Introduction to Point Set Topology, (Part I)
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Module - 19
Lecture - 19
Baire's Category Theorems

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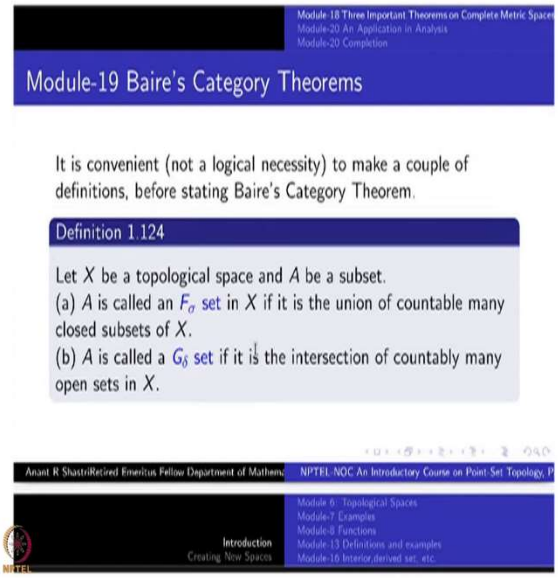

Welcome to module 19 of Point Set Topology, part I. Last time, we had announced that we will do three important theorems in metric space theory. The two of them we have already done last time. The third one is Baire's Category Theorem. Before we even state this theorem, it is convenient to make a few definitions. Of course, one can make the statement without these definitions and so on that is not a logical necessity.

The definitions only help in reducing the number of words we use ultimately. So, let X be a topological space and A be a subset. So, these notions I am use introducing inside any topological space, not necessarily metric space; remember that. A is called an F_σ set in X, if it is the union of countably many closed subsets of X. This definition is cooked up just to take care of these kind of sets because they are not closed you know, in general, an infinite union of closed sets may not be closed.

But, however, in this theory what happens is countable union of closed sets becomes very important, but you cannot call them closed sets. So, we have to put a name for them usually F is used for closed sets at least in German (or is it French?) topology in those days, F was used for closed set. So, the sigma is for countability, you know like countable sum. So, F_σ stands for countable union of closed subsets. Any set which can be written as countable union of closed subsets will be called F_σ.

Similar to this one and dual to that A is called a G_δ set, if it is the intersection of countably many open sets. So, that is like De Morgan law, ok? So, the explanation for the name G_δ is same thing G was used for an open subset and δ for intersection; so, G_δ is for countably many open subsets and then take the intersection.

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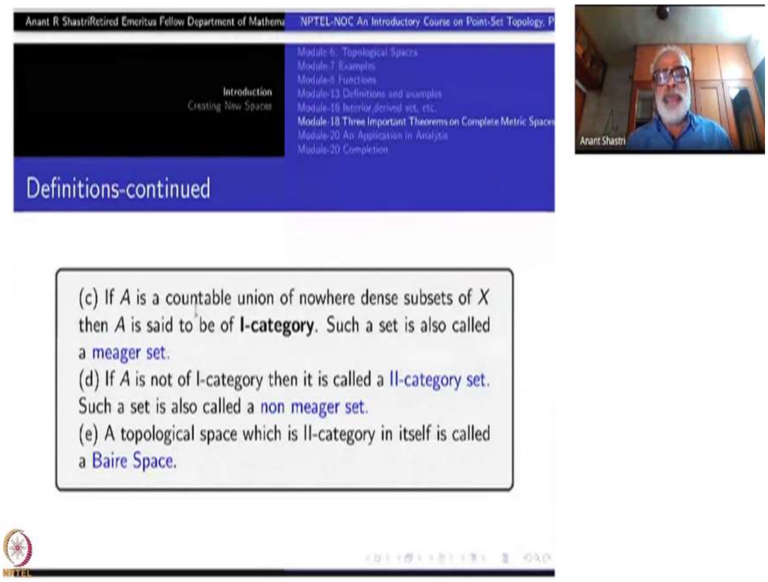

Let us continue this definitions - if A is a countable union of nowhere dense subsets. Now, that is a stronger word here not just closed subsets, nowhere dense subsets, then, A is called a 1st category set or belonging to the 1st category or just say 1st category. A set is said to be 1st category, if it is a countable union of nowhere dense subsets of X.

So, everything is happening in the ambient space X, ok? If we change X, the nature may change. Such a set is also called a meagre set. If A is not of 1st-category, then it is called 2nd category. Just to make distinction between these two things that is all, ok? Such a set is also called non-meager because it is not meagre, that is all. Now, there is one more terminology here that people are using. If a topological space itself is of 2nd category, you call it a Baire space ok? After the mathematician Baire ok, who introduced these ideas.

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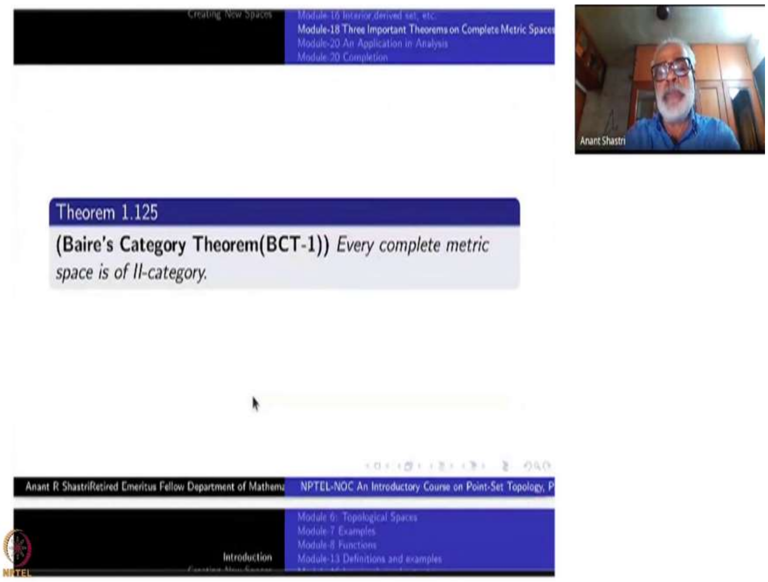

Now, let us state Baire's category theorem. It becomes very simple because of this terminology. Every complete metric space is of 2nd category. What is the meaning of this? It is not 1st category. What is the meaning of that? It is not the countable union of nowhere dense subsets of X. The only condition is that X is a complete metric space now ok?

So, statement becomes very easy that is the whole idea. Now, I have put this as BCT - Baire Category Theorem 1, because there are several versions of this one, ok? So, I have picked up one of them, simplest, very simplest in terms of these definitions. So, this is: every complete metric space is 2nd category.

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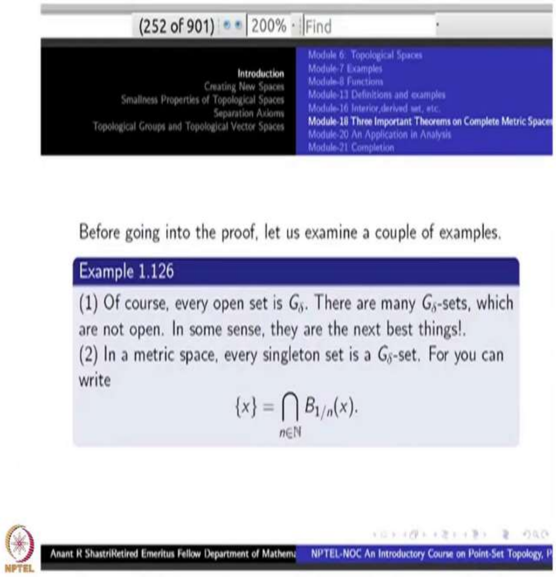

Before going into the proof, let us examine a couple of examples. Of course, every open set is G_δ. There are many G_δ sets which are not open. In some sense, they are the next best things to open sets. See, in topology we always keep studying open sets right, but in a metric space G_δ sets also become important, ok?

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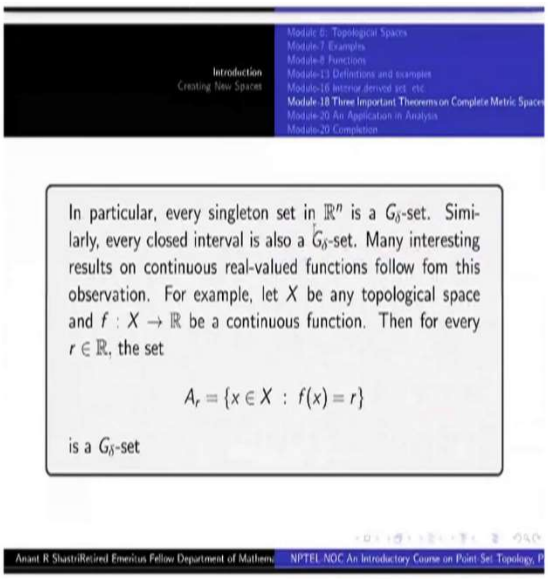

In a metric space especially every singleton set is a G_δ set for you can write singleton x as intersection of balls of radius $1/n$ with centre x as n varies over all positive integers, right. In particular, every singleton set in $\mathbb{R}^n$ is also a G_δ, that is precisely what we have done. In every metric space this is true alright. Similarly, every closed interval is also a G_δ set because you can write it as say, $[a, b]$ is the closed interval; you can take $(a - 1/n, b + 1/n)$ and then take the intersection ok.

So, many interesting results on continuous real valued functions follow from this observation namely, let X be any topological space and f from X to $\mathbb{R}$ be a continuous function. Then, for every $r \in \mathbb{R}$, the set of A_r consisting of all points x belonging to X such that $f(x)$ is equal to r. This is a G_δ set why? Because singleton r is a G_δ set.

You can take the inverse image of all those countably many open sets right. They will be open, when you take the intersection it will be the intersection singleton r. Do you understand what is going on here? Since $\{r\}$ can be written as intersection of $(r - 1/n, r + 1/n)$ right? You take the in inverse image of that, they are say G_{n_r}, intersection of that will be precisely A_r.

So, take any continuous function into $\mathbb{R}$, ok? So, inverse image of open set is open inverse image of G_δ is G_δ. That is all what I am trying to say it here, more generally, alright.

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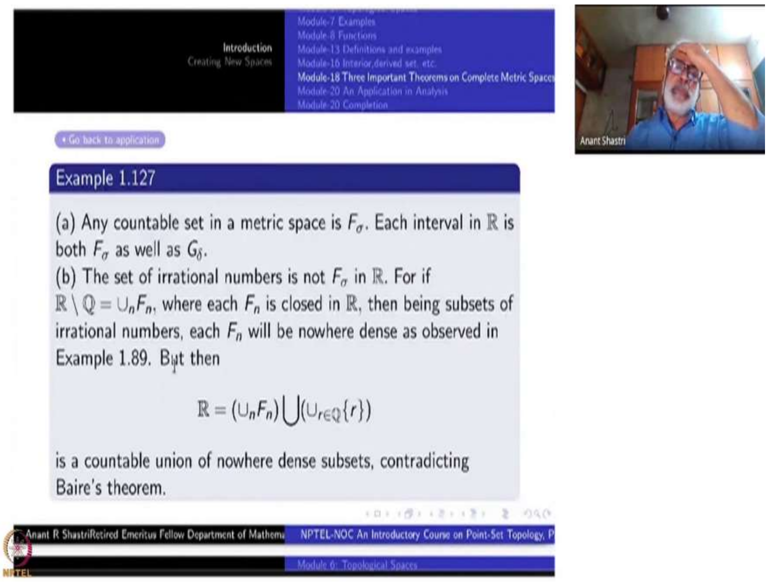

So, there are some other more interesting examples. Any countable set in a metric space is F_σ . You know by taking complements, because each singleton a closed set this is a countable union. Each interval in $\mathbb{R}$ is both F_σ as well as G_δ. So, just like G_δ, similarly we can take F_σ also.

But now comes an interesting one. The set of irrational numbers is not F_σ, ok? So, I have given you examples, but there are not everything is F_σ or G_δ. The set of irrational numbers is not F_σ inside $\mathbb{R}$. For suppose, it is like this namely $\mathbb{R} \setminus \mathbb{Q}$ is the irrational numbers; suppose you write it as union of countable union of closed sets, ok.

So, that is the meaning of this is F_σ, right? Then being subsets of irrational numbers, we know that each F_n will be nowhere dense ok, as observed in example 1.89. Let me just show you this example which you have done earlier, the fourth one here, F is a closed subset of $\mathbb{R}$, and is contained in either $\mathbb{Q}$ or $\mathbb{R} \setminus \mathbb{Q}$. Then, it is nowhere dense right?

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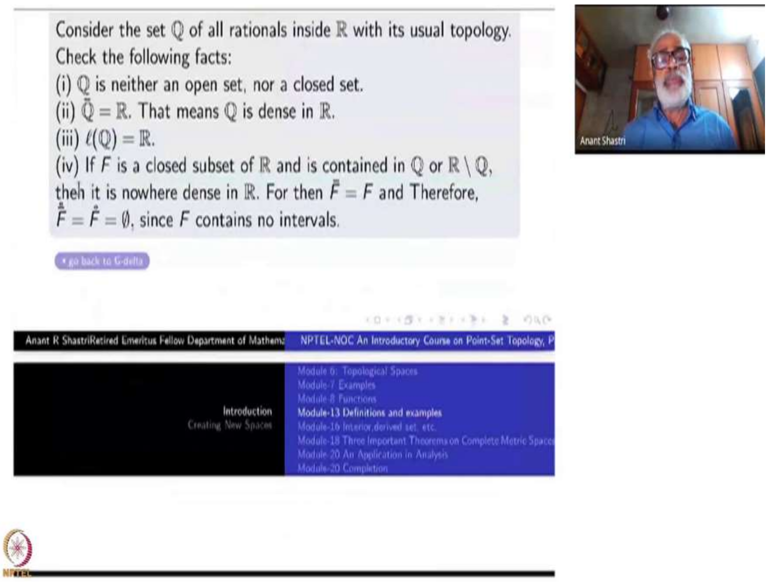

We have seen this one. For F is a closed subset, then $\bar{F} = F$, interior of this one is same thing as interior of $\bar{F}$, So that means that F contains no intervals. Any subset of irrationals contains no intervals. So, this is what we had seen earlier. So, I am just recalling it. This is an important example here ok? So, if you write $\mathbb{R} \setminus \mathbb{Q}$ as union of F_n's, first of all each F_n is closed is the assumption, then each F_n becomes nowhere dense.

But then you can put more you know another countable family of sets namely all singletons ok, singletons are anyway nowhere dense right? They also do not contain any intervals, these singleton are coming from $\mathbb{Q}$, But now whole of $\mathbb{R}$ is a countable union of closed sets and nowhere dense sets. So, that shows that this $\mathbb{R}$ is written as union of nowhere dense sets. It means $\mathbb{R}$ is 1st category in our definition right, but $\mathbb{R}$ is a complete metric space.

So, Baire's theorem just says that, every complete metric space is 2nd category. So, I have given you an application you know very simple mind application of Baire's theorem to show that that the set of irrational numbers cannot be written as a countable union of closed sets. Of course, with rational numbers are F_σ no contradiction, there is no harm ok?

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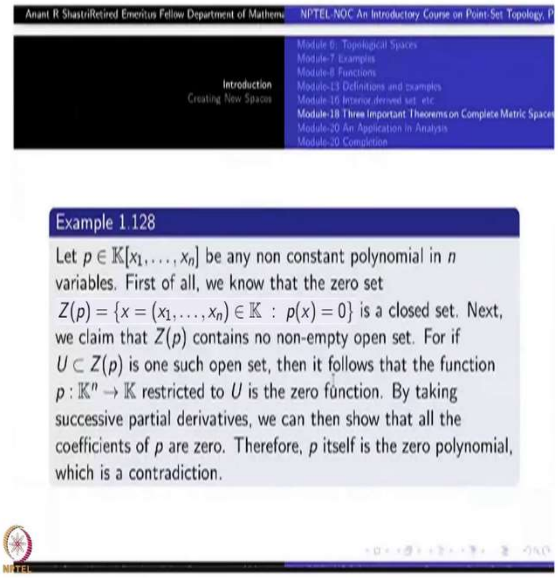

I will give you one more example these are the easy consequences of of Baire's theorem, but that is not the end of Baire's theorem we will see. So, let us take another example here take a polynomial in n variables. Of course, if you take zero polynomial that is not interesting. So, non constant polynomials, let us take one ok?

Look at all the zeros of that, the zero set $Z(p)$ of polynomial p; all $x = (x_1, x_2, \ldots, x_n)$, when you evaluate p on that: $p(x) = 0$; that is a closed set because p is continuous alright. We claim that $Z(p)$ contains no non empty open set, it is nowhere dense ok. So, this is elementary calculus. As soon as there is an open set of $\mathbb{R}^n$ (or $\mathbb{C}^n$ does not matter,) contained inside a set, you can study the polynomial on that set ok? Which is identically 0 by definition, that is it is contained in $Z(p)$.

But a zero function on an open set has all its partial derivatives 0. If you compute partial derivatives cleverly you can compute all the coefficients of this polynomial; not only in one variable in any variable, any number of variables. You have to do all the partial derivative various partial derivatives, That means what?

All the coefficients are 0; that means, p itself is a zero polynomial, but we started with a non-constant polynomial, ok. So, the zero set of any polynomial is nowhere dense. What is the consequence? The entire $\mathbb{R}^n$ or $\mathbb{C}^n$ cannot be the union of countably many zero sets of polynomials ok. So, that is the consequence by Baire's theorem.

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The zero set of any nonzero polynomial is nowhere dense. It follows that $\mathbb{K}^n$, $\mathbb{K}$ could be $\mathbb{R}$ or $\mathbb{C}$ does not matter, cannot be written as a union of countable many zero sets of non-zero polynomials ok?

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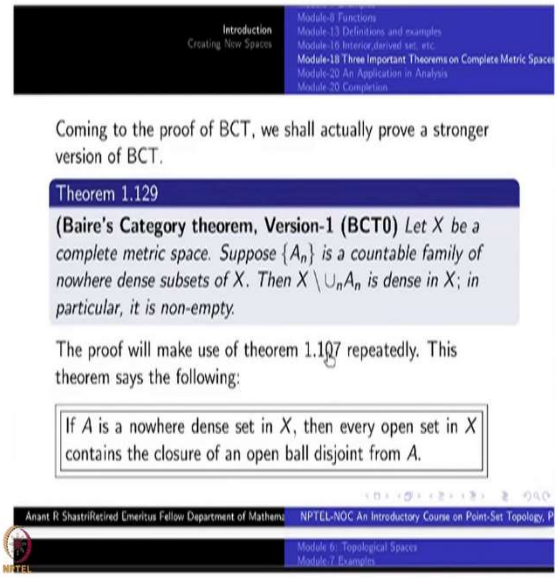

So, let us prove this theorem now. We shall actually prove a stronger version of this theorem. The stronger version I have called BCT0 because it sits over all other versions ok? Take a metric space which is complete. Take a countable family of nowhere dense subsets. The complement is actually dense that is a statement. $X \setminus \cup A_n$ is dense. In particular it is non-empty. An empty set cannot be dense because closure of an empty set is empty ok? That it is non-empty same thing as BCT1. That we have seen because if it were empty then X would have been union of A_n's that would mean that X is 1st category right?

But the statement is X is 2nd category. So, BCT0 implies BCT1 very easily. So, we are going to prove the stronger statement alright. Once again, we have done the groundwork already.

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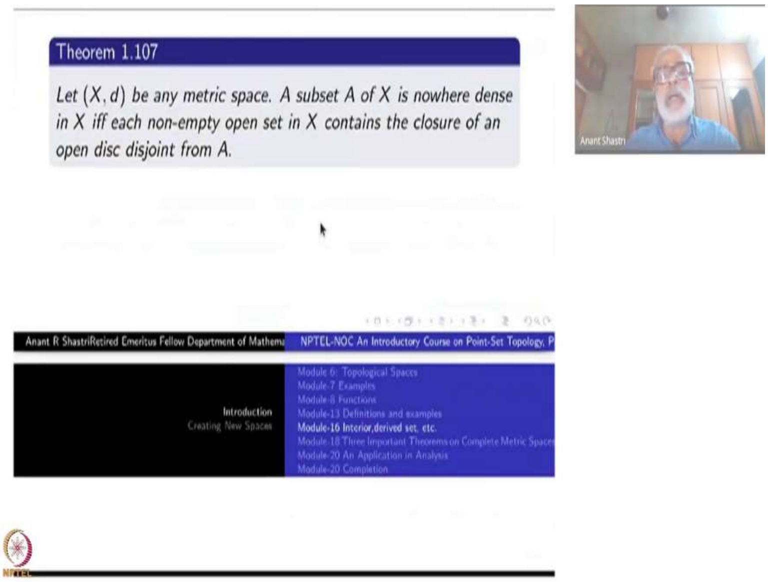

So, this is theorem 1.107. Let us take a look at this one ok? (X, d) is a metric space. A subset A of X is nowhere dense if and only if, each non empty open set in X contains the closure of an open disc disjoint from A. So, there will be some $B_r(x_0)$, closure of that intersection with A will be empty ok? And this will be contained inside any given non-empty open set. So, this theorem I am going to use again and again ok. So, I have to go back now. Yeah, we have had this statement here, but I wanted to show that what actually we have done, we might have forgotten it.

If A is a nowhere dense subset of X, then every open set in X contains the closure of an open ball disjoint from A. So, this is a statement I am going to use again and again ok?

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So, this is precisely the kind of thing that we are going to be. This is dot, dot, dot, dot, dot is the nowhere dense set. This single line, thin line indicates an open set. Inside that, I can find a ball of some positive radius such that the closure is disjoint from all these points dot, dot, dots ok? A nowhere dense set compared with an open set.

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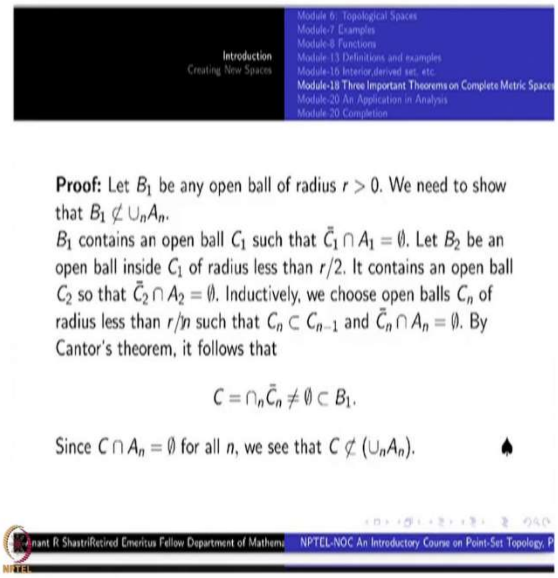

Let B_1 be any open ball of positive radius you start with. Instead of any open set, if I prove this one for B_1 which is an open ball then it will follow for every open set, I will produce something inside B_1. So, starting with an open set, I can start with B_1 instead right? That is why that B_1, you know, an open ball of radius r.

We need to show that B_1 is not contained in the union of A_n's. That is enough, ok? This B_1 is not contained inside union of A_n that is all I want to show. First of all B_1 contains an open ball C_1 such that the $\bar{C_1} \cap A_1 = \emptyset$, because A_1 is nowhere dense.

So, this is the first time I am applying this theorem 1.107, ok. Now, I have got C_1. Let B_2 be an open ball inside C_1 of radius less than $r/2$, ok? I am making sure that the radii are going down, down, down to 0 by putting $1/2$ here ok. B_2 is contained inside C_1; This C_1 is a ball, but I want a ball of radius smaller than $r/2, r/3$ whatever ok? It contains again, now apply the theorem again, it contains an open ball C_2, such that $\bar{C_2} \cap A_2 = \emptyset$.

Now, you know the game. Inductively, suppose you have chosen C_n of radius less than r/n such that C_n is contained C_{n-1} and $\bar{C_n} \cap A_n = \emptyset$. Once you have that, inside the C_n, you will get another one and so on. So, you keep taking them.

Now, we apply Cantor's intersection theorem. To what? To these C_n's, $C = \cap \bar{C_n}$. So, these are all closed subsets, their diameters are less than $2r/n$, right? So, as n tends to infinity, they to go to 0.

So, therefore, this intersection is actually a single point. It is non empty is all that I require. ok? Since, I want to apply Cantor's theorem I have put $r, r/2, r/3, \ldots, r/n$ so on. So, this will go to 0. Therefore, C the intersection is a singleton. But I want only non-empty that is ok. They are all contained in B_1, ok. So, this singleton is inside B_1, but what is this point? This point is in C and $C \cap A_n$ is empty for all n, ok?

Why? C_n it is contained in $\bar{C_n}, \bar{C_n} \cap A_n = \emptyset$. So, $C \cap A_n = \emptyset$ for all n right. So, this point is in none of the A_n's. So, this means B_1 is not contained in the union of A_n's.

See if you wanted to prove that something is dense, what you have to do? Take any non empty open set, it should intersect that set. Here, we wanted to show that $(\cup A_n)^c$ is dense. So the open set should not be contained inside the $\cup A_n$. So, that is what I have proved.

Take any open set B_1, ok, it is not contained inside union means what? The compliment intersects B_1. Therefore, what we have proved is that the compliment of $\cup A_n$ is is dense. Be sure that we have non-emptyness is not just what we have proved. We have actually proved that $X \setminus \cup_n A_n$ is actually dense. So, that is the proof of this ok?

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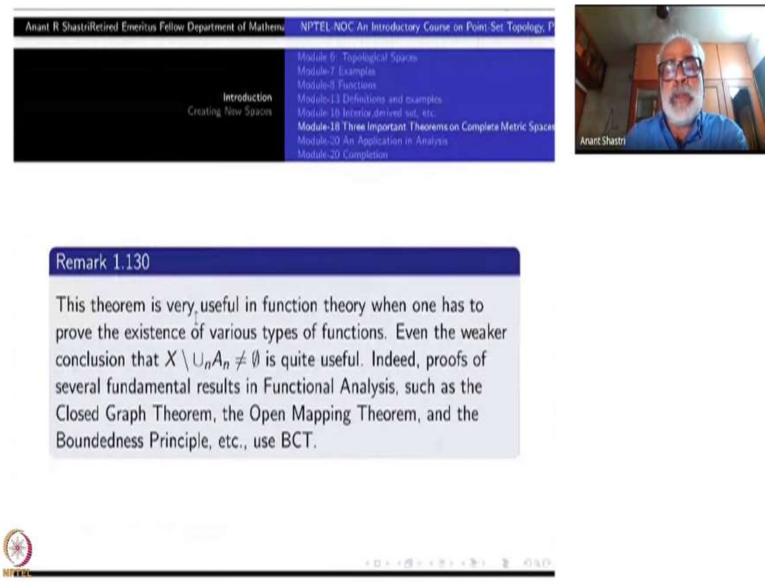

So, let me make a few remarks here. This theorem is very useful in function theory when one has to prove the existence of various types of functions. So, this theorem says something is non-empty; that is how it is used ok? Some how you get a complete metric space, you cook up a complete metric space then you cook up a sequence of closed subsets which are nowhere dense in it. They will not cover the whole space means there is something left out.

So, that is an existence theorem. So, that is the way it is used in many existence theorems ok? Indeed proofs of several fundamental results in functional analysis use this theorem. I will quote some of them which are very very fundamental namely, closed graph theorem, open mapping theorem and boundedness principle and so on and so forth.

So, all these things come in elementary functional analysis itself. The first course in function analysis seems you will have all these theorems ok. They are all using Baire's category theorem to prove, alright.

So, all these things come in elementary functional analysis itself. In the first course in function analysis you will have all these theorems ok. They are all using Baire's category theorem in the proof alright.

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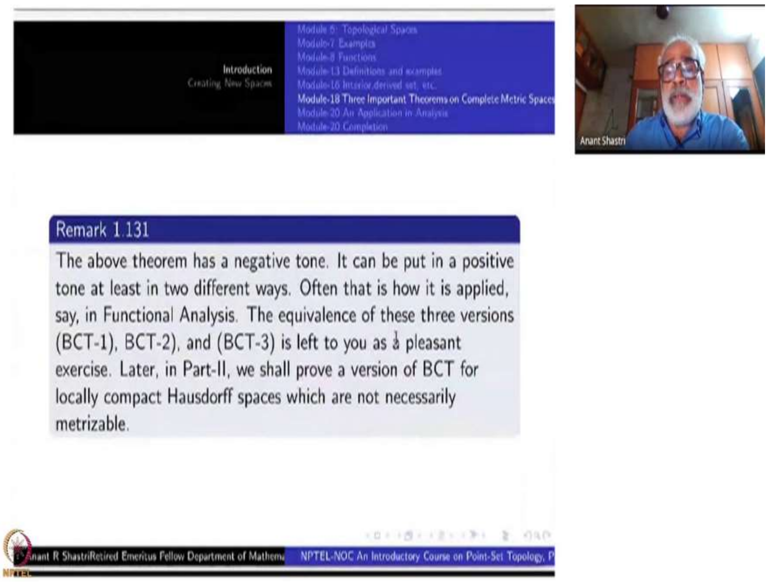

So, I will repeat this one. The above theorem has a negative tone; that means, 2nd category itself, definition is it is not 1st category. The 1st category is what it cannot be something. So, there are too many negations there.

But it can be put in slightly a positive tone as follows. So, I have given you those tones. And then often it is how this positive versions are here. So, let us have those versions; Later on, in part 2 we shall prove a version of this Baire's category theorem for locally compact Hausdorff spaces which are nothing to do with metrizability, there is no metrics ok.

So, here are those versions.

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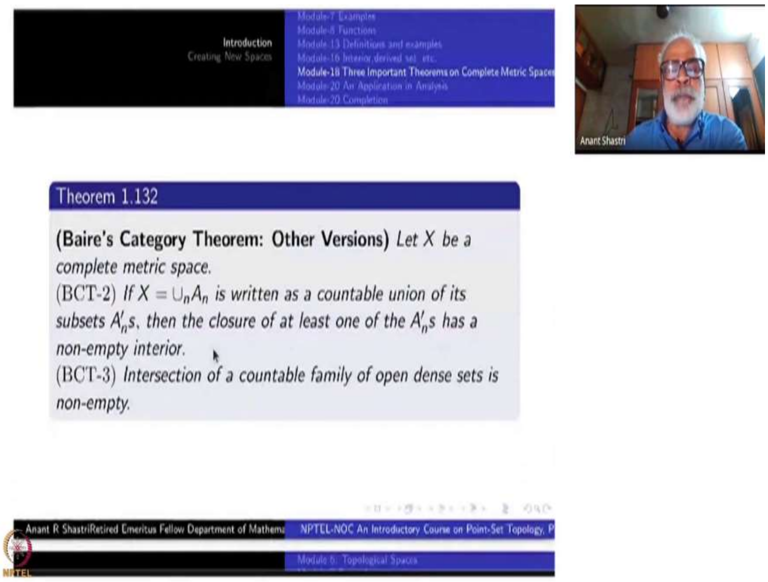

Let X be a complete metric space that is that is standard assumption there is no other. So, BCT2 says that suppose X is $\cup_n A_n$ written as a countable union A_n's, then at least one of A_n 's has its closure with non empty interior. See the hypothesis on A_n's is deleted, space is just a countable union.

But, then you conclude closure of one of them has empty interior, A_n is nowhere dense for some n, ok. So, it is just the other way around you. So, here there is no negation here. If you write like this closure of one of them is empty interior. It is a positive tone.

Another one is: intersection of a countable family of open dense sets is non-empty. So, this is the way it will be used. So, there is one element they want. So, that is that is the existence theorem. So, the proof of that 1 implies, 2 implies, 3 implies 1 they is they are equivalent ok? is very easy for you. But 0 is a stronger statement which will imply all of them ok.

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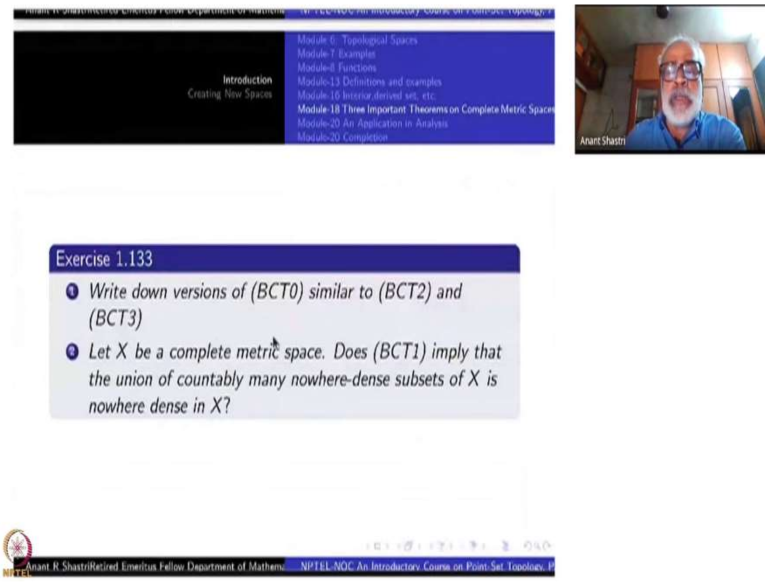

So, here is an exercise, namely, write down versions of BCT0 also similar to BCT2 and BCT3 ok. Taken a complete metric space, does BCT1 imply that union of countably many nowhere dense subsets of X is nowhere dense? Pay attention to the statement there it does not say this, union of countably many nowhere dense sets does not fill up the whole space; this is the weaker version.

Complement is actually dense is the stronger version, but here it is said that the union itself is nowhere-dense ok. So, you have to see whether this is true ok? It is not stated does not mean that it is not implied. So, I am asking whether this is implied by the statement. Think about them. So, that is that is the exercise you have to think about it that is all. So, let us stop here.

Thank you.