Galois' Theory
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Lecture No 12
Kernel of homomorphisms and ideals in K[X],Z

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Alright, so in few minutes back we saw ring homomorphisms, algebra homomorphisms, examples and so on. Now let us continue little bit more. In

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group theory one studies now how do you get an isomorphism from a given group homomorphism?

How do you test a given group homomorphism is injective or surjective, and such corresponding things for the ring homomorphisms or K-algebra homomorphisms I am going to state only and the proofs I will leave it to the participants.

They are exactly along the same lines as one do in a Group Theory. So for example, so when do we say, that how do you test that given ring homomorphism R to S, ring homomorphism ϕ injective? So ϕ is injective if and only if the kernel ϕ which is by definition, all those elements of the ring are, which are mapped to 0, that is $\phi(x)=0$.

So this is a subset of R and it is not just a subset, it has some more properties. I will come in a minute, so ϕ

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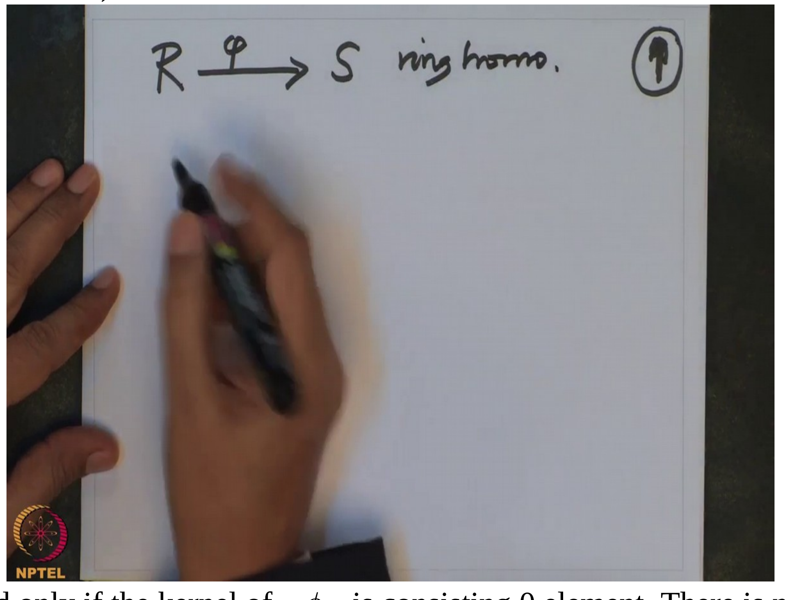

is injective if and only if the kernel of ϕ is consisting 0 element. There is no

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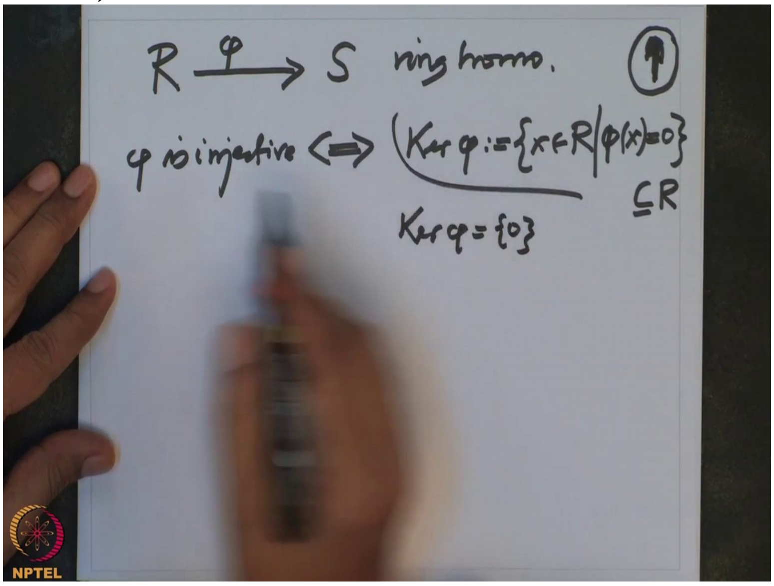

non-zero element of R goes to 0 under ϕ, then ϕ is injective.

Now what are the more properties, what are the more properties of this kernel? That it is an ideal, kernel ϕ is an ideal in R. So now I have used this word, new word, ideal in R, so that simply means that is first of all, under addition it is a subgroup. So kernel ϕ under addition is a subgroup of R, (R ,+) that is one

And another property is

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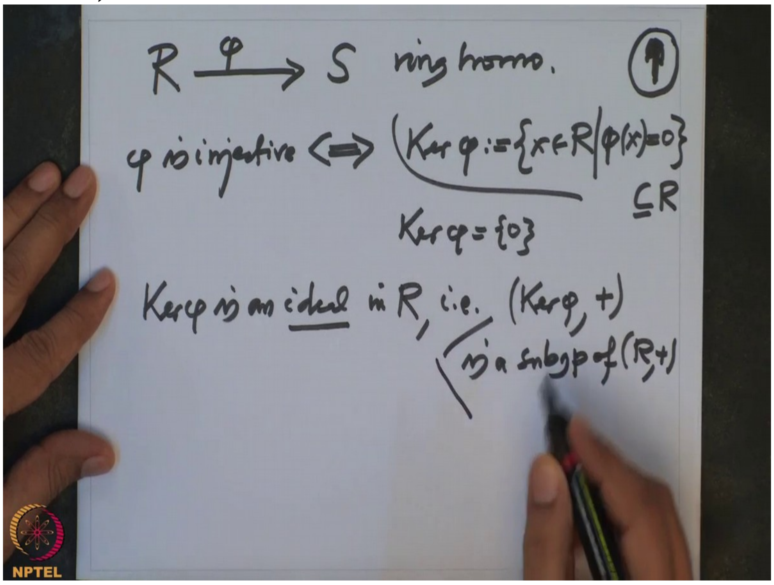

if I have arbitrary element a in the ring R and x in the kernel, then $a\,x$ is also in the kernel.

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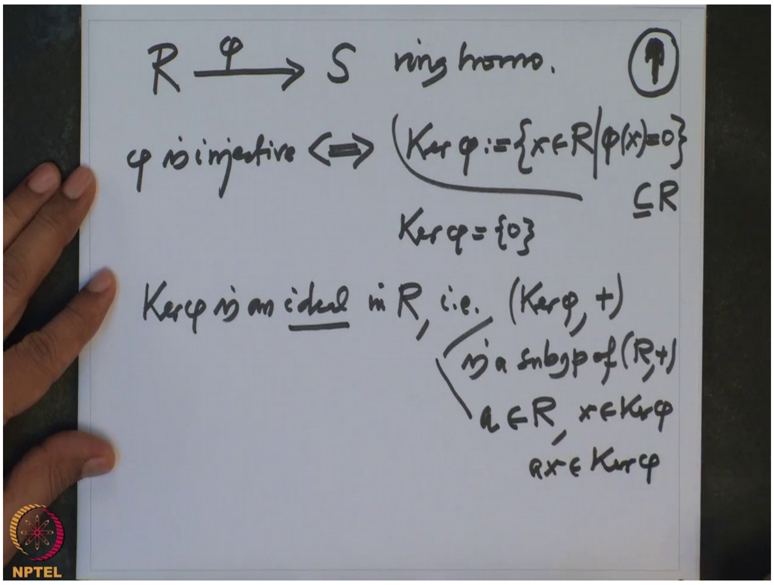

And because we have in a commutative ring, we are not going to be bothered about left ideal, right ideal and so on. Similarly here only it is enough that we multiply on the left. So it is better that we assume now rings are commutative always. Ok that is an ideal.

I will soon give examples of ideals in some rings. And like injectivity how do you test the surjectivity? It is easy. That as usual, for map between the two sets we denote image of ϕ to be all those elements $\phi(x)$ as x varies in R. Take all the elements of R and take all their images. This is the subset of S which is called the image of S.

And note that this is a sub-ring of

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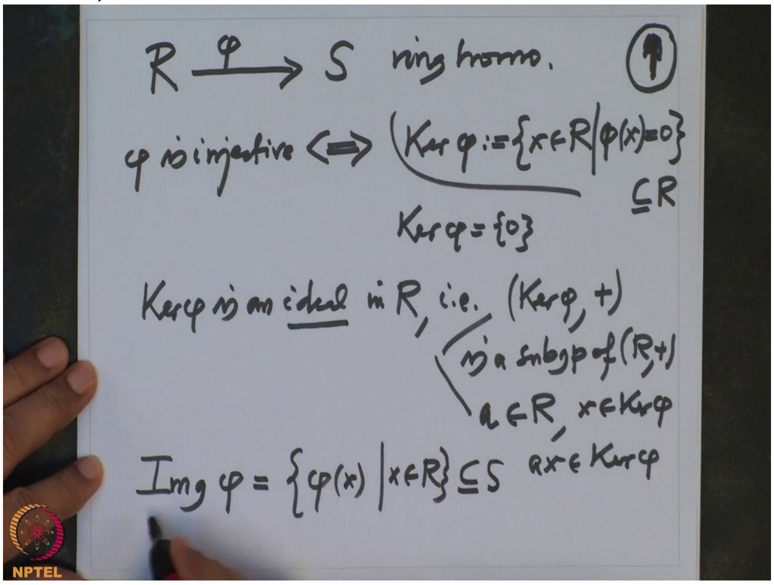

S. So let me remind you sub-ring means

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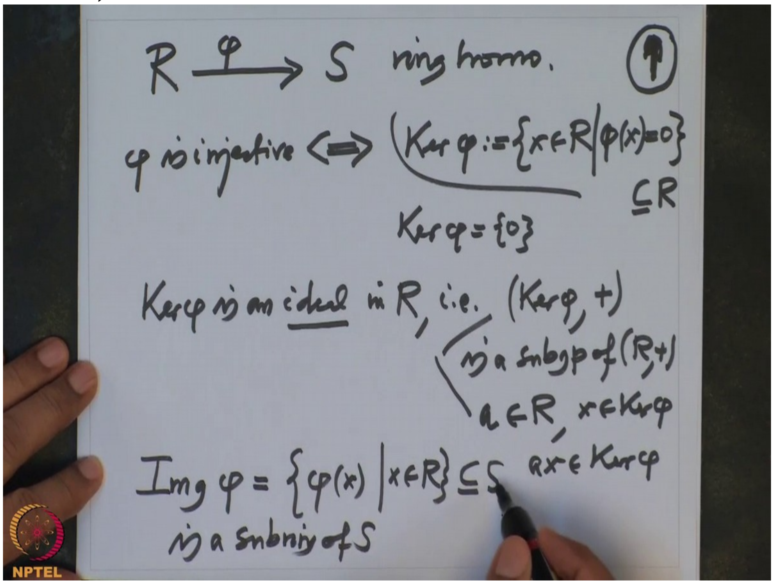

with respect to the same binary operations, plus and multiplication of S, this subset induces, induced on this subset is also a ring. And in particular that 1 also belongs here. So for that, let me remind you because our assumption is very important. 1_R goes to 1_S .

So that means

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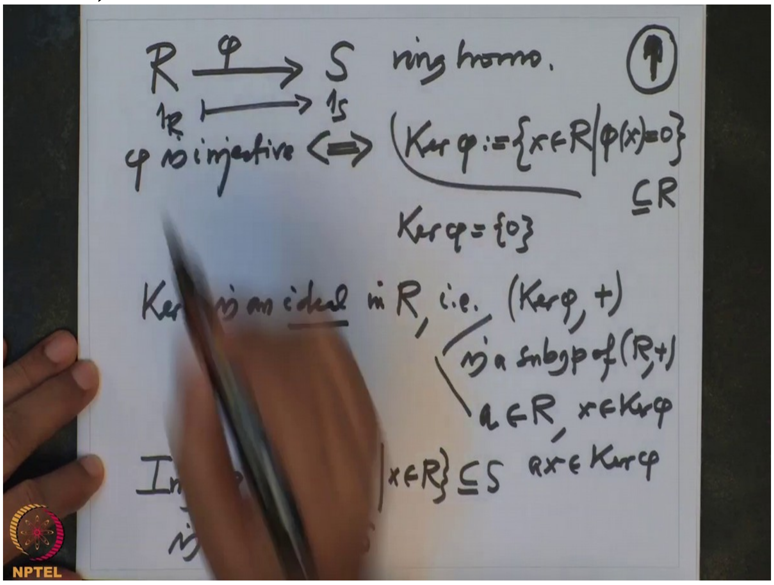

1_S is in the image. So 1_S belongs here and the remaining properties are just followed from the fact that it is a ring homomorphism,

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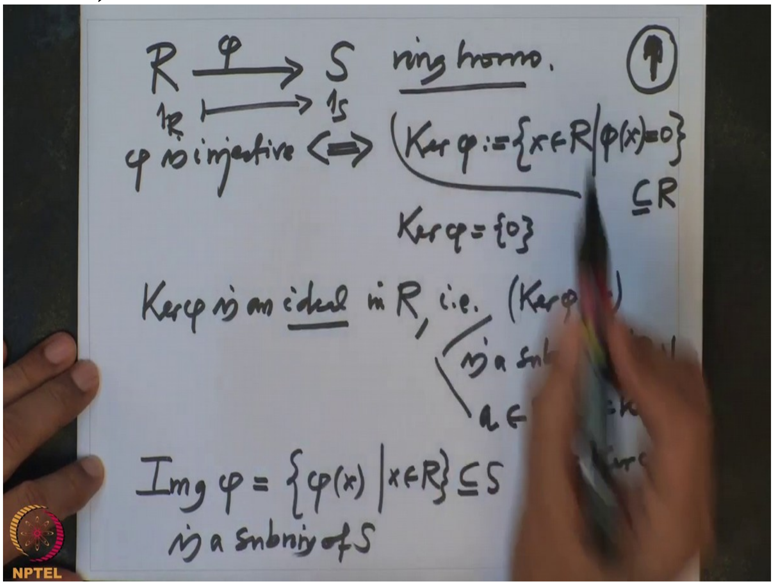

that is all. So if you, if you want to, and what is the isomorphism of the ring? Isomorphism of the rings, of ring that means

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ring homomorphism which is bijective.

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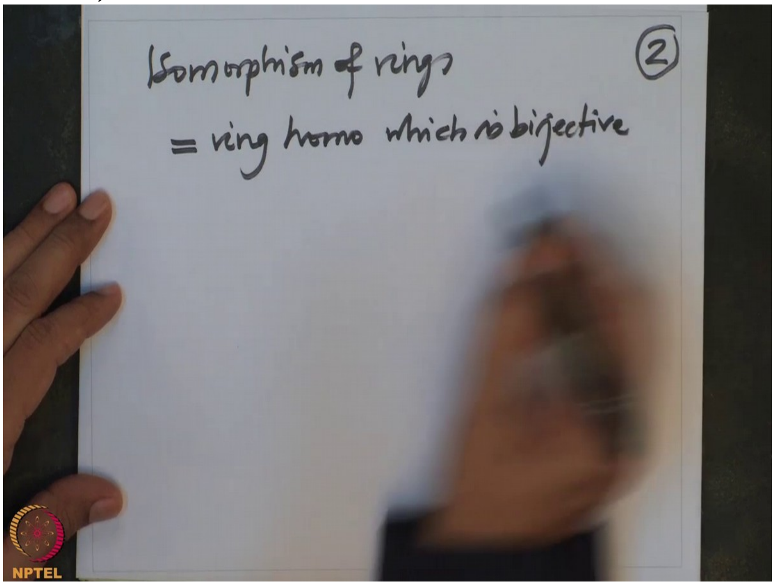

This is equivalent to saying that ϕ ; so isomorphism ϕ from R to S is a

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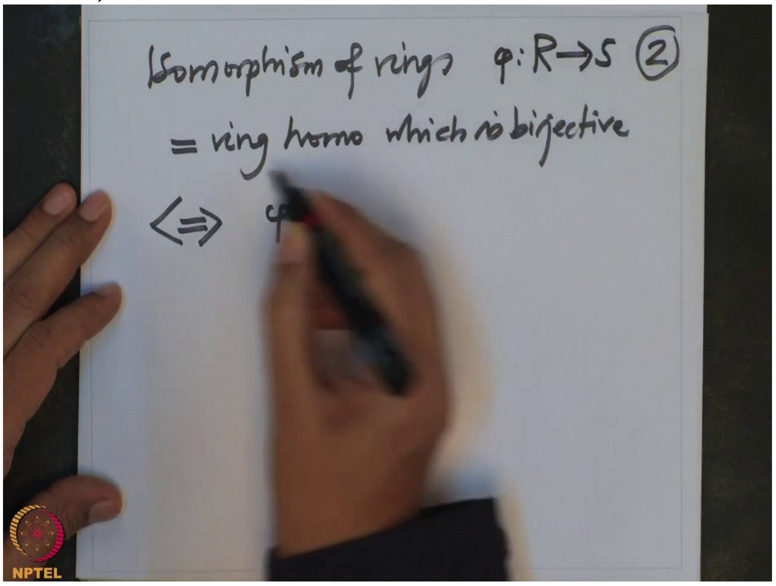

ring homomorphism which is bijective.

That means ϕ and ϕ inverse both are ring homomorphisms.

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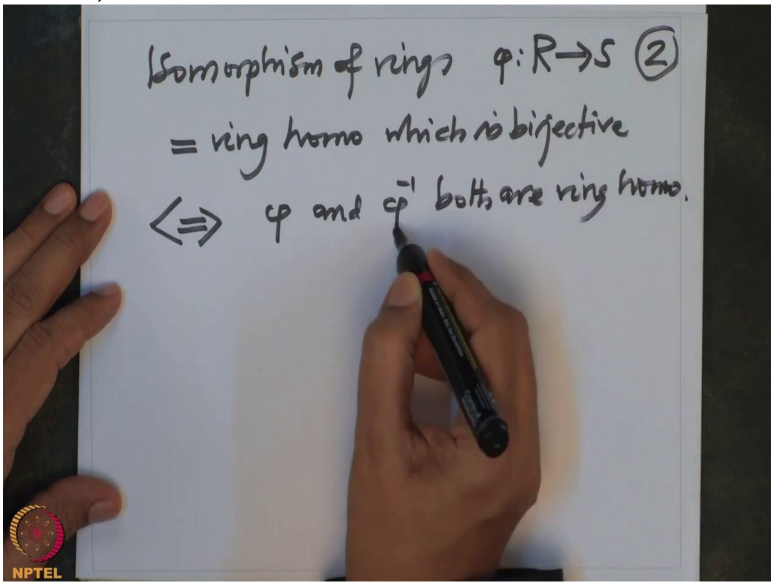

Remember because of this bijectivity ϕ inverse makes sense and both are ring homomorphisms. Then one says it is an isomorphism. And

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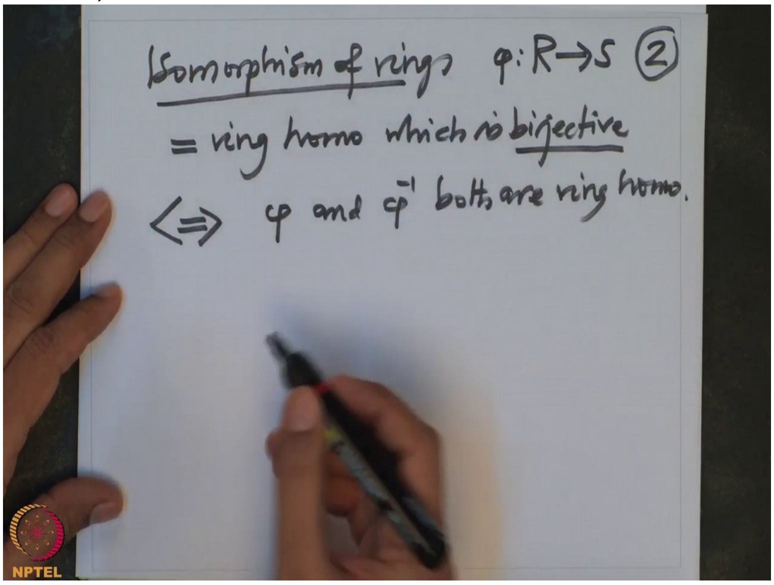

this is usually denoted by R to S ϕ and to show that it is bijective, it is

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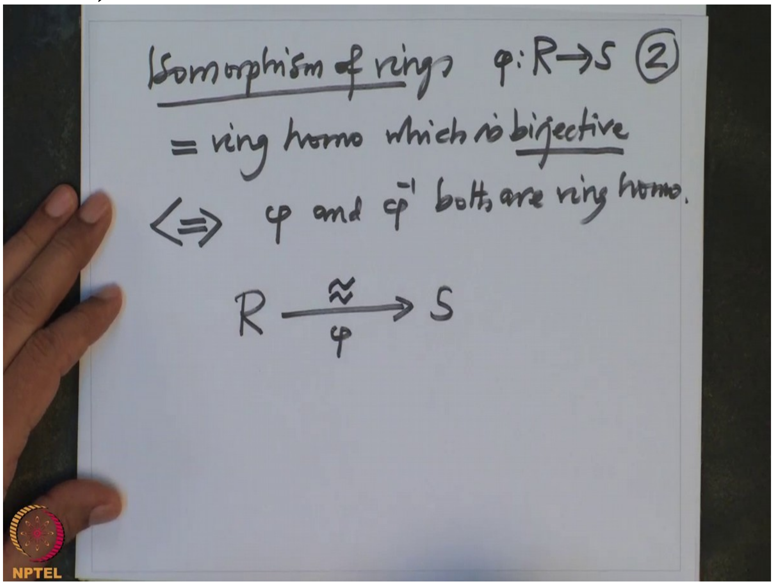

denoted like this. So this is a typical notation one will use for the isomorphism of the rings. And

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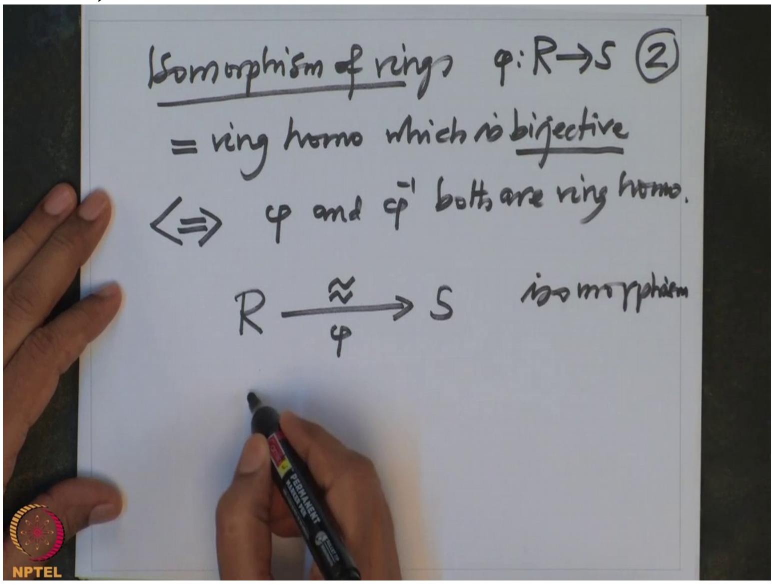

for injectivity normally one writes R hook arrow S, so this means injectivity

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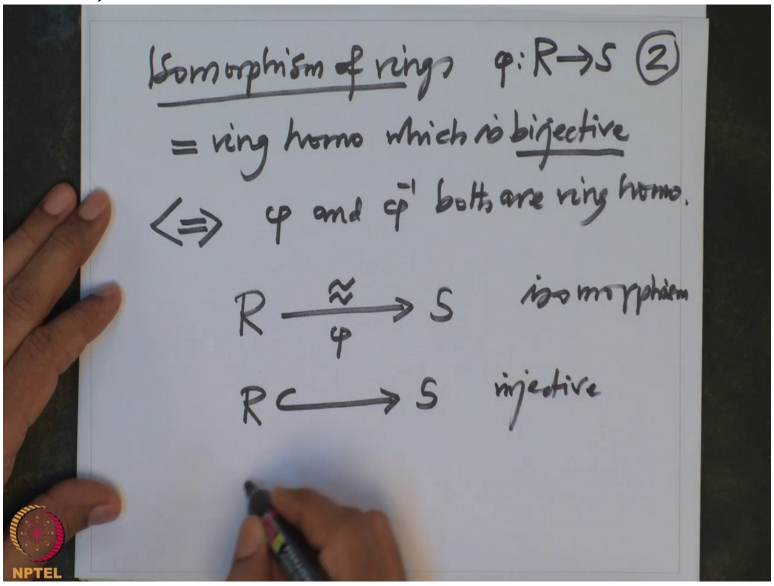

and R to S surjectivity is denoted by putting 2 arrows. This is surjectivity. This is the standard notation we will adopt

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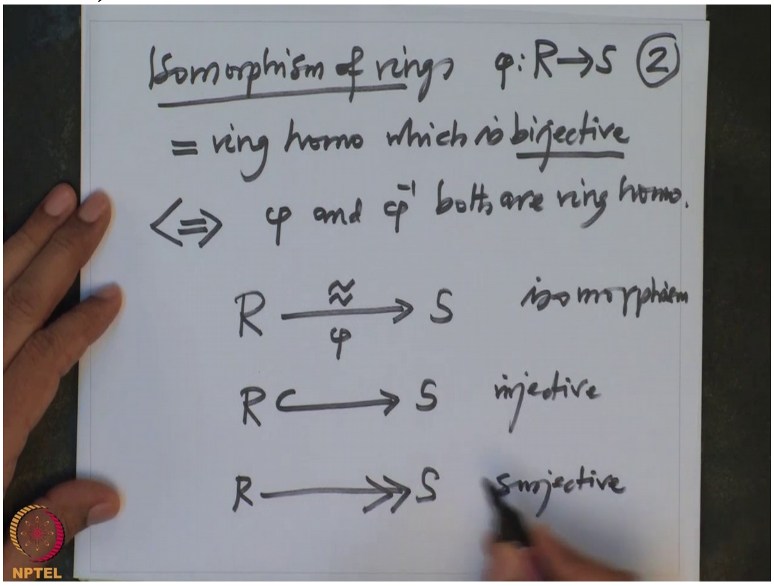

in this course. So now let us see some examples of ring ideal isomorphisms and so on.

So,

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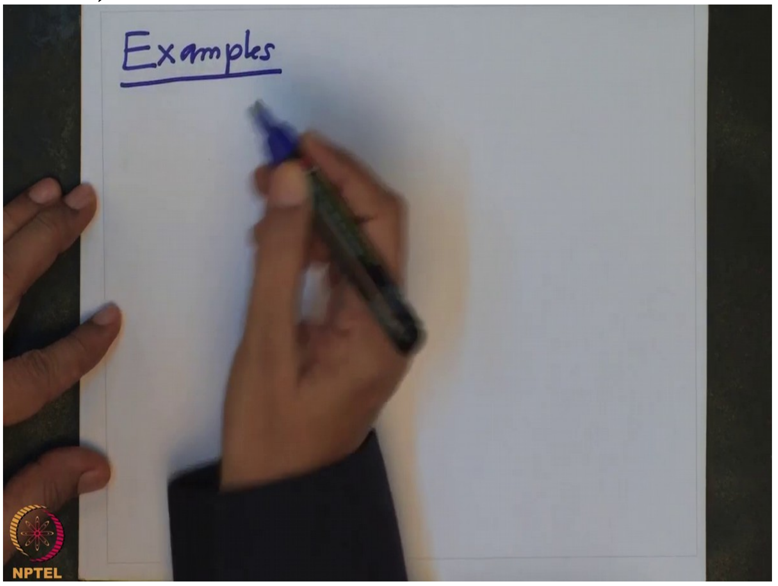

so first of all I want to give some examples where you can describe the ideals. So first, one can describe all ideals in $\mathbb{Z}$.

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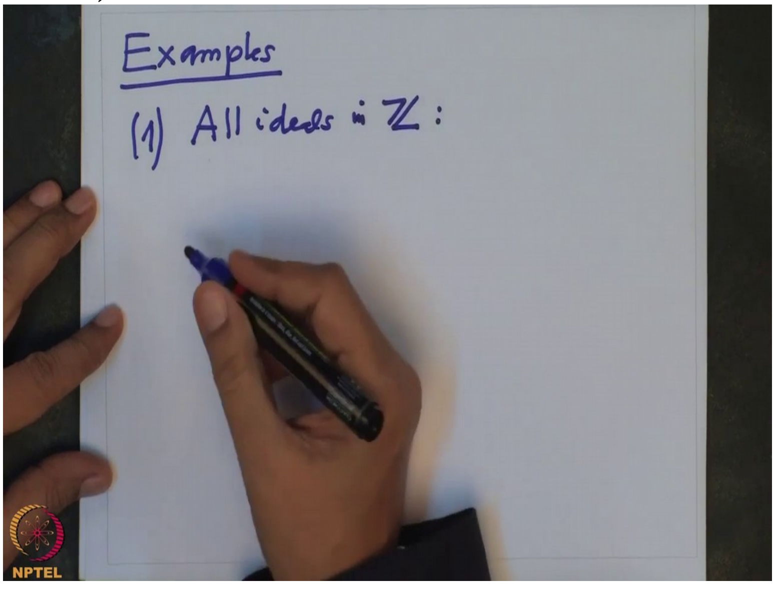

So how do you do that? So take any, the ideals

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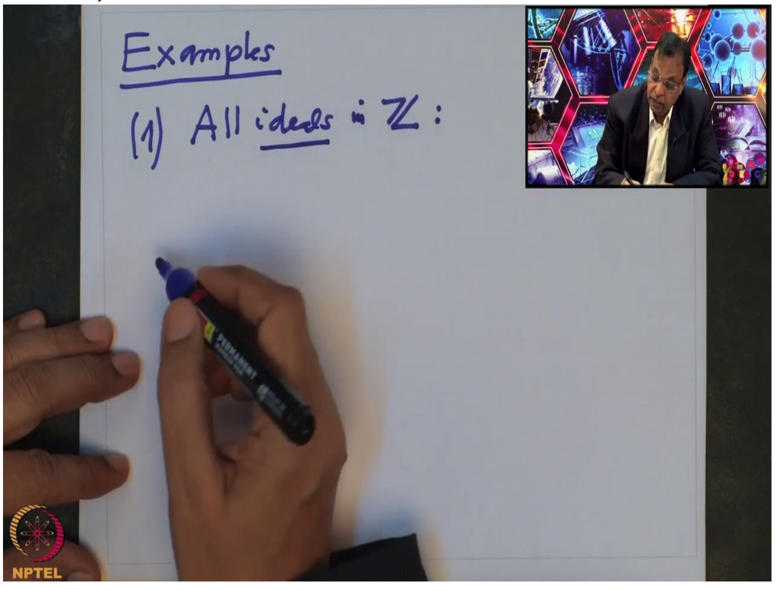

they are usually denoted by the Gothic letters A, B, C etc.

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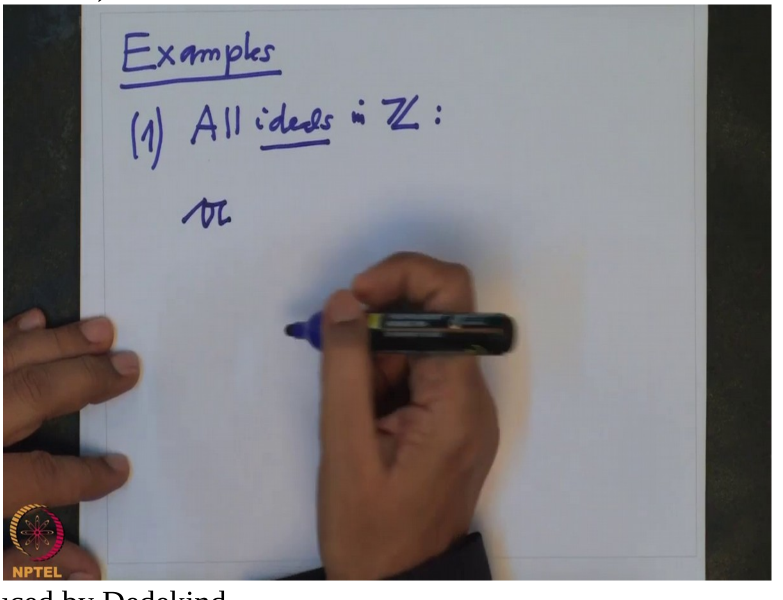

These are introduced by Dedekind.

Dedekind was a German mathematician and therefore he used German letters which are, which is, which are Gothic letters to denote ideals in the rings.

So suppose A is an ideal in $\mathbb{Z}$.

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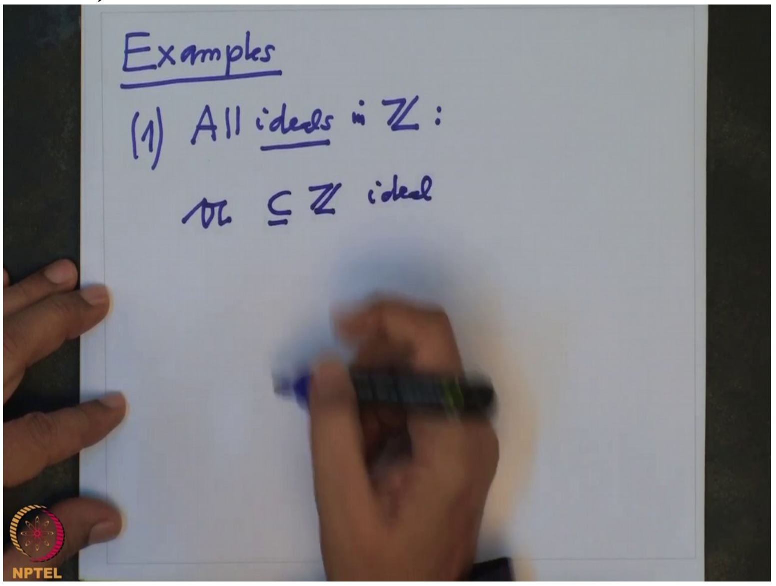

Then I want to show that it is generated by a single element. What does that mean? First of all, before I started this, I should have given examples in arbitrary ring. So let us take R be any ring and x be any element in R.

Then if I take all multiples of this x, R multiples of x that obviously I would denote by R times x. This is by definition, all ax where a is varying in R.

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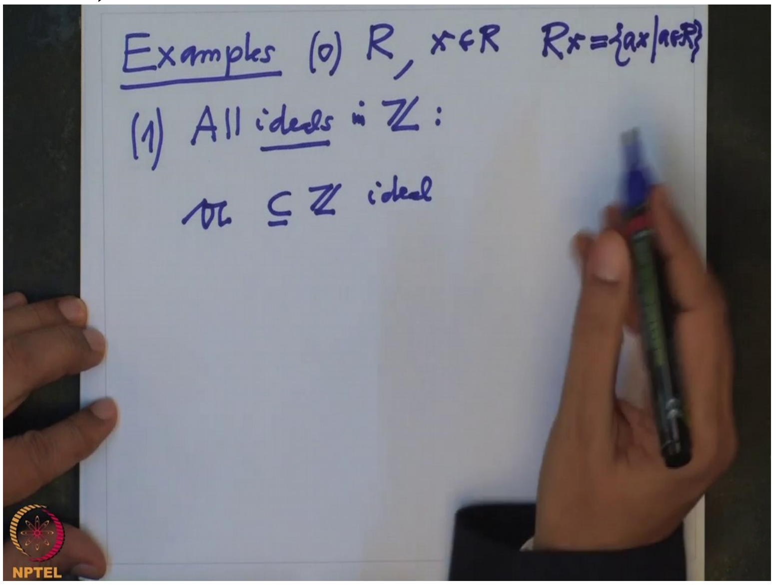

And I want to check that this is an ideal in R.

And

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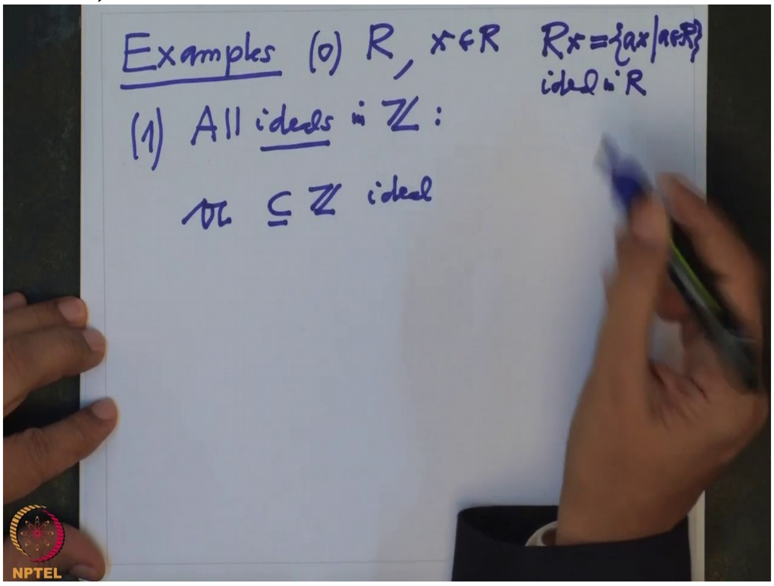

how do I check that? So I have to check that it is an subgroup under addition, which is clear. Because if I take 2 elements like this, $ax+bx$, then because of our rules this is nothing but a plus b when I add first and then multiply.

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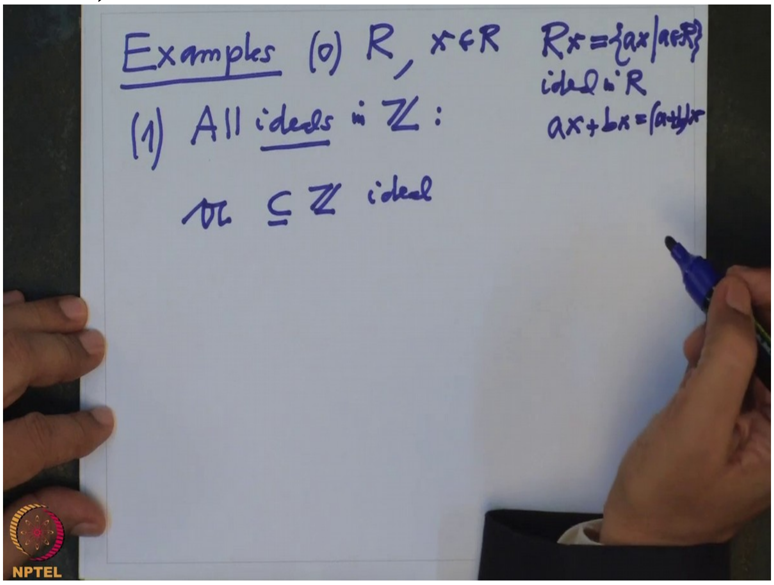

So therefore it is again a multiple of x. So it is closed under addition.

Similarly it is closed under sub, negation because minus of a x is same as minus a into x.

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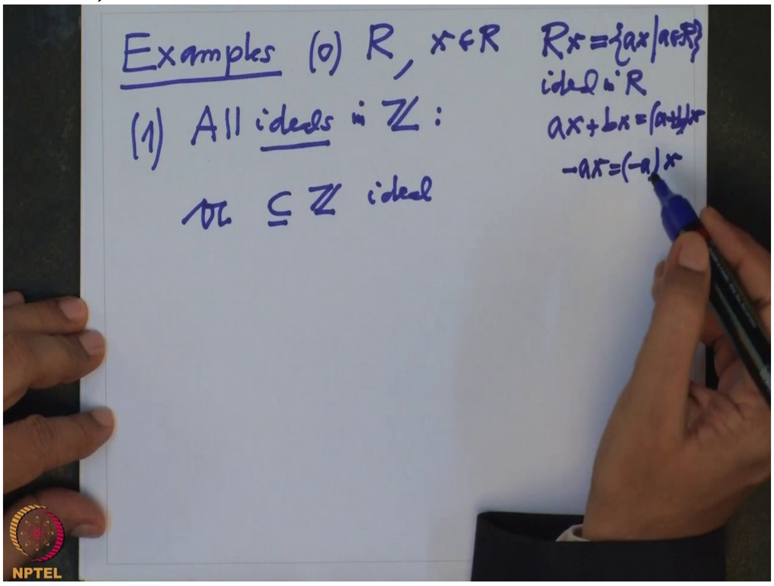

So it is a subgroup. And it is clearly closed under scalar multiplication of R because if I take c times a x it is same c a times x. So what we have checked is this is an ideal in R. This ideal is called principal ideal generated by x,

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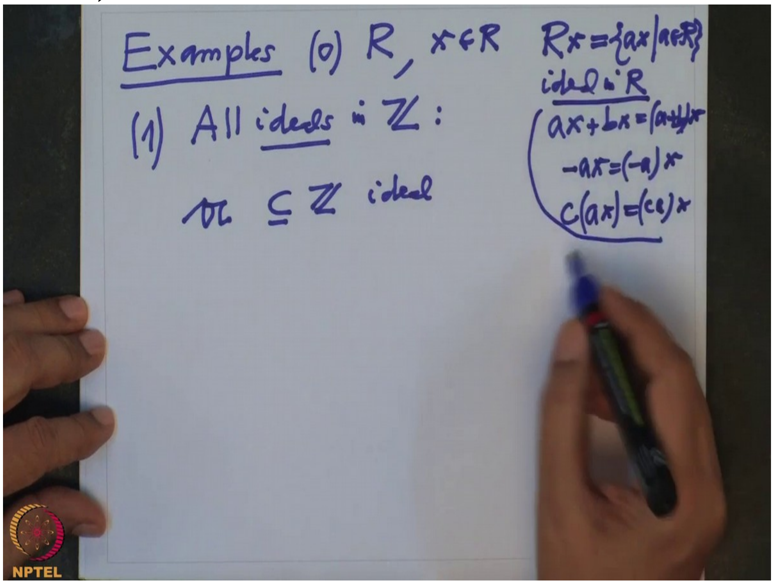

principal ideal generated by x.

And note that this x

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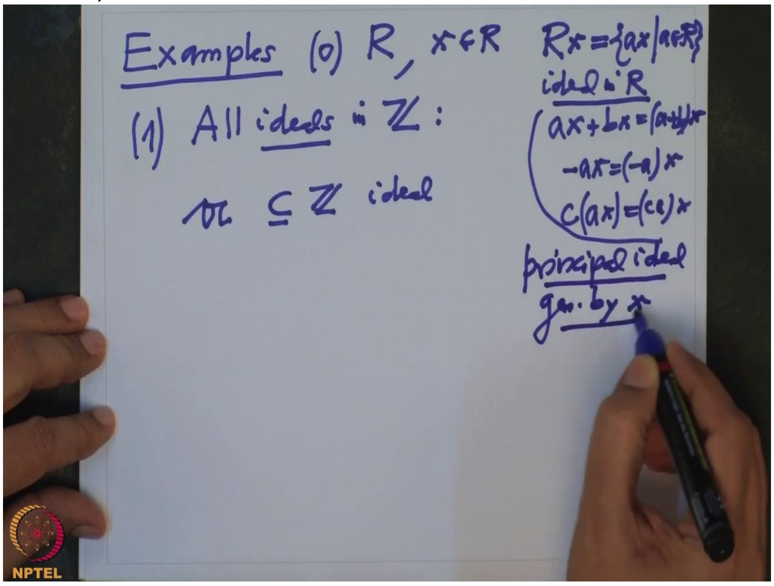

may not, this x may not be uniquely determined

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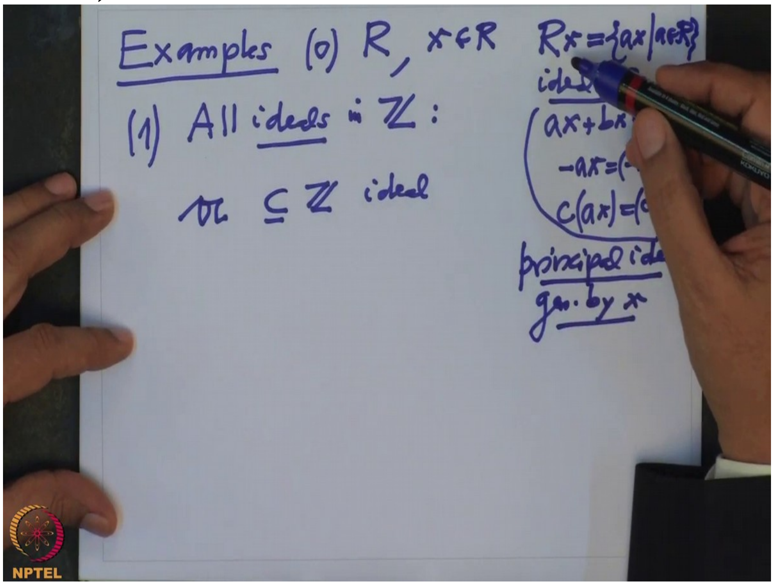

by this ideal, if you give the ideal then you cannot, you need not get back x. For example minus x is also generating in the same ideal. So Rx is same as $R(-x)$. Remember, this means all R multiples of $-x$,

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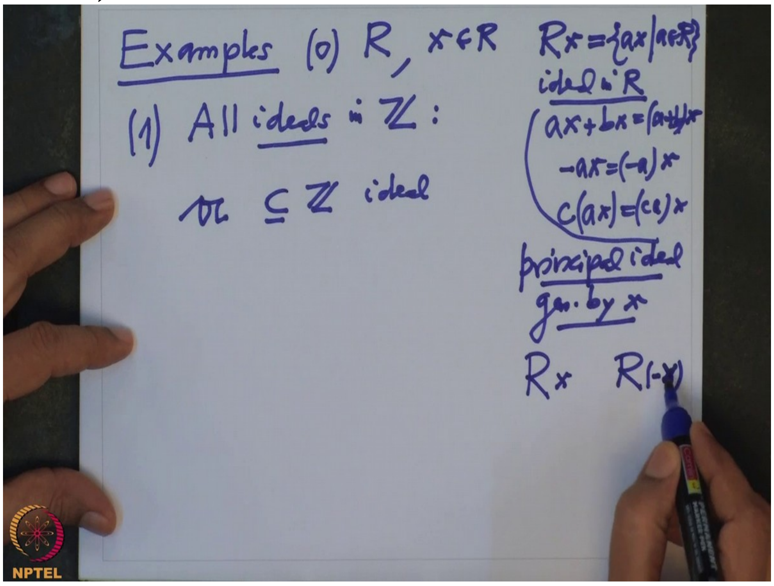

minus x is a additive inverse of x. So these two ideals are same.

So therefore

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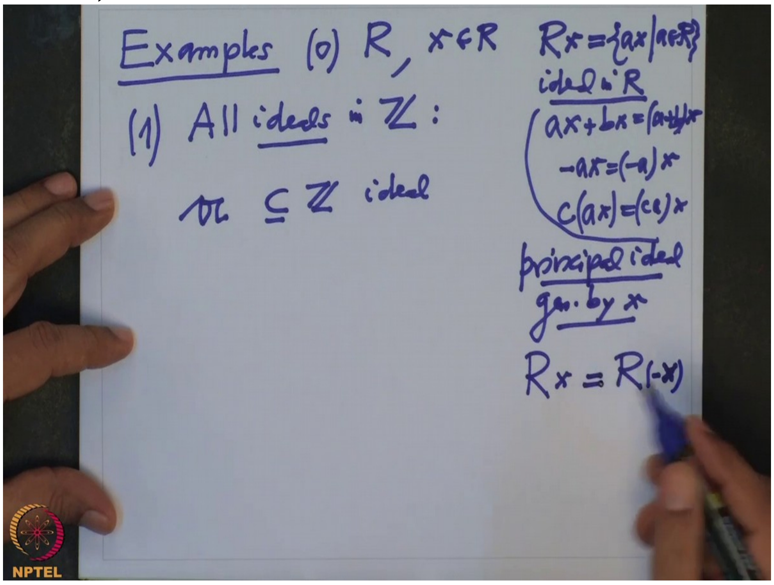

in particular, not only this, in particular also, this is also same as R times, R multiples of any ux, where u is a unit in R. Remember we had notations for units in R.

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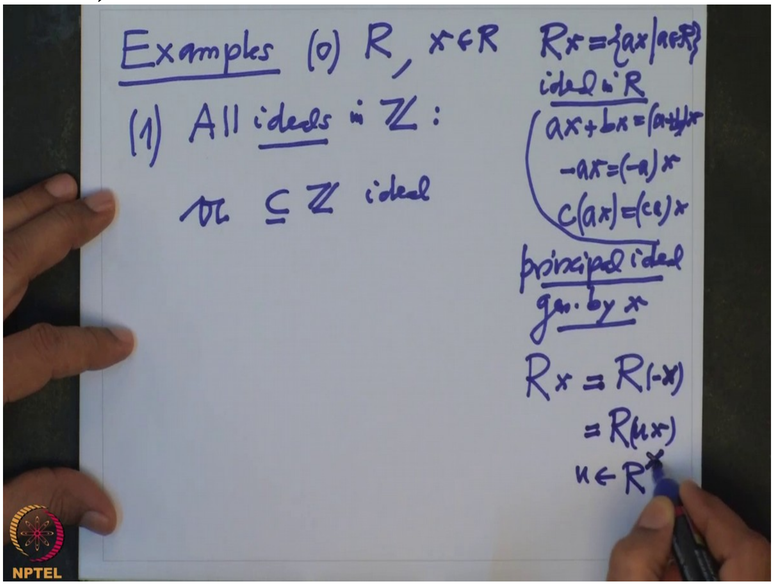

These are set of al invertible elements with respect to multiplication.

So these 2 ideals are, so u and, x and u x generate the same ideal. So you cannot

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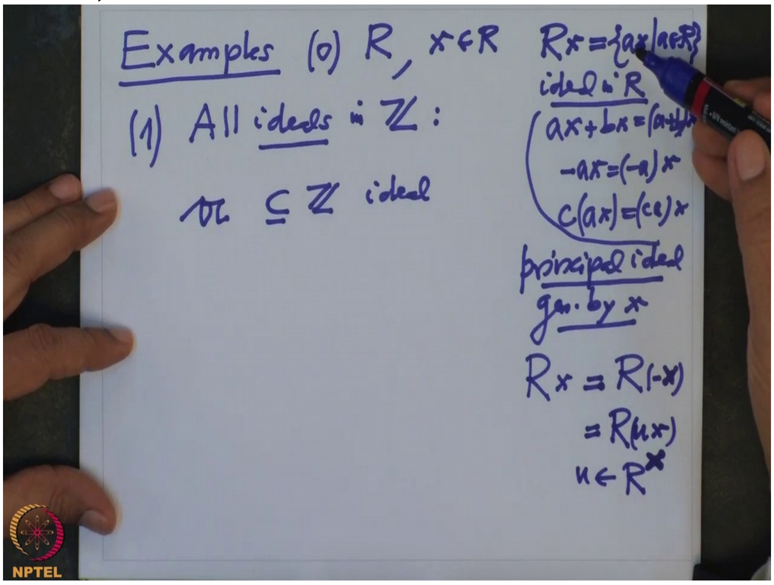

recover a generator from an ideal. But if it is generated by single element then one calls it a principal ideal generated by x. And the first statement I wanted to make that all ideals in $\mathbb{Z}$ are principal.

So we have a good knowledge of ideals in the ring $\mathbb{Z}$.

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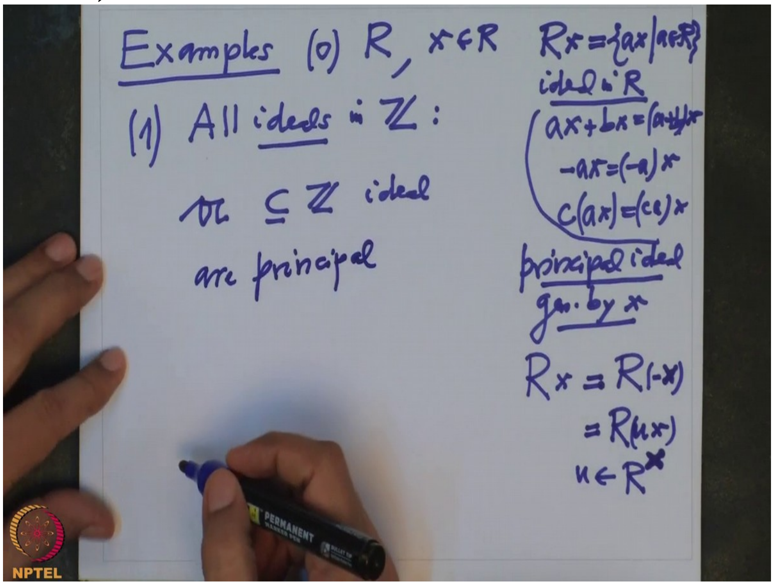

And how do you check this? The important tool to check this is what we have mentioned earlier the division with remainder. So to prove it is principal, we may assume A is 0, non-zero because A is a 0, 0 is always

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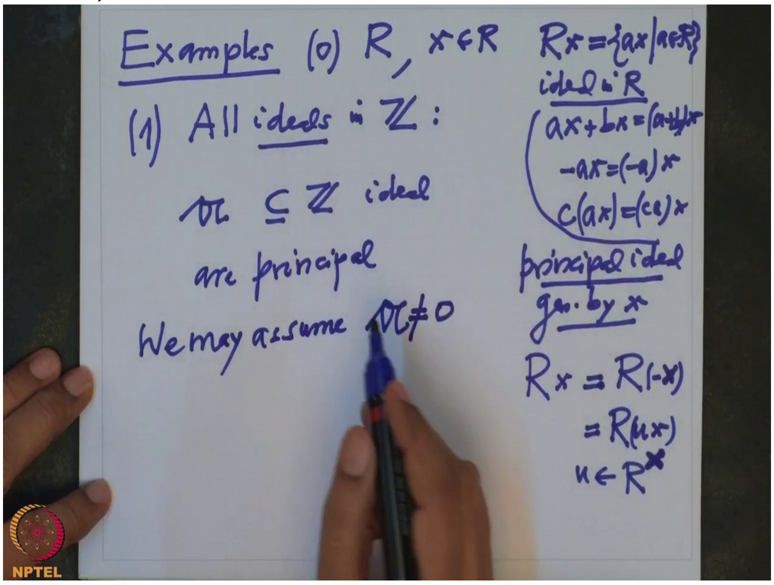

an ideal in a ring, and it is clearly generated by 0. So it is principal.

So similarly I can also assume A is not R because R is also

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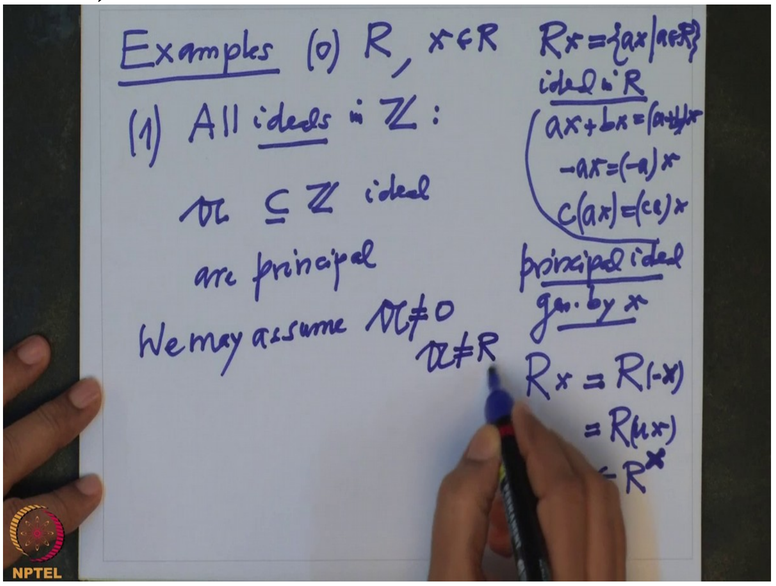

an ideal, R is generated by the element 1. So these are called

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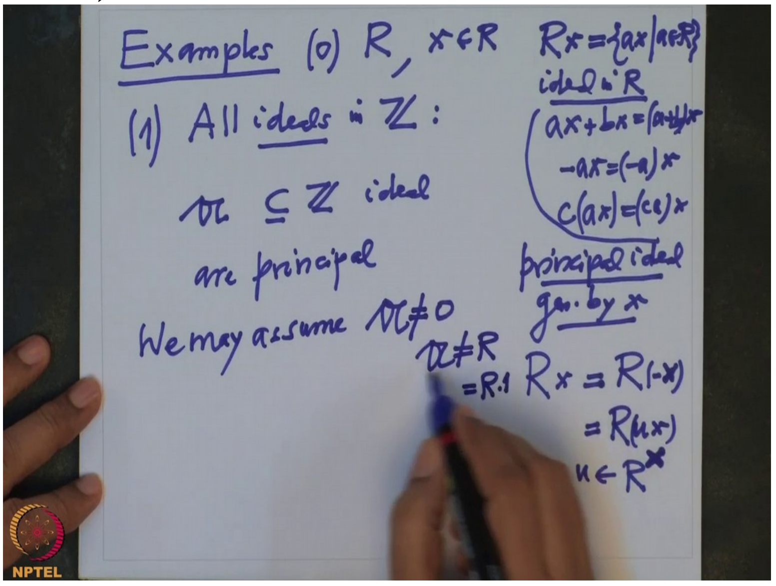

trivial ideals. Trivial ideals is 0 and the whole ring, they are obviously ideal in any ring. They are called as trivial ideals.

R is usually called a unit ideal and 0 is called a 0 ideal. So to prove it is principal, I will assume A is non-zero and A is also R. If anyone of the case happens then there is nothing to prove. Now we have a proper subset of $\mathbb{Z}$

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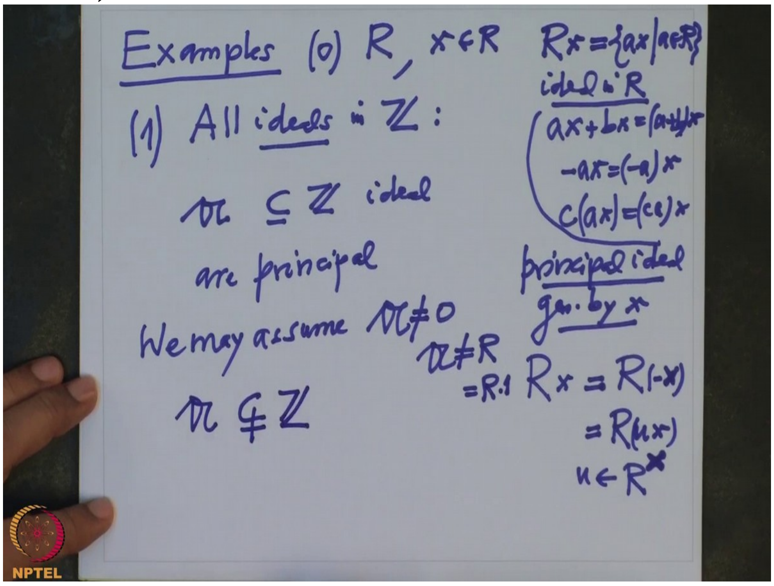

and I look now, I take the minimum element. So let us take $A \cap \mathbb{N}$ then this is non-empty.

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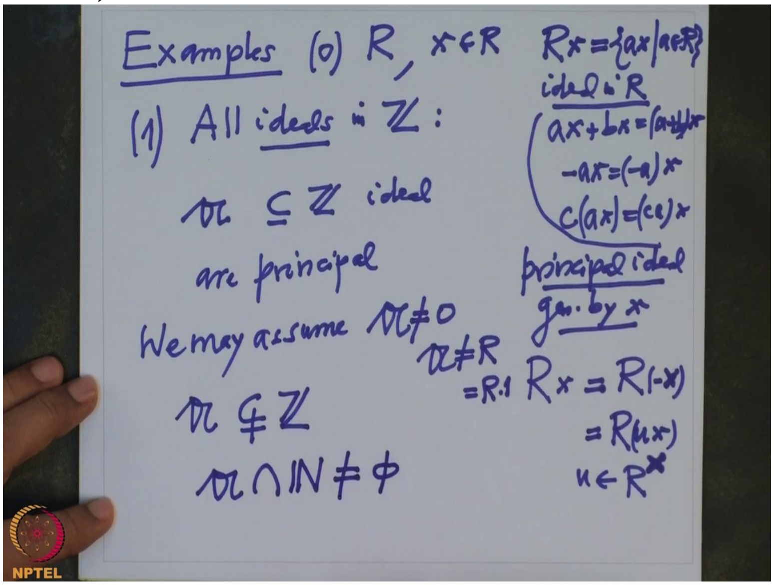

Because I know that A has at least 1 non-zero element, if it is positive already that element is in $\mathbb{N}$ and A, therefore it is non-empty. If that element is negative, then I take negative of that element which is also in A because A is an ideal and therefore in any case, $A \cap \mathbb{N}$ is empty.

And now I want to use the basic property of natural numbers which says that every non-empty subset of natural numbers has the minimum.

This is called well-ordering property of $\mathbb{N}$, of $\mathbb{N}$ that says that every non-empty

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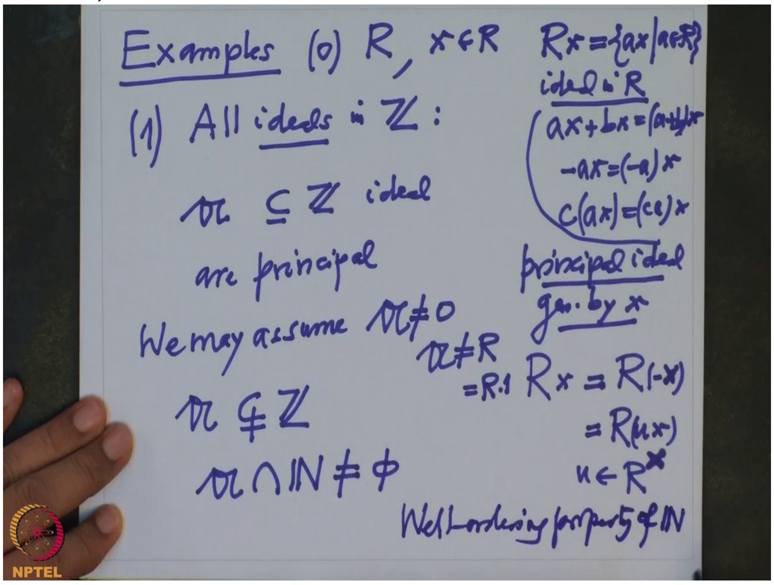

subset of $\mathbb{N}$ has a minimum. So I apply to this subset, so choose $n \in A \cap \mathbb{N}$ which is the minimum, minimum of $A \cap \mathbb{N}$.

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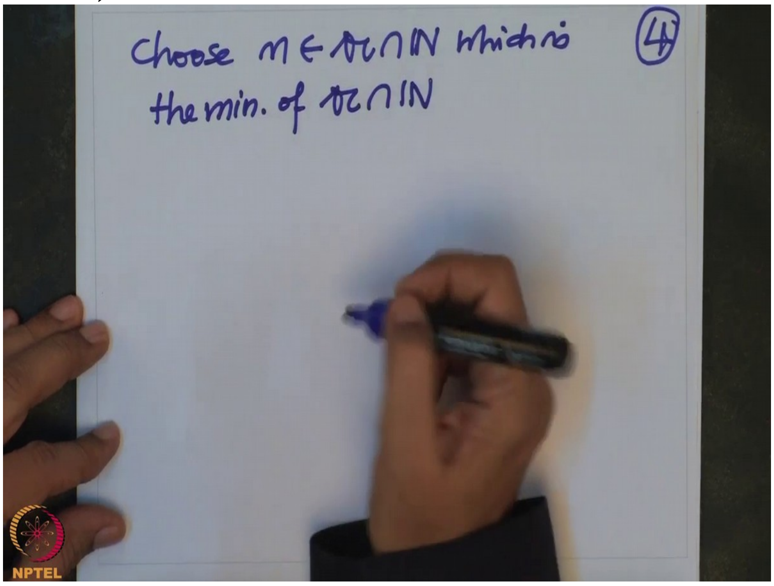

And claim is A is nothing but all $\mathbb{Z}$ multiples of n.

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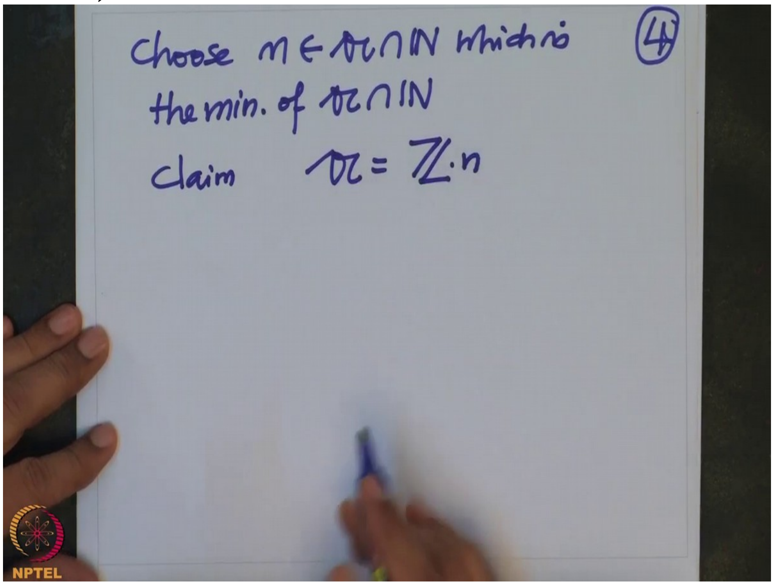

That is what we want to prove.

And that will finish our proof that A is the principal ideal. So obviously this inclusion is obvious

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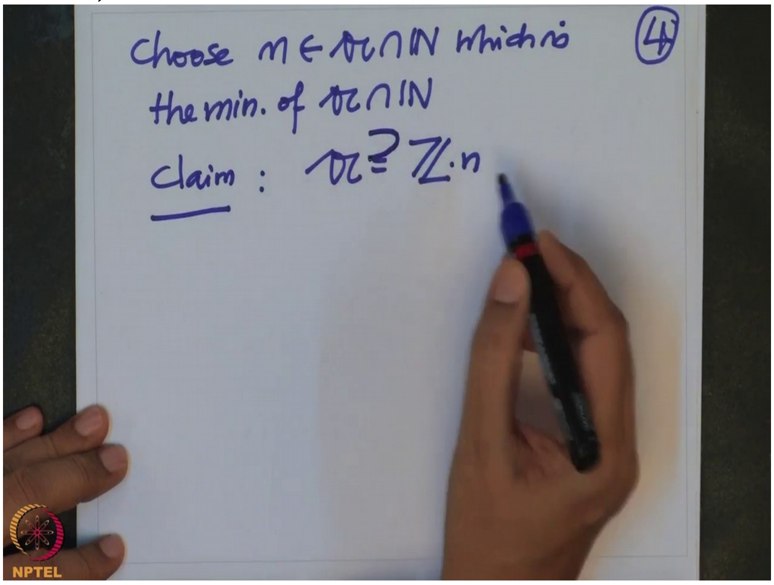

because we chose this n to be in the ideal A. So because it is an ideal, all $\mathbb{Z}$ multiples of n are also in that set, because it is an ideal.

Second condition of the ideal, that if I take the arbitrary element in the ring and arbitrary element in the ideal then the multiplication is also again in the ideal. So this is clear. I want to prove the other way.

So start with any element a in A. And I want to show that this a

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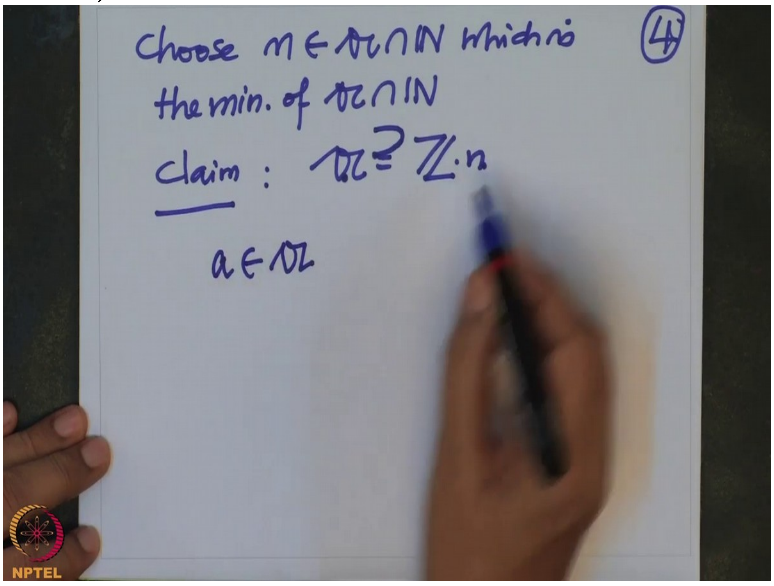

is the multiple of this n, $\mathbb{Z}$ multiple of n. So what we can do with a and n? a and this n, we are given. Obviously this a is, n is non-zero

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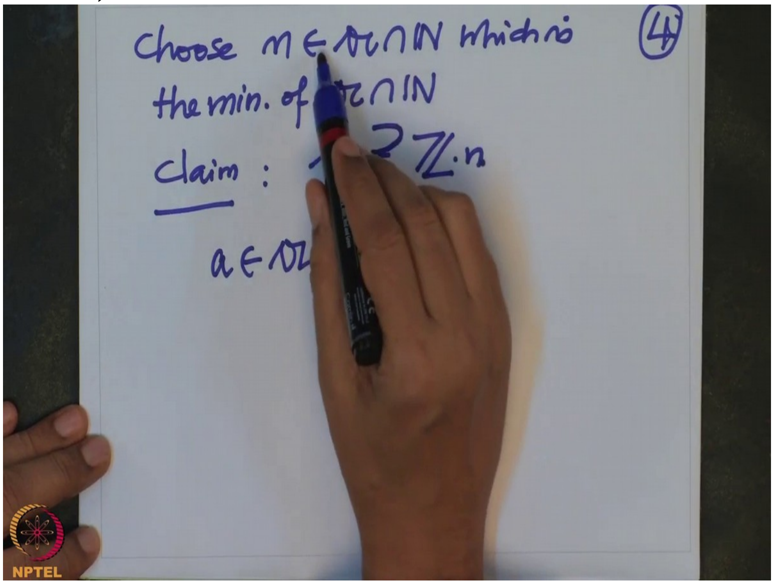

because A, $A \cap \mathbb{N}$ we are assuming, we have chosen a non-zero element in $A \cap \mathbb{N}$, so this is a minimum.

And now we can divide A by n so therefore write, by division algorithm write a as q times n plus remainder r

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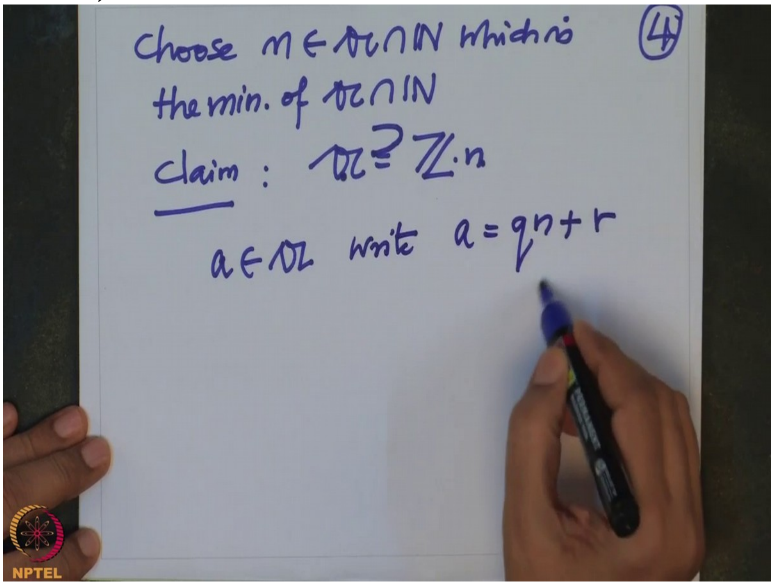

where q and r are elements in ℤ

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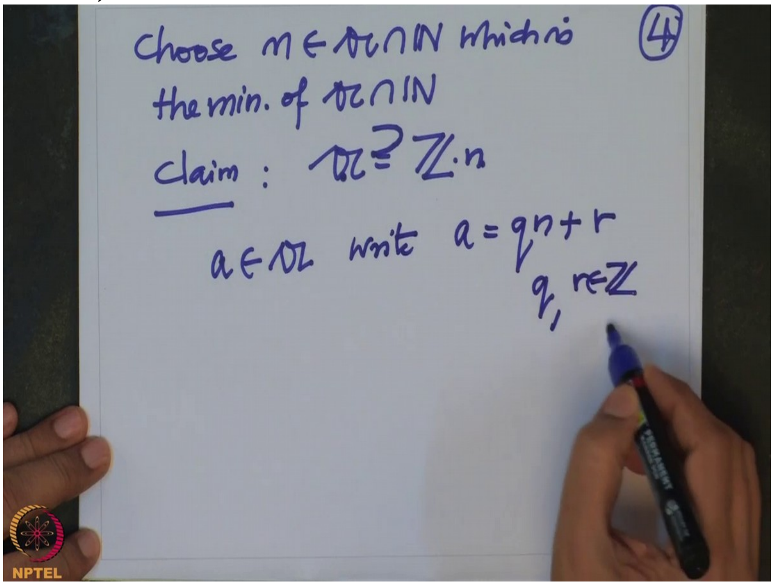

and r is strictly less than n.

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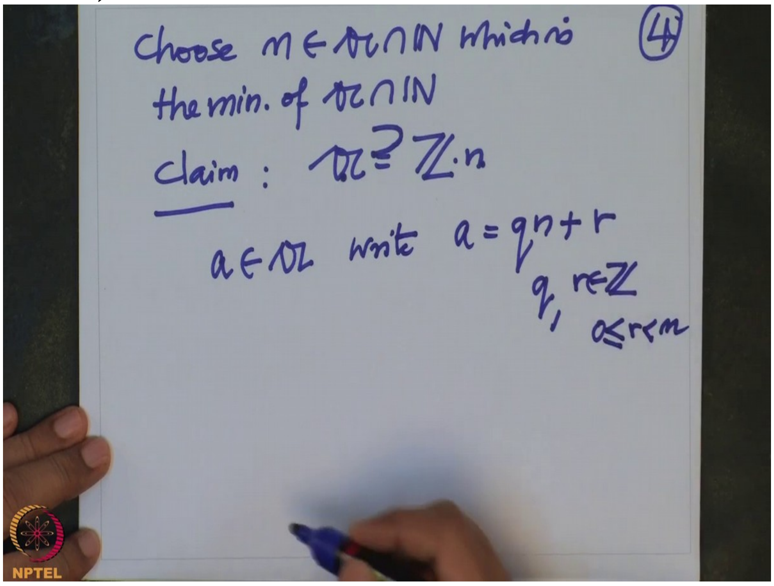

Now look here, this equation says that this a is in A, this n is in, this multiple is in $\mathbb{Z}$ multiple therefore it is in the ideal A so when I shift it to this side, a times q n, this is r but this is in the ideal A because of the definition

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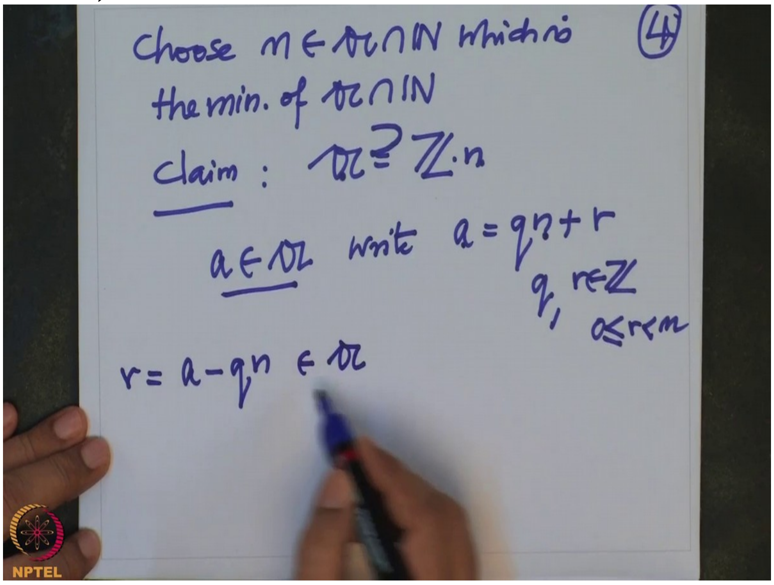

of the ideal and therefore r has to be 0.

If r is not 0,

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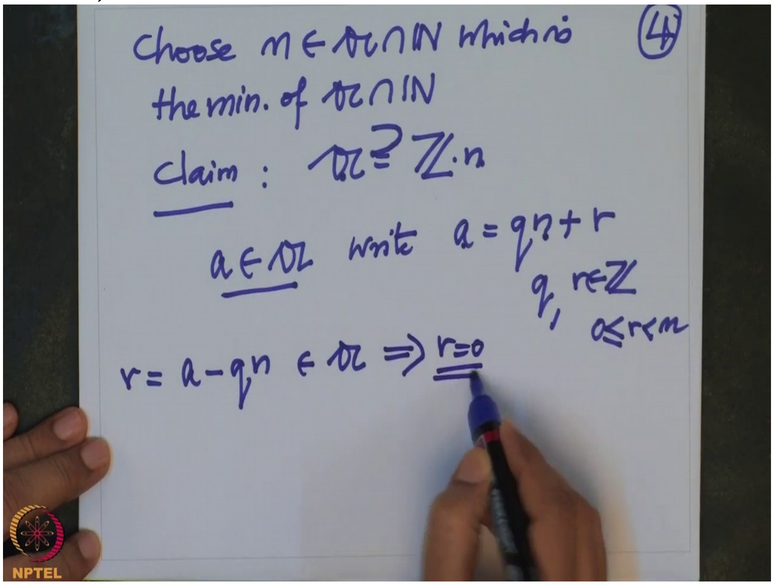

that will contradict the minimality of n. So just to be very sure that we have taken, remember we have chosen

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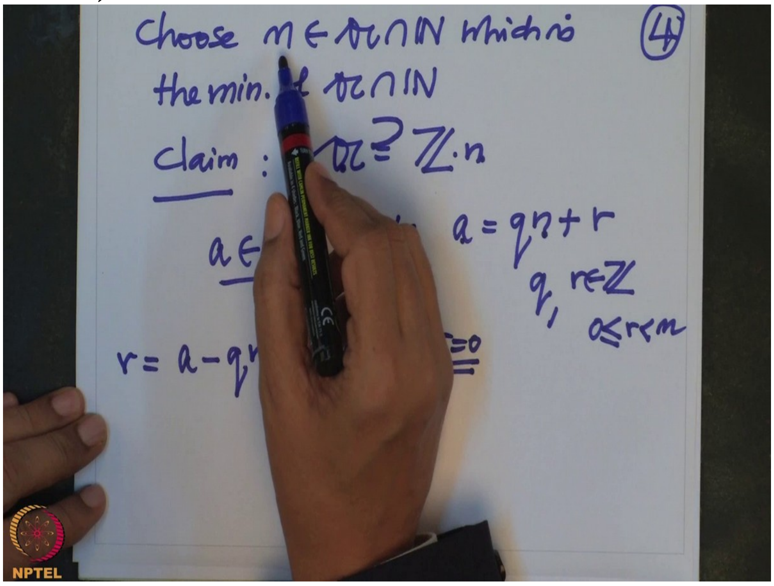

n in this which is the minimum in this minus 0 I should have said.

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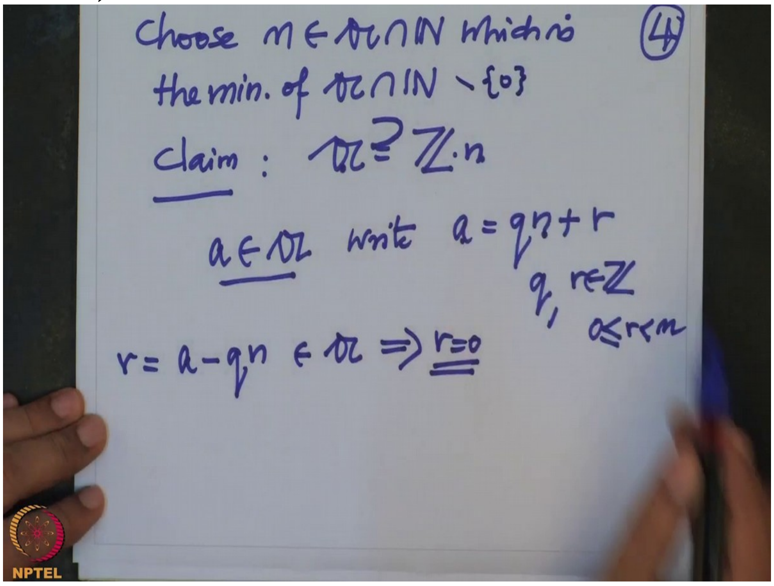

That is, that was what, that is why we need to assume a is non-zero. So therefore this proves that all ideals in $\mathbb{Z}$ are principal and remember the only tools we have used in this is the division with remainder.

And not only that if I choose now, what are the choices, how many, can I recover back from this ideal

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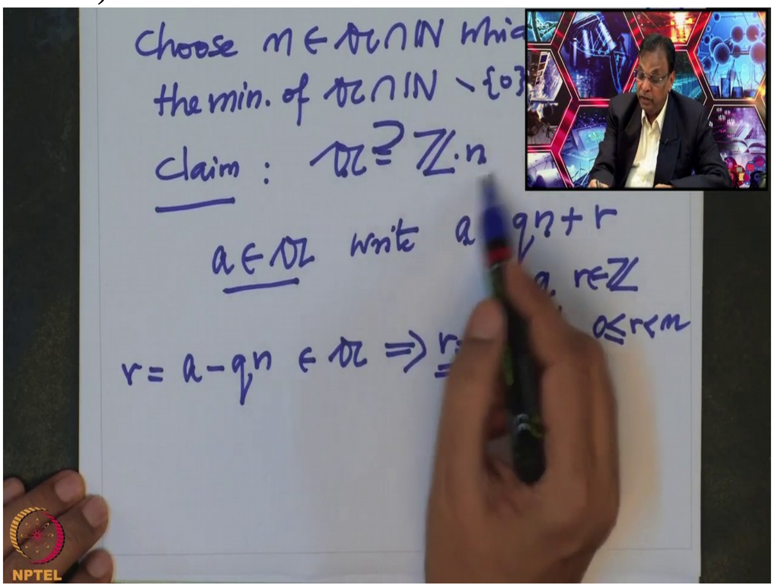

this n, the generator, there are only 2 possibilities because we have, in this ring as a property that the units are only 1 and minus 1. These are the only 2 units.

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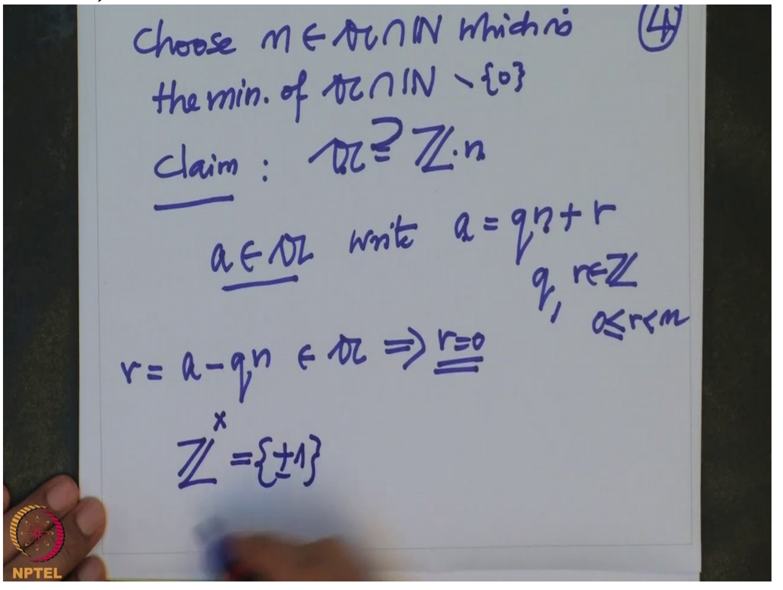

So therefore the only possibility is the $\mathbb{Z}$ multiples of n is same as $\mathbb{Z}$ multiples of minus n.

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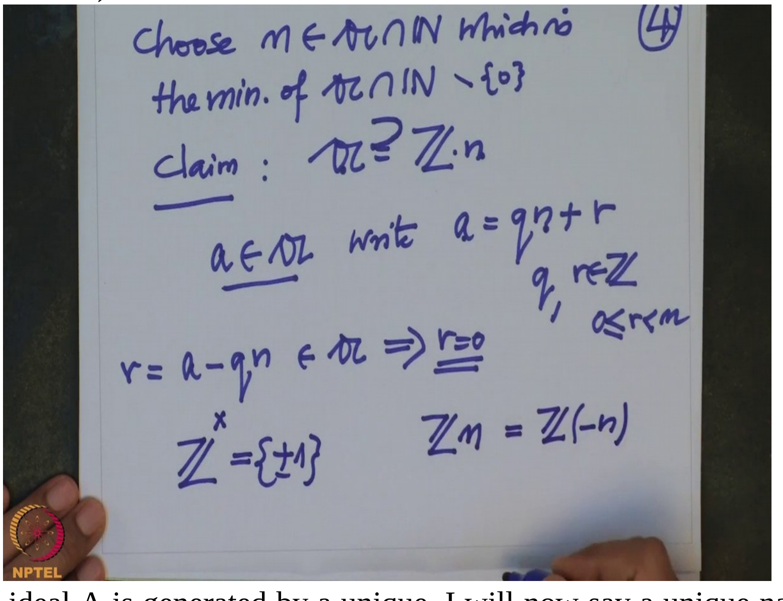

So therefore this ideal A is generated by a unique, I will now say a unique natural number n, unique natural number n

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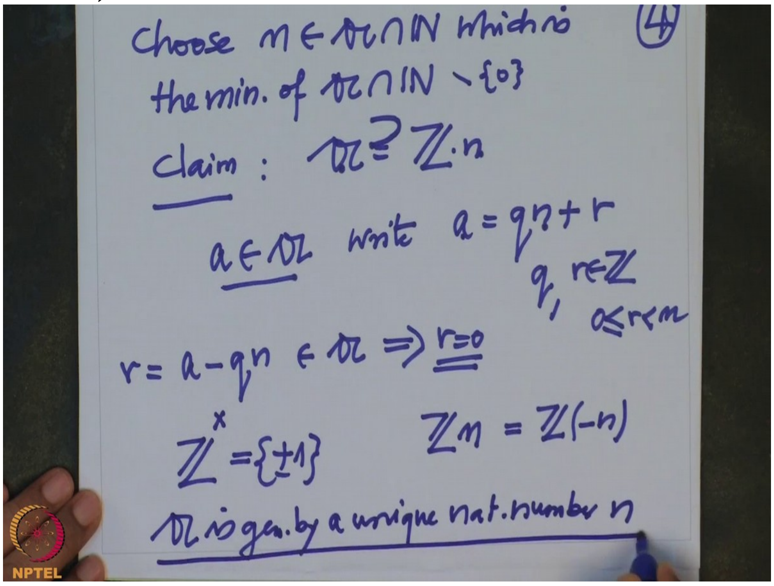

n because this gives the possibility to remove the negative ones. So and nothing special about this ring.

So I will state without proof and one should check this, I will just say similarly, check that in the polynomial ring, $K[X]$ where K is a field, every ideal A is generated by a unique monic polynomial $F(X) \in K[X]$.

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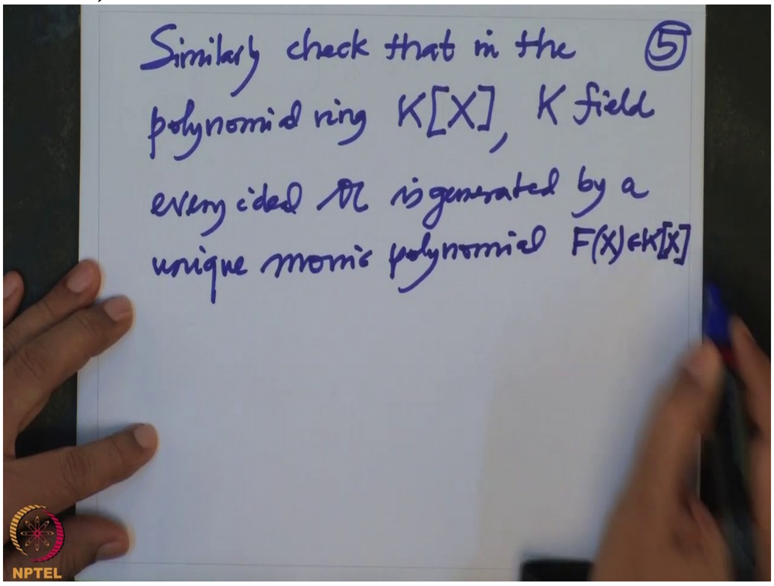

Same proof, we just take, because there we have used the division algorithm. Here also division algorithm is available because K is a field,

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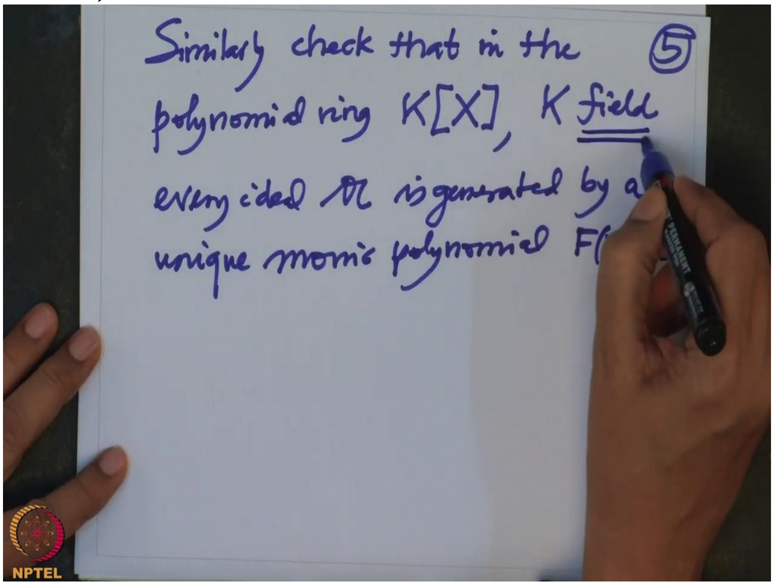

Ok. So before I go on, at least one example where not all ideals in all rings are principal. For example, in the ring $\mathbb{Z}$ polynomial ring, its coefficient in the ring $\mathbb{Z}$, the ideal A which is generated by 2 and X, what does this mean?

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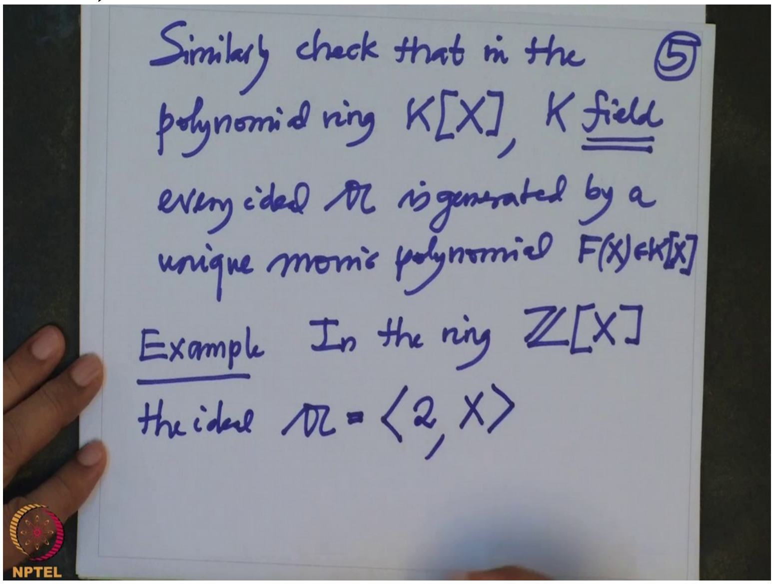

This is indeed an ideal and also check that this ideal is not principal. This I will leave it to you to check,

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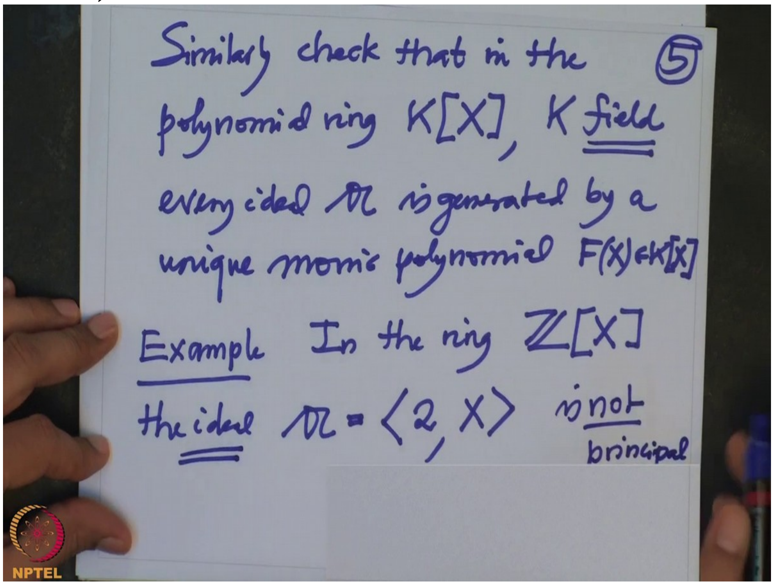

Ok.

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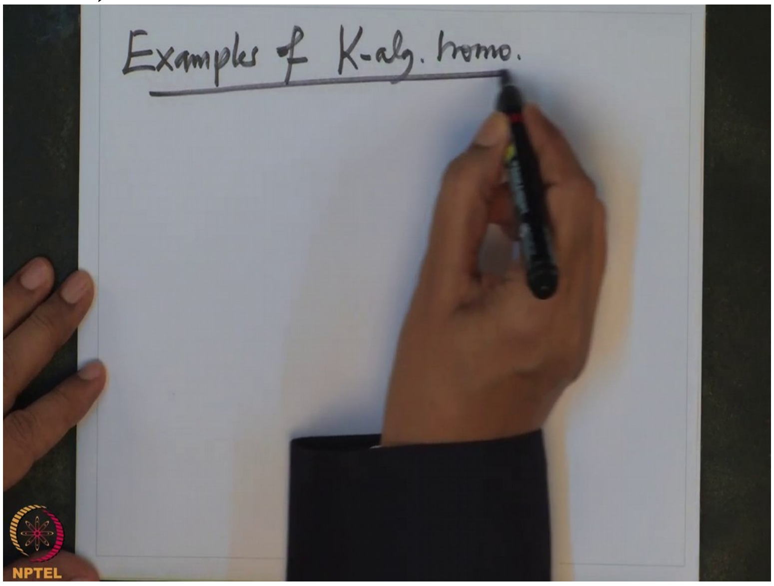

So, on the way we are going to find more examples of ideals, right. So I want to give few more examples of algebra homomorphisms. So for example, examples of K-algebra homomorphisms.

So, so you have 2 K-algebras, so one K-algebra, so K is a field I will take. Though it is not really necessary that K is a field

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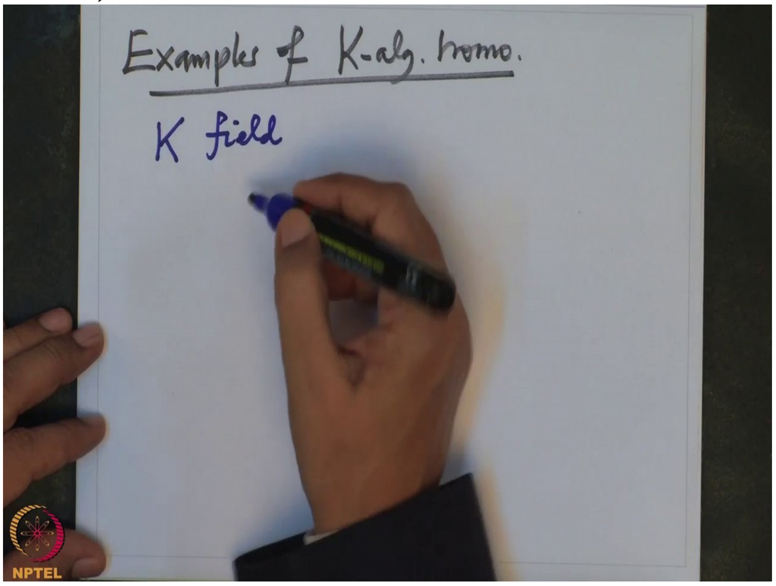

but if I keep assuming more general thing then this course will get more and more delayed and more and more going away from our aim, so let us assume that K is a field. And the typical K-algebras that we are going to deal is $K[X]$, a polynomial ring over K in one variable X.

So this is clearly K-algebra. So typically

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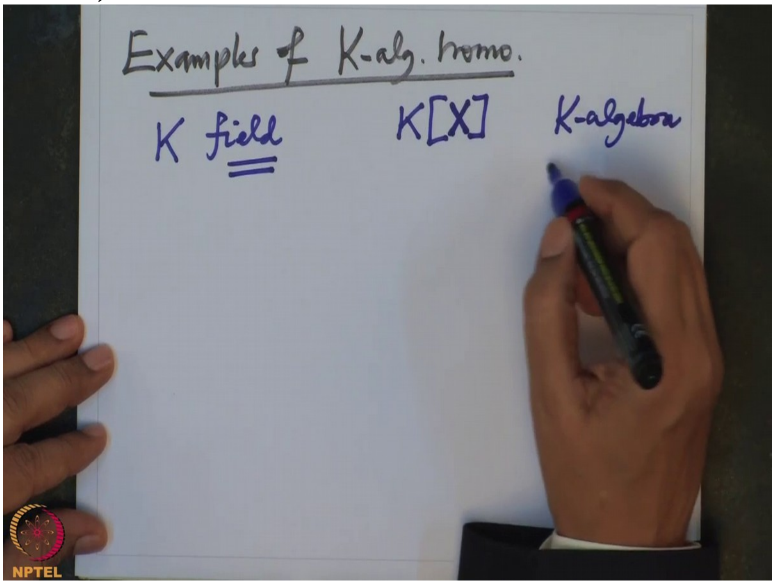

you should keep in mind, take K equal to $\mathbb{Q}$ and these are all polynomial with rational coefficients. Or K equal to $\mathbb{R}$ or K equal to $\mathbb{C}$. These are

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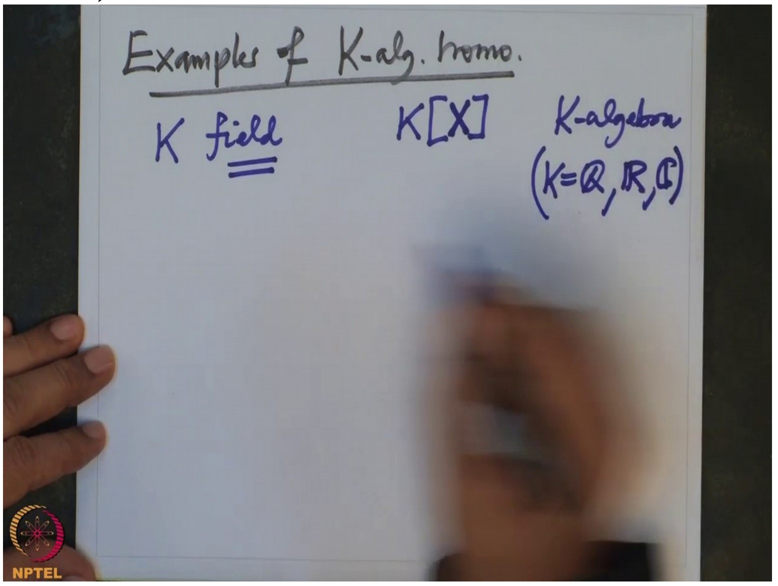

the typical examples.

And now take K itself is a K-algebra, K is also a K-algebra with the same scalar multiplication as the multiplication in K. So I

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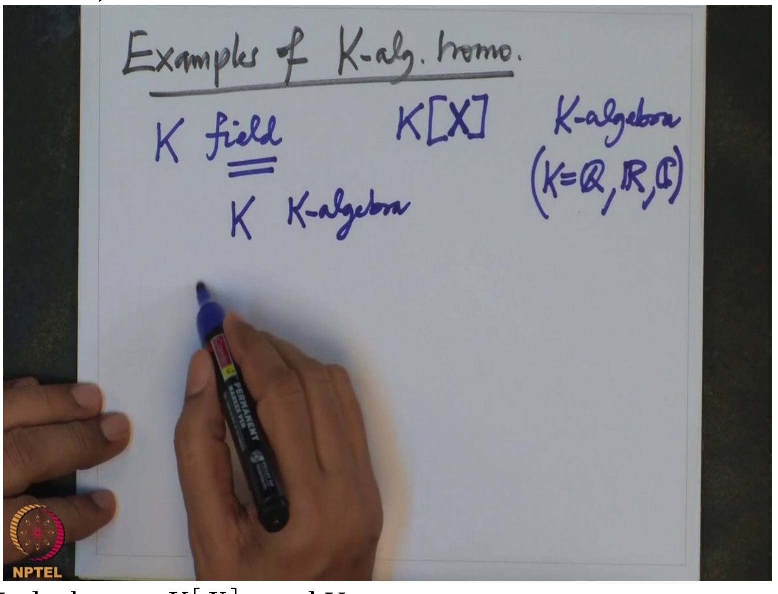

have these two K-algebras, $K[X]$ and K.

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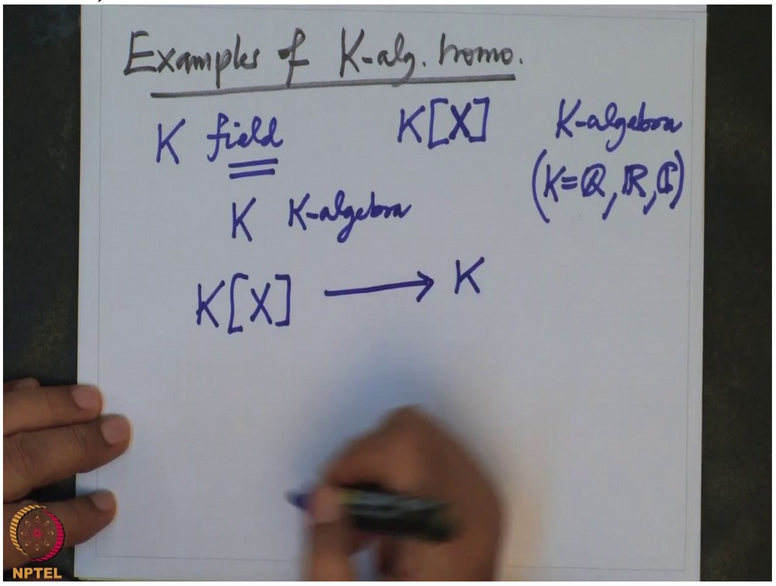

And I want to give a map between these two. It should be K-algebra homomorphism and K-algebra homomorphism means it is a ring homomorphism and it is K linear map.

So given any element a in K this epsilon a

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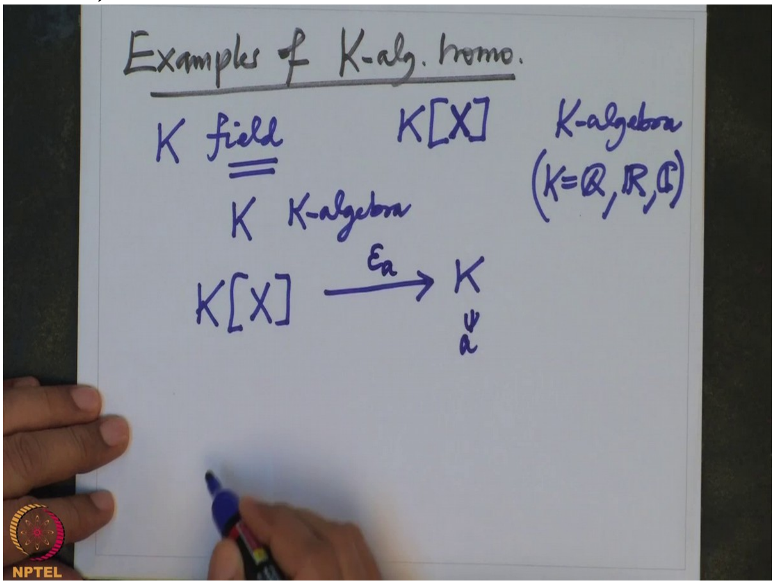

or E a, E is for evaluation, evaluation at a. What is this map?

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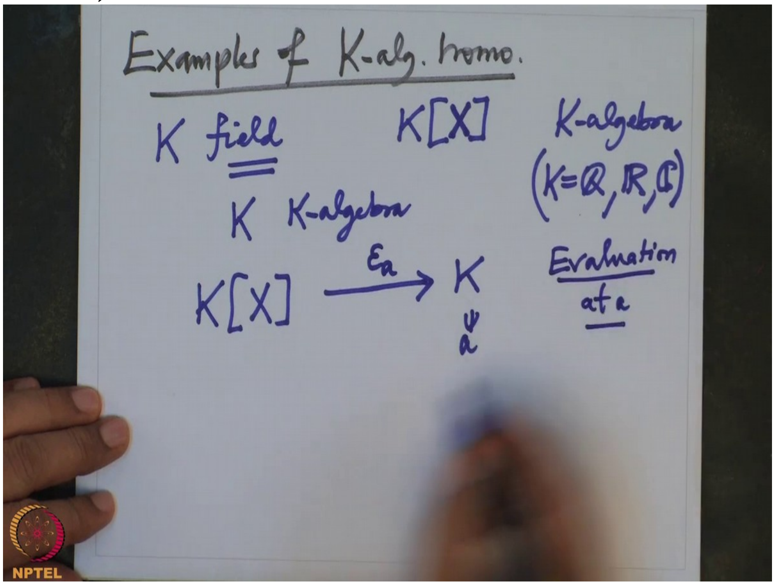

Take any polynomial F and map it to F of a. F of a is a times a 0 plus a times a 1 plus so on. Just put, wherever there is X, put that is a. And the next term will be; I should actually write the coefficient first. So a 2 times a square etc etc a n times a power n.

This makes sense

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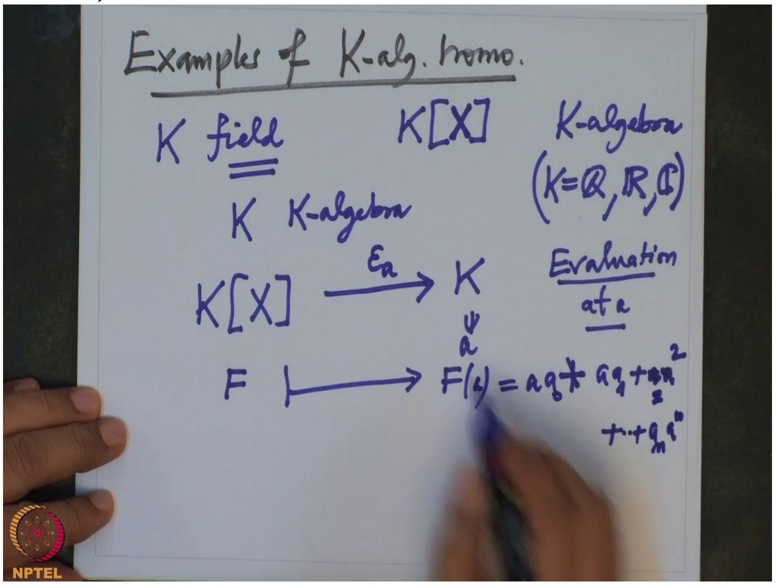

in K now because all these are the elements of K. That is precisely why we needed a concept of K-algebra.

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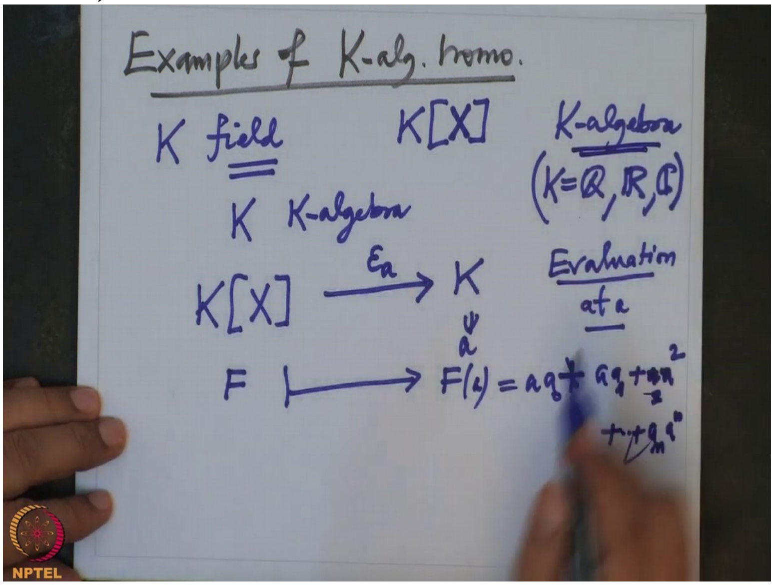

So this is precisely what we call it F of a. And we are interested in, given an element, given a polynomial F we want to find a so that F of a is 0, so that means precisely we are interested in finding out the kernel of y. This is what we are interested in.

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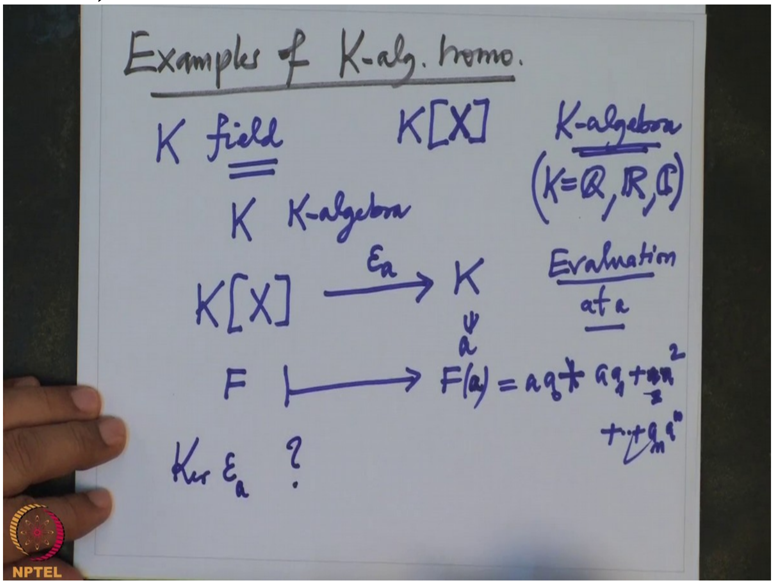

And particularly knowing, so it is an ideal, we know kernel is an ideal. This is ring morphism so this is an ideal. And also we know it is a principal ideal. Also we know that this ideal is generated by a unique generator, unique monic generator and that is what we are going to formalize for the, and the little bit more general situation.

But before I do that I also need a construction of a new ring by ring ideal. So remember from $\mathbb{Z}$ and given any integer n, natural number we have constructed a new ring called $\mathbb{Z}$ modulo n.

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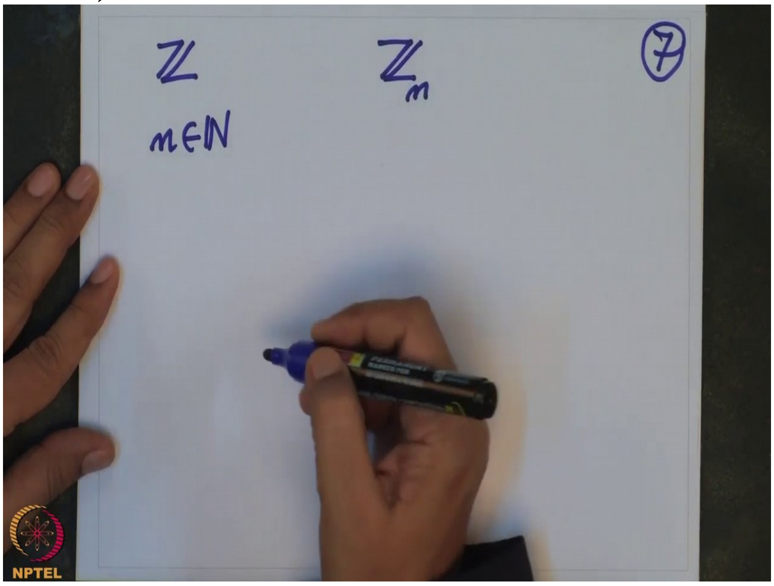

This ring was constructed by using $\mathbb{Z}$ and n by taking the operations addition modulo n and multiplication modulo n.

Similarly

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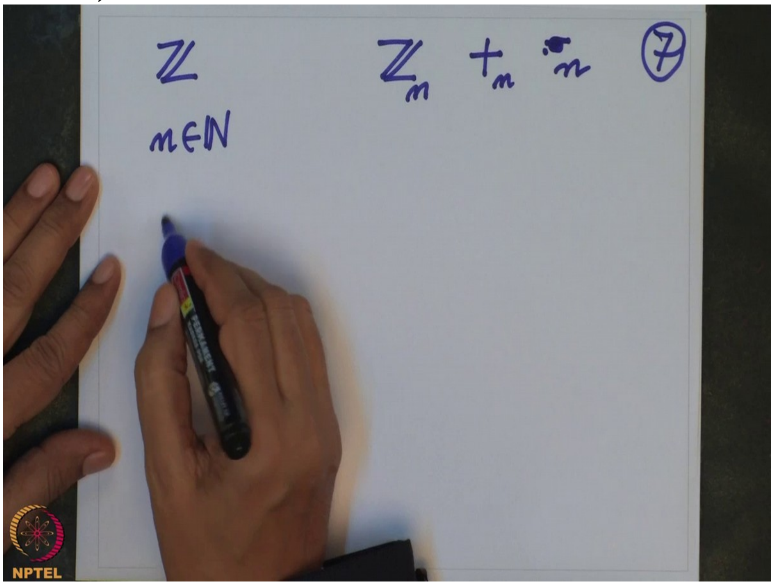

given any ring R and given any ideal A, I am going to construct a new ring which will be denoted by R modulo A. So the new ring, the addition there I will use a similar construction.

That means I will introduce an equivalence relation which will be defined by A and take the equivalence classes and on the equivalence classes I define addition and multiplication.

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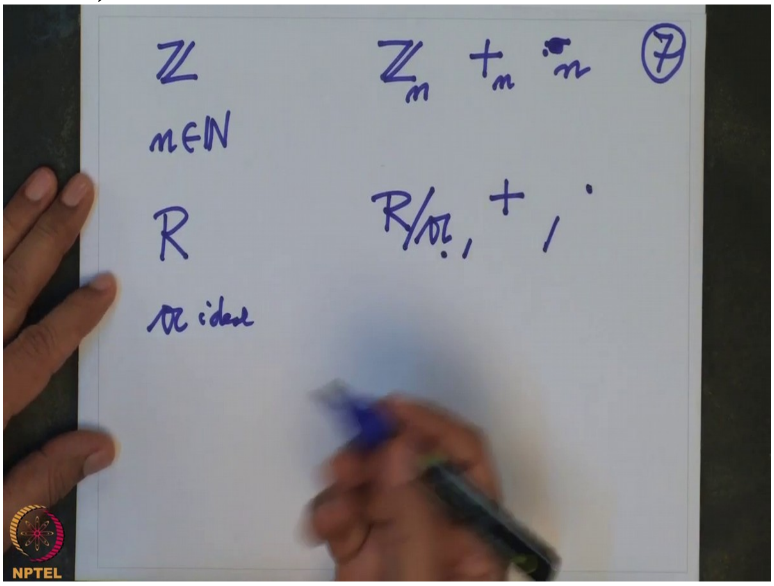

And how do I do that? It is very simple. Take any x in, take, define an equivalence relation by x, y in R then we say that this is a definition. x is related to y

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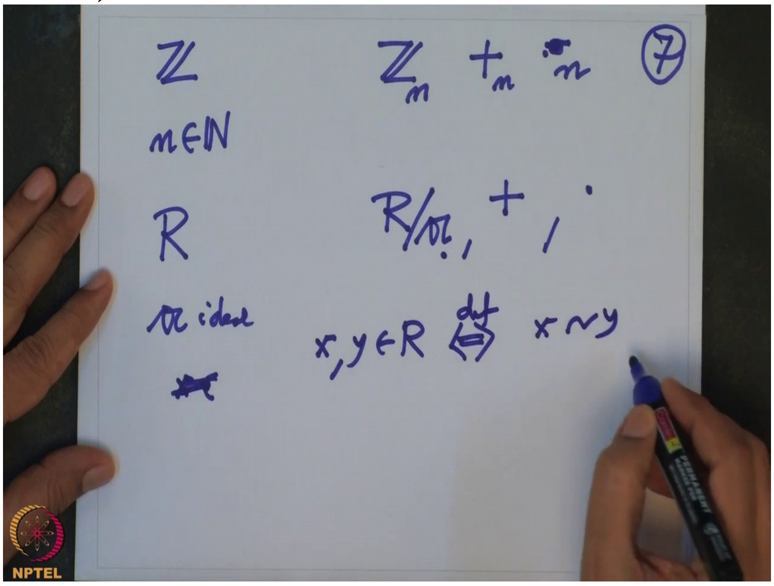

under, A should come somewhere notation, so I will just write simply this. So this means x minus y should belong to the ideal A.

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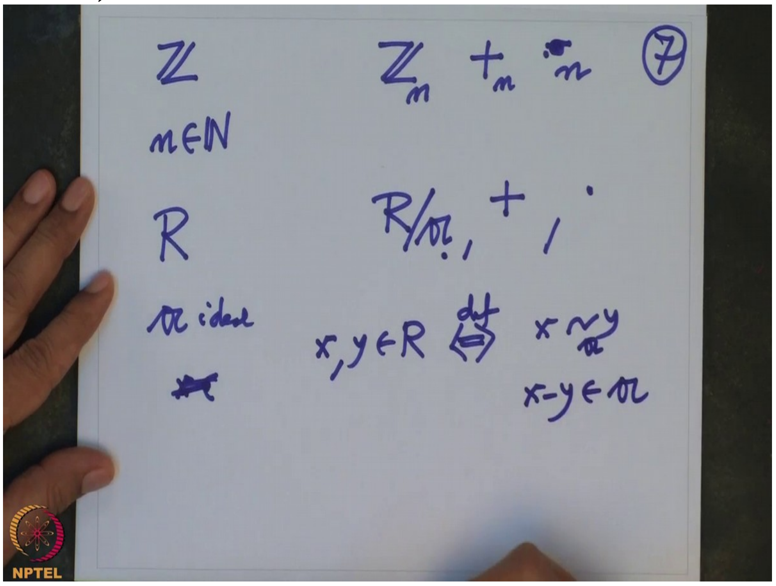

And now we will check that, check I will just say that check that this is an equivalence relation on R

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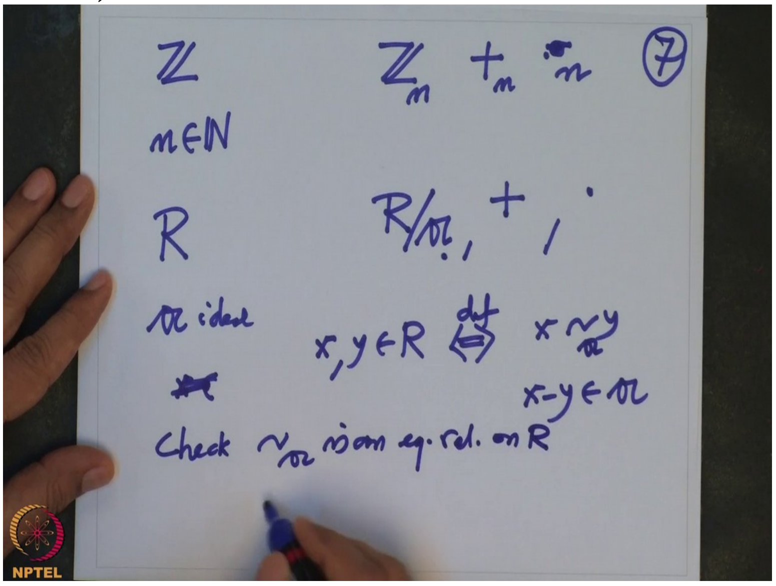

and let us denote the equivalence classes x suffix A here, equivalence classes, class of x and obviously you have to add

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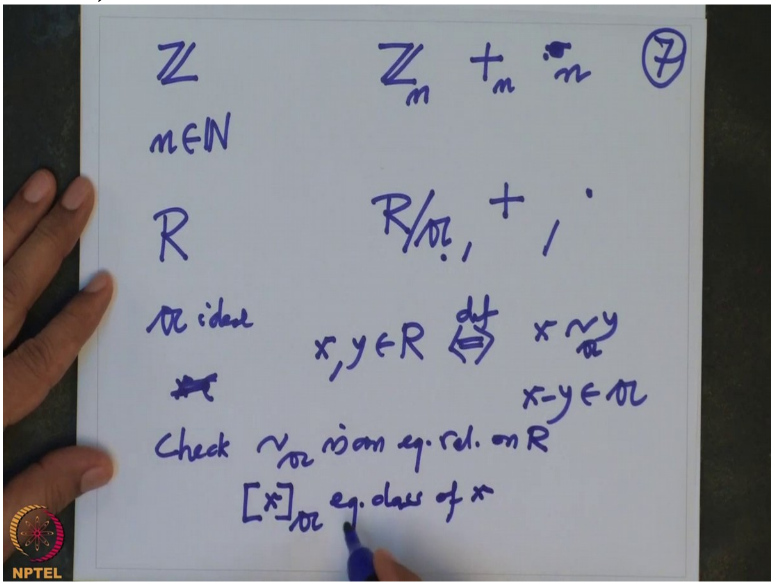

these, add as usual and take its equivalence class under A and then you check that these operations are well-defined and same, everything is same. If one gets stuck, go back to study the example.

So from this ring R we have this new ring. So R we have passed on to this new ring R by A and also we have a natural map here. This is, I denote by π_A ,

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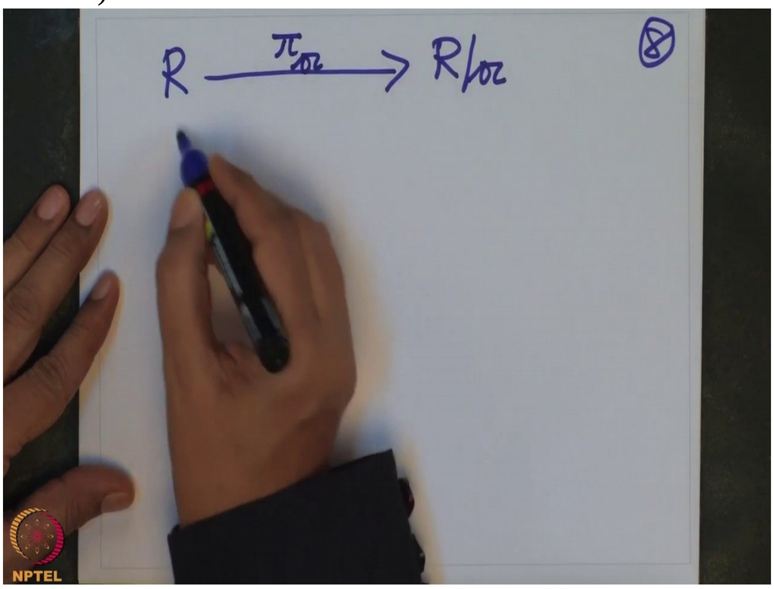

this is x going to equivalence class of x under A

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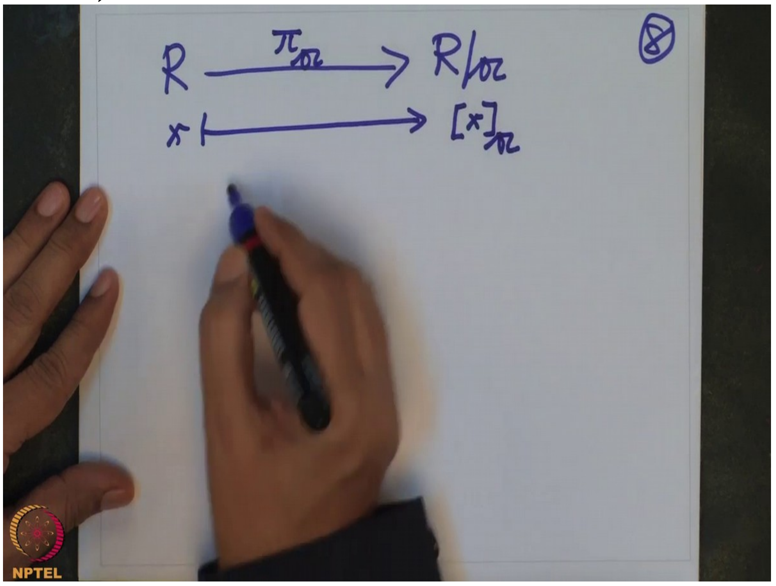

and also I will leave it for you to check that this is a ring homomorphism, ring homomorphism with kernel, the given ideal A and image, image of π_A is the whole R by A, that means

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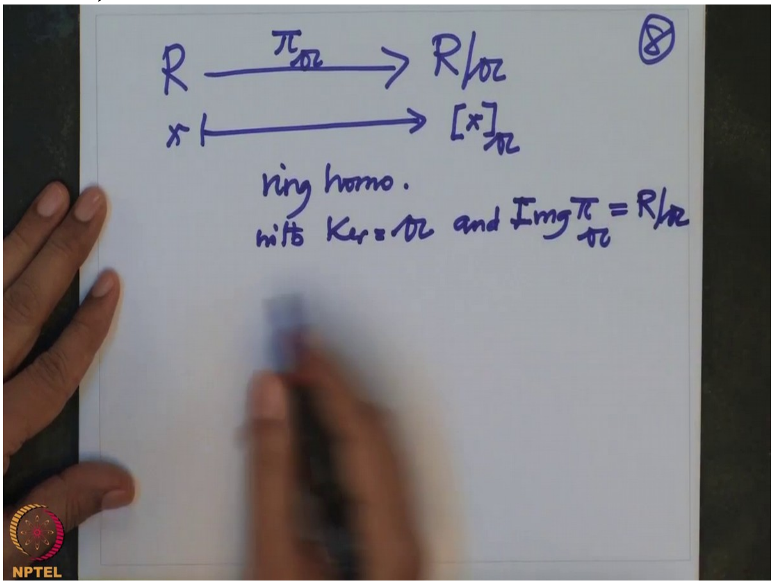

this.

In other words, this π_A is the canonical surjective ring homomorphism from R on to R by A.

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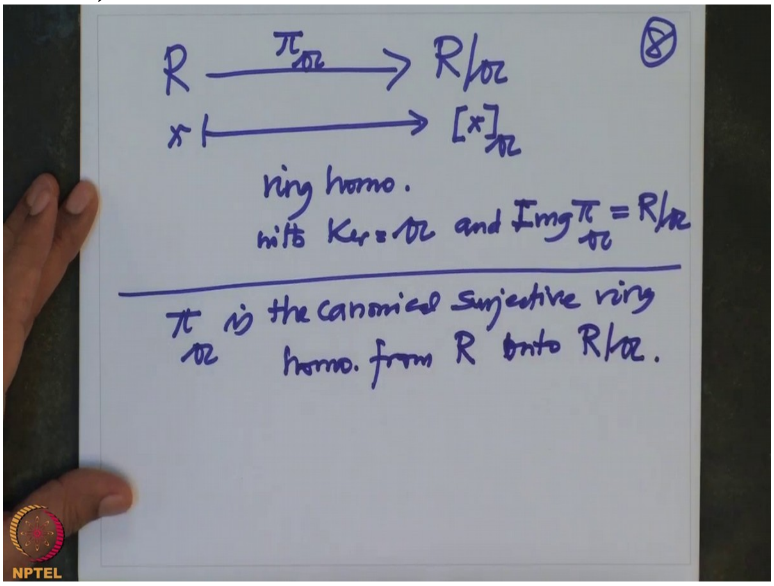

So that is a kernel. Now we are in much better position to describe what algebraic element means and what algebraic extensions are.

But this program I am going to carry out in the next lecture. And just to summarize what we have seen today's lecture is rings, ring homomorphisms, algebras, algebra homomorphisms, kernel, image and so on.

And now in the next lecture we will introduce the concept of algebraic elements. We will introduce the concept of field extensions which are algebraic and also define transcendental elements and so on. And use this further to one of our main step, the beginning step that given a polynomial over arbitrary field, we extend the field in such a way that all the

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zeroes of this polynomial lie in the bigger field.

This we do it for single polynomial first and then we do it for many polynomials and as a consequence we will deduce the fundamental theorem of algebra namely that is the field $\mathbb{C}$ is algebraically closed. With this I will stop today, thank you.