module 8: Functions; Module 11: Definitions and examples; Module 16: Baire's theorem, etc.; Module 18: Three Important Theorems on Complete Metric Spaces; Module 20: An Application in Analysis; Module 20: Completion. The video feed shows Anant Shastri, a man with glasses and a beard, wearing a blue shirt. The text box contains the following text:

**Remark 1.131**  
 The above theorem has a negative tone. It can be put in a positive tone at least in two different ways. Often that is how it is applied, say, in Functional Analysis. The equivalence of these three versions (BCT-1), BCT-2), and (BCT-3) is left to you as a pleasant exercise. Later, in Part-II, we shall prove a version of BCT for locally compact Hausdorff spaces which are not necessarily metrizable.

At the bottom of the slide, there is a footer with the text: Anant R. Shastri Retired Emeritus Fellow Department of Mathematics, NPTEL-NOG: An Introductory Course on Point-Set Topology, P.

So, I will repeat this one. The above theorem has a negative tone; that means, 2nd category itself, definition is it is not 1st category. The 1st category is what it cannot be something. So, there are too many negations there.

But it can be put in slightly a positive tone as follows. So, I have given you those tones. And then often it is how this positive versions are here. So, let us have those versions; Later on, in part 2 we shall prove a version of this Baire's category theorem for locally compact Hausdorff spaces which are nothing to do with metrizable, there is no metrics ok.

So, here are those versions.

(Refer Slide Time: 25:01)

The screenshot shows a presentation slide with the following content:

- Table of Contents:**
  - Introduction: Creating New Spaces
  - Module-7: Examples
  - Module-8: Functions
  - Module-13: Definitions and examples
  - Module-16: Interior, derived set, etc.
  - Module-18: Three Important Theorems on Complete Metric Spaces
  - Module-20: An Application in Analysis
  - Module-20: Completions
- Video Feed:** Anant Shastri
- Theorem 1.132:**

**(Baire's Category Theorem: Other Versions)** Let  $X$  be a complete metric space.

(BCT-2) If  $X = \cup_n A_n$  is written as a countable union of its subsets  $A_n$ 's, then the closure of at least one of the  $A_n$ 's has a non-empty interior.

(BCT-3) Intersection of a countable family of open dense sets is non-empty.
- Footer:**
  - Anant R Shastri, Retired Emeritus Fellow, Department of Mathematics
  - NPTL-NOC: An Introductory Course on Point-Set Topology, P
  - Module 8: Topological Spaces

Let  $X$  be a complete metric space that is that is standard assumption there is no other. So, BCT2 says that suppose  $X$  is  $\cup_n A_n$  written as a countable union  $A_n$ 's, then at least one of  $A_n$ 's has its closure with non empty interior. See the hypothesis on  $A_n$ 's is deleted, space is just a countable union.

But, then you conclude closure of one of them has empty interior,  $A_n$  is nowhere dense for some  $n$ , ok. So, it is just the other way around you. So, here there is no negation here. If you write like this closure of one of them is empty interior. It is a positive tone.

Another one is: intersection of a countable family of open dense sets is non-empty. So, this is the way it will be used. So, there is one element they want. So, that is that is the existence theorem. So, the proof of that 1 implies, 2 implies, 3 implies 1 they is they are equivalent ok? is very easy for you. But 0 is a stronger statement which will imply all of them ok.



(Refer Slide Time: 26:38)

The image shows a presentation slide titled "Exercise 1.133" with two questions. The slide is part of a course on Point-Set Topology by Anant R. Shastri. The slide content is as follows:

**Exercise 1.133**

- 1 Write down versions of  $(BCT_0)$  similar to  $(BCT_2)$  and  $(BCT_3)$
- 2 Let  $X$  be a complete metric space. Does  $(BCT_1)$  imply that the union of countably many nowhere-dense subsets of  $X$  is nowhere dense in  $X$ ?

The slide also includes a navigation bar at the bottom with the text: "Anant R. Shastri/Retired Emeritus Fellow Department of Mathemat... NPTEL-NOC: An Introductory Course on Point-Set Topology, P...

So, here is an exercise, namely, write down versions of BCT0 also similar to BCT2 and BCT3 ok. Taken a complete metric space, does BCT1 imply that union of countably many nowhere dense subsets of  $X$  is nowhere dense? Pay attention to the statement there it does not say this, union of countably many nowhere dense sets does not fill up the whole space; this is the weaker version.

Complement is actually dense is the stronger version, but here it is said that the union itself is nowhere-dense ok. So, you have to see whether this is true ok? It is not stated does not mean that it is not implied. So, I am asking whether this is implied by the statement. Think about them. So, that is that is the exercise you have to think about it that is all. So, let us stop here.

Thank you.