Calculus for Economics, Commerce & Management

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Lecture - 33
Derivative tests for convexity, concavity and points of inflection, higher order derivative conditions

So, we said that geometrically it is not possible to have a function discontinuity at a point and be convex or concave in that interval. So, that was the picture we saw. So, let us go ahead and look at how the notion of derivative helps us in analyzing the convexity and concavity of a function. So, here is what is called the first derivative test. So, note that we will defining the function to be convex or concave, we have not looked at the continuity or any other property of the function. We just defined geometrically that the chord joining any 2 points should remain above the graph of the function. That we called as the concave up and so on.

So now we are going to give conditions which will ensure that in a portion of the domain function is concave up or concave down. So, the first derivative test says, suppose a b is a function, f is a function on a interval a b to R domain as a function is a open interval a b.

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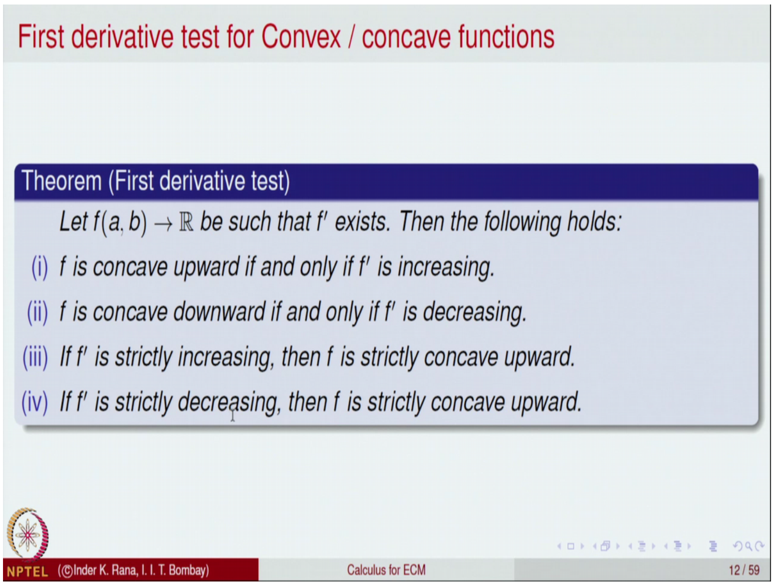

And dispose at the derivative exist in this interval in this interval a b, then the following holds one, f is concave upward if and only if f dash is increasing. So, here is a property which is if and only if and only if means both ways namely given that f is concave upward will imply it is derivative is increasing. And conversely if the derivative is increasing that will imply the function is concave upwards, there is what if and only if means it is a 2-way condition if f is concave upward, then f dash is increasing and conversely if f dash is increasing then f is concave upwards. So, this is stated as f is concave upward if and only if f dash is increasing.

The second condition is the opposite of this. Namely, f is concave downward if and only if f dash the derivative function is a decreasing function. So, again it is a if and only if condition. So, f is concaved downward if and only if f dash is decreasing. The third condition is only one-way condition, it says that if the derivative of a function is strictly increasing; now the difference comes here to as f dash decreasing, f dash increasing this condition is about strict increasing. So, if the function on a b is such that is derivative exists, and is strictly increasing. Then one can say that conclude that f is a strictly concave upward function. And similarly, if f dash the derivative is strictly decreasing, then it is concave. So, it should be concave downward. So, it should be this should be called as concave downward.

So, if f dash is strictly decreasing, then f is strictly concave downward. So, that is a conclusion. We will not be proving these theorems, they have proofs are slightly technical. So, will assume these proofs, and go ahead and apply them in our examples.

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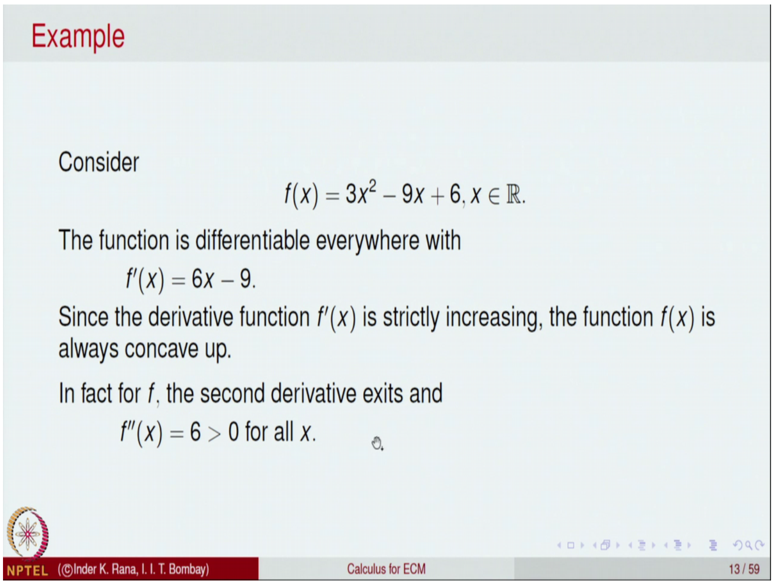

So, let us look at an example f x is equal to 3 x square minus 9 x, plus 6 x belonging to y. So, domain of the function is a whole of real line, this is a polynomial function. So, it is continuous everywhere, and of course, it is also differentiable everywhere. So, no problem of different non-differentiability and such things. So, it is differentiable everywhere in the domain which is a whole of real line.

So, let us conclude the derivative of this. So, derivate will be 6 x minus 9. So now, to analyze for local maxima minima, one used to put first derivative equal to 0. So, we are not looking into that now, we are looking at the sign of f dash of x. Whether it is f dash of x it is increasing or a decreasing function. So, let us look at f dash of x which is equal to 6 x minus 9. So, that means what? So, that means, f is strictly increasing because since the derivative function is strictly increasing, how do you know 6 x is strictly f dash of x is a strictly increasing function? Because it is a linear function, and for linear function it is slope y equal to mx plus c, the slope is 6 which is positive. So, whenever the slope is positive the function is strictly increasing. That is one way of saying that this derivative function is strictly increasing.

Or if you like you can compute the derivative of this derivative function. So, compute the second derivative, the second derivative is f double dash of x is equal to 6 which is bigger than 0. So, since the derivative of this function f dash is strictly bigger than 0, it should be a strictly increasing function. So, second derivative test will give you that. So, that will either way you can conclude, that this function is concave up, because it is derivative function is strictly increasing. So, here is a second derivative test also could have been applied to conclude that the function is strictly increasing.

Second derivative test we are not actually done in the previous thing. So, here is a observation which motivates one to say that probably that is true. And in fact, that theorem is true. So, that theorem is the second derivative test for convexity and concavity of a function.

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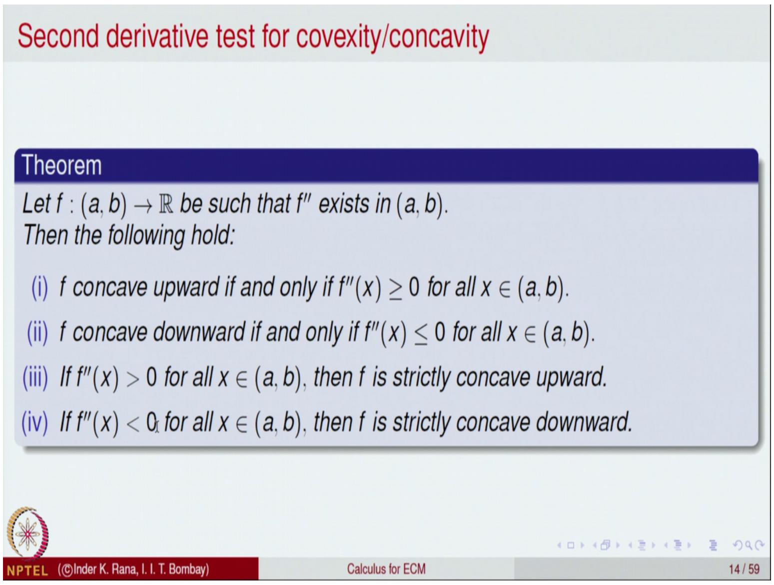

So, when concave up is also called convex, and concave down is normally called concave.

So, that is what convexity and shouldn't be confusing, in the term convex concave up is also called convex. And concave down is normally just called concave. So, there is some of the books follow that terminology. So, let us take a function f defined on a open interval a b taking values in R such that it second derivative exists in whole of the interval a b. So, we are assuming that the function has second derivative in the whole of interval a b. Then the following hold 1, f is concave upward if and only if the second derivative is bigger than 0. So, once again it is if and only if conditions f is concave upward if and only if second derivative is bigger than or equal to 0. So, 2-way condition f concave upward implies f double dash is bigger than 0. And conversely f double dash bigger than or equal to 0 implies f is concave upward.

The second parallel thing would be let us it is concave downward if and only if f double dash is less than or equal to 0. In all of in all of interval a b. So, this is the condition about the property of the function or the derivative in whole of a b. So, if and only if f is concave upward if and only if the double a second derivative is bigger than 0 is positive, for all points in a b in the domain. And similarly, concave downward if and only if the second derivative is less than or equal to 0 for all interval all points in interval a b.

And here is a one-way condition which says that if f double dash is strictly bigger than 0, as it happened in the previous example, if f double dash is strictly bigger than 0 for all points x in a b, then f is strictly concave upward. And similarly, we have the other condition that if f double dash is strictly less than 0, then so, these 2 are only one way that if we know some property of the derivative namely second derivative is bigger than 0, or second derivative is less than 0, for all points in that interval that is important, then we can conclude that either f; f is strictly concave upward or concave downward depending on which property is true.

So, f double dash strictly bigger than 0, in all points in a b will imply, then it is strictly concave upward, and f double dash less than 0 for all points will imply f is strictly concave downward. So, this theorem is useful in certaining to find out when a function is can strictly concave up or concave down. And this is applicable when the second derivative of the function exists. We had also defined what are called the points of inflection for a function. So, what is the point of inflection for a function point of inflection were the points where the function changes it is nature from strictly concave up to strictly concave down, or strictly concave up to strictly concave up. So, those are the points which are called points of inflection.

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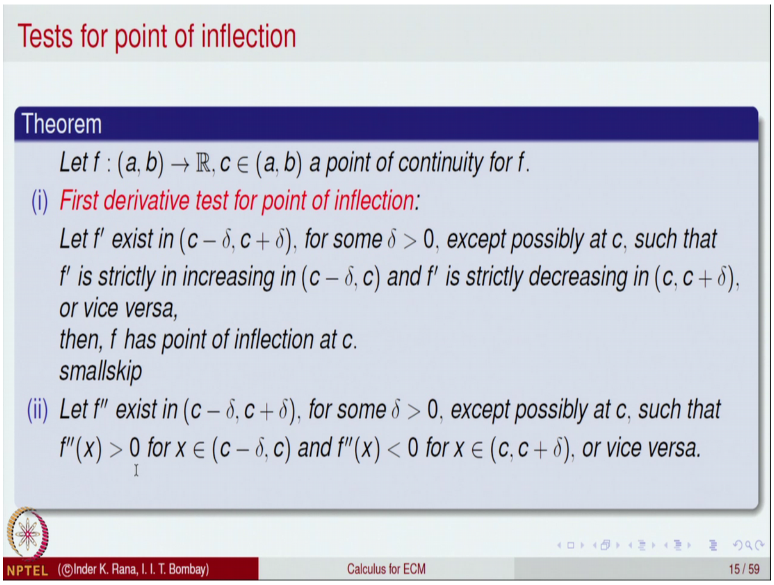

So, here is a test for points of inflection. So, let f be a function on a interval open interval a b and c is a point inside, and it is a point of continuity for f.

Then we say the first derivative test for this point of inflection says, what we want to say we want to say that on the left of the point, it should be one nature namely either strictly concave up, and on the other side it should be strictly concave down. And we already have a test for strictly concave up and strictly concave down; so will put those conditions appropriately. So, suppose the first derivative exists in a interval around c; so c minus delta to c plus delta, except possibly at c.

We are do not we are not concerned with a point c, we want to analyze that point to be whether is a point of inflection or not. But we need that continuity point. So, f should be continuous at that point c, f need not be differentiable at the point c, but it should be differentiable at all points in enable on the left as well as on the right of c.

So, let us look at what is the condition, that if f dash is strictly increasing in c minus delta to c, that is strictly increasing on the left and is strictly decreasing on the right. So, what will that imply f is strictly in increasing on the left, will imply the function is concave up on the left. And strictly decreasing will imply it is concave down on the right. So, that is put these to put together, will say that the function changes it is nature from strictly concave up to strictly concave down. So, that should be a point of inflection. So, this is what is called the first derivative test, for the point of inflection. And similarly, so, if these conditions are satisfied, then c will be a point of inflection. And similarly, the other way round condition that if second derivative is bigger than 0 on the left side right. So, this is the first derivative this is a second derivative condition from that, and strictly second derivative is less than 0 or vice versa.

So, basically the condition that on the for a point of inflection on the left side, the function should be either strictly concave up, and on the right side should change it to strictly concave down or other way round, this is the first one gives a condition in terms of the first derivative and second one gives a condition in terms of the second derivative. So, either one can be used depending on whether the function is as a first derivative and second derivative and so on. So, let us look at some examples to illustrate this point. So, let us look at the example of f of x is equal to 1 over of 1 plus x square. See this for this function is defined for all points x in R, because denominator is never going to be equal to 0 1 plus x square is not equal to 0 x square is always positive so.

So, this is a function to find out analyze this function, let us look at the first derivative. So, how will you compute the first derivative? To compute the first derivative, one has to apply the quotient rule formula.

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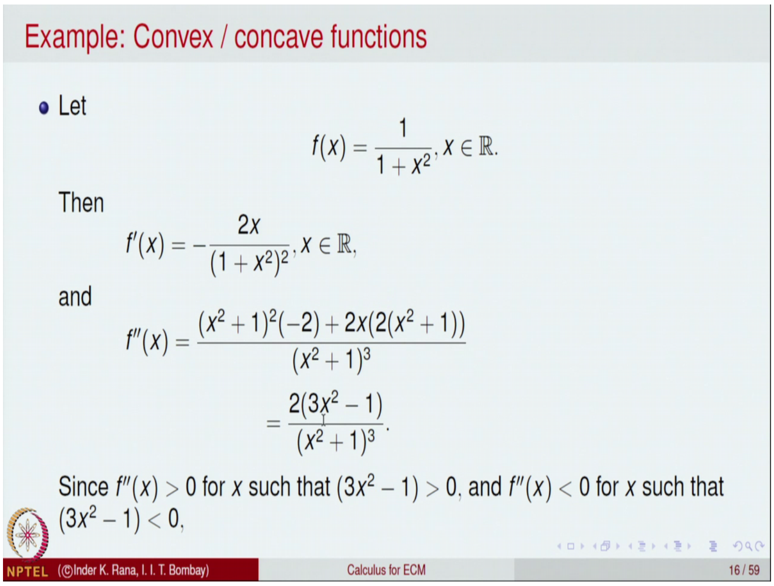

So, 1 plus x square it is derivative right it is not equal to 0, and it is derivative is 2 x. So, it is derivative is equal to 2 x. So, we can apply the quotient rule formula. So, f dash of x will be equal to numerator is 1. So, that is minus derivative of quotient rule is minus g dash x divided by g square. So, it is derivative of 1 plus x square is 2 x. So, minus 2 x divided by 1 plus x square whole square. So, that is a derivative of f dash we want to differentiate this once again. So, f double dash of this will be again apply the quotient rule formula. So, that will be equal to 1 plus x square, square whole to the power square that will become out 4, once you put the derivative where and apply the appropriately, one power will cancel out it will be x square plus 1 to the power 3.

So, we are strongly urged to check, that if f dash is equal to this second derivative is equal to this. So, let us assume for the time being that this calculations are correct, and let us go ahead. So, once you simplify that x square minus 1 square x square minus 1 1 power will cancel. So, this has should have been 4 here, right square square is 4. So, one power cancels out and you get equal to 3. So, once that is there this is the final form of the second derivative. So, exercise for you to check that if this is the first derivative. This is the second derivative by using the quotient rule formula.

So, second derivative will be bigger than 0, say denominator is always positive. So, this 2 times so, sign of second derivative depends upon the sign of the numerator. So, numerator is 3 x square minus 1. So, this will be bigger than 0 will imply that second derivative is bigger than 0. So, 3 x square minus 1 is bigger than 0, will imply f double dash of x is bigger than 0, and that less than 0 will imply this is second derivative is less than 0.

So, 3 x square minus 1 bigger than 0, means 3 x square bigger than strictly bigger than 1. So, that means, x square strictly bigger than 1 over 3, so x square bigger than 1 over 3 gives you 2 values.

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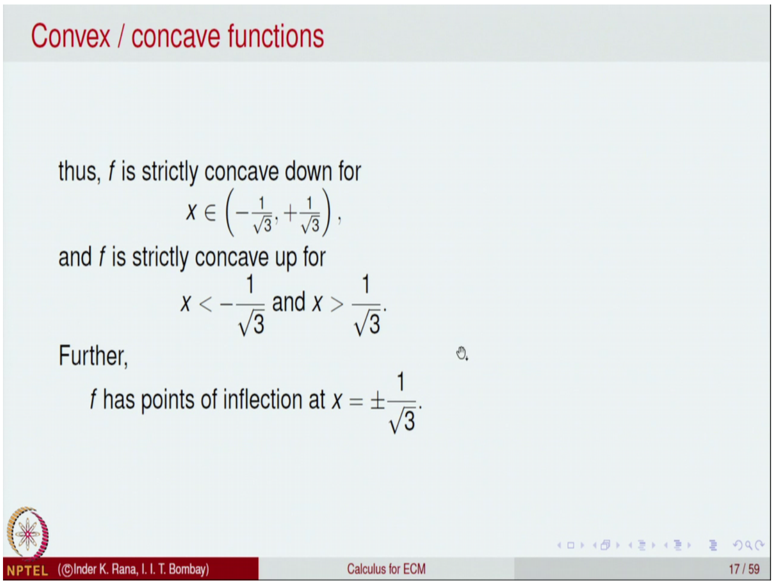

So, you have that f is strictly concave down for the interval minus 1 by square root 3 to plus 1 by square root 3. And outside that interval the function will be concave up. So, f is strictly concave up, when x is less than minus 1 by 3 and x is bigger than 1. So, these conditions are coming from the fact that x square minus 1 x square minus 3. So, that is what we had 3 x square minus 1 is bigger than 0 or less than minus 1 is bigger than 0 or less than 1.

So, that gives us the interval, where the second derivative is going to be less than 0. So, in this portion the function is concave down, and on the left side of this and on the right side of this the function is second derivative is positive. So, that will give you it is strictly concave up. So, once that is established; that means, the function is changing it is nature at the point x is equal to minus 1 by 3 and x is equal to plus 1 by 3. So, at the point on the left of x 1 x on the point x equal to minus 1 by 3, on the left of this the function is concave up. And then on the right of it becomes concave down it stays concave down, and then again at the point 1 over square root 3 it changes his nature and becomes concave up again. So, this there are 2 points of inflection, one is minus 1 over square root 3, the other is plus 1 over square root 3.

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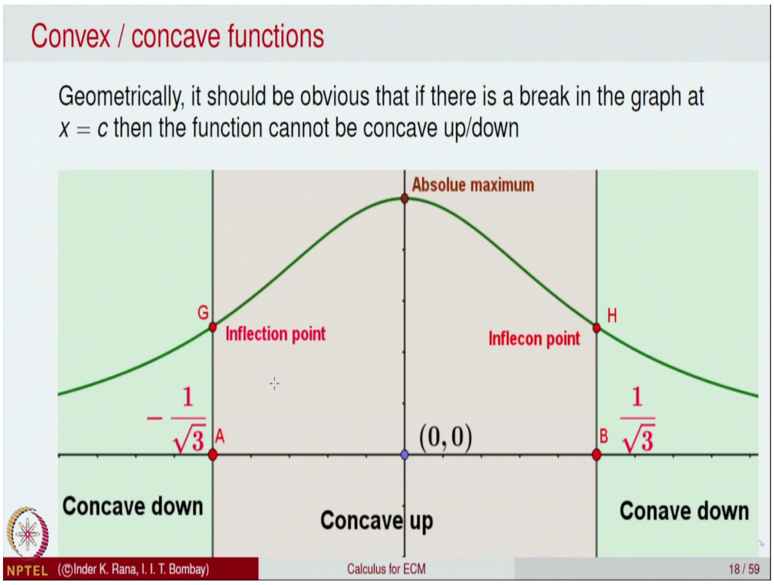

So, using this one can give a better picture of the function. So, this is a better picture of the function, namely in the portion minus 1 over square root 3 plus square root 3 these are the points of inflection. So, in between so, this is the in between point where the function is concave up also it is a wrongly written here this picture is wrongly mentioned. So, this is concave down because in this portion. It is in the portion minus 1 over square root 3 to plus 1 over square root 3 the function is concave down.

So, in the picture in the graphical representation ignore this part. So, this this should be on the left side. So, it is concave up in the green portion, concave up in this green portion, and concave down in this portion. So, this is how one sketch is so, convexity and concavity are more useful in sketching the graph of a function more accurately. So, that is the application of one can also give higher order conditions for the point of inflection. So, they are as follows. So, they are necessary conditions. So, before like local maxima minima first derivative equal to 0 gave you the conditions for possible points of local maxima minima. Similarly, for points of inflection so, necessary condition for point of inflection is if the second derivative exists, and as a point of inflection let see, then second derivative must be equal to 0.

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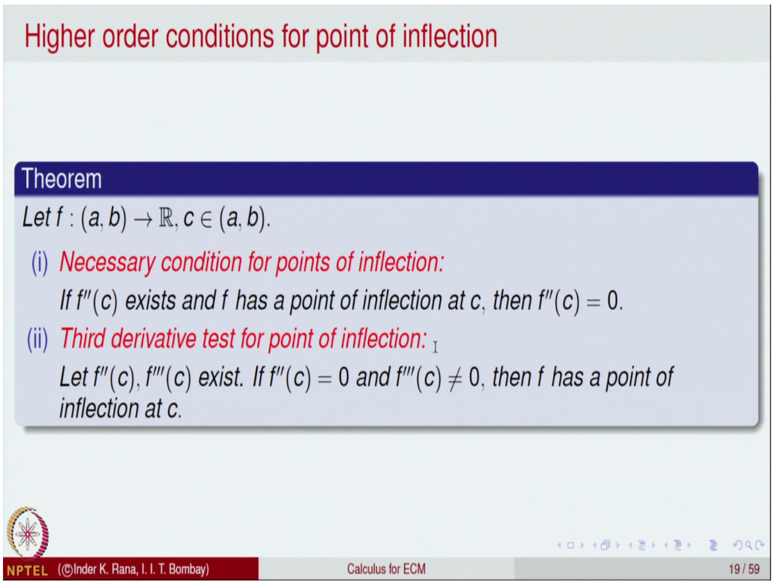

This is very much similar to the condition for local maxima minima, there we had that if the first derivative exists and there is a point of local maxima or minima, then the first derivative is 0.

A corresponding theorem similar theorem holds as a necessary condition for points of inflection, it says that f double dash exists, and it is a point of inflection both is conditions are true then second derivative will be equal to 0. So, that means what? That means, if the second derivative of a function exists to locate the possible points where the function can have points of inflection, you have to solve the equation f double dash x is equal to 0, and find out those points, and then analyze whether these points are point of inflection are not by our previous conditions.

So, we can have a third derivative test for point of inflection like you had the second derivative test for local maxima minima. So, this is a necessary condition you are located the points, and supposing you are able to find the third derivative, let the third derivative of the function second derivative and the third derivative both exists, second derivative is equal to 0. So, that is a necessary condition for point of inflection. So, that has to be satisfied. So, if the second derivative is equal to 0, but the third derivative is not 0, then the point c will be a point of inflection.

So, this necessary condition in is in terms of the second derivative, if the second derivative exists the points where points of inflection concave are the points where the equation f double dash equal to 0 are satisfied. Among these points if you want to check whether which are points of inflection are not, and if you are lucky enough to have the function having say third derivative, then you look for the condition that the third derivative or those points should not be equal to 0. So, the points where it is not equal to 0 will give you points of inflection.

So, these are necessary and sufficient conditions in terms of higher order derivatives for a function to analyze points of inflection, right.

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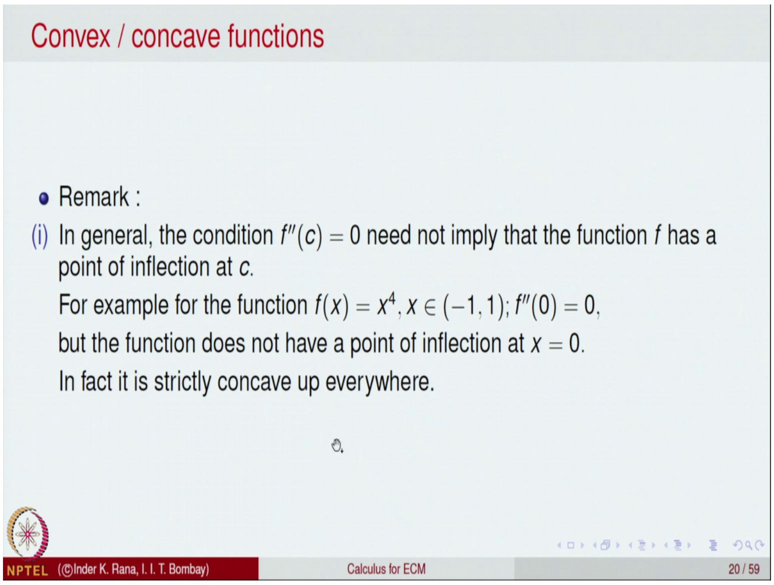

So, here are some remarks questions, that the condition that second derivative equal to 0 need not imply the function as a point of inflection at c. So, like we said for the local maxima minima first derivative equal to 0 at a point gives you possibility of that point to be local maxima or minima. It does not say it should be. So, because that is only a necessary condition. Similarly, f double dash of c equal to 0, only says that c possibly can be a point of inflection. One has to check by the conditions by the definition that this is a indeed a point of inflection, it may not be. So, not all the points where the second derivative is equal to 0 will be points of inflection.

So, here is an example look at the function f x is equal to x to the power 4, in the interval minus 1 to 1. It is first derivative is equal to 4 x cube, and second derivative will be equal to 12 x square. So, 12 x square equal to 0 gives you second derivative to be equal to 0 at the point 0. So, second derivative is 0 at the point 0, but second derivative is not a point of inflection, because for this function second derivative is everywhere positive. So, it is concave up function. So, it is not a point of inflection actually x is equal to 0 a is a point of absolute minimum for the function f of x is equal to x 4. So, the condition that second derivative is equal to 0 is only a necessary condition it need not be it does not say all such points will positively be points of inflection as this example illustrates. So, as we said it is a strictly concave up everywhere concave up.

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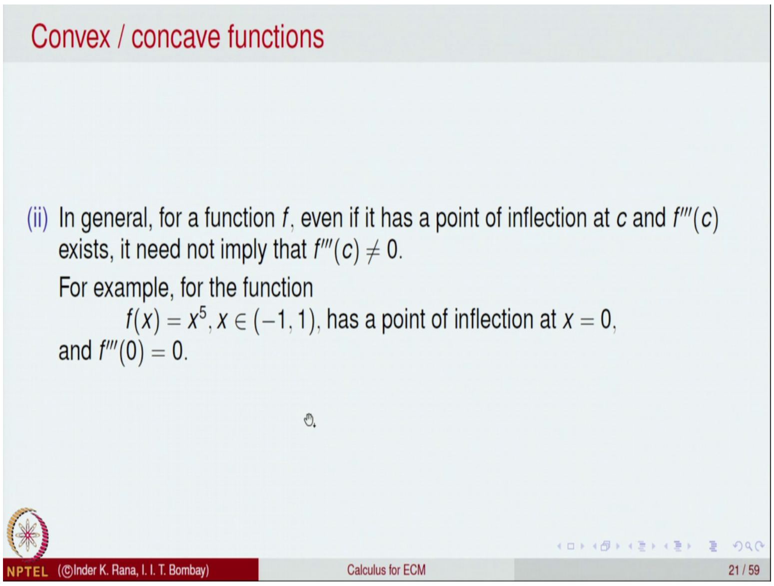

In general, for a function, even if it has a point of inflection at c, and the third derivative exists it need not imply third derivative is not equal to 0. So, that condition was only a sufficient condition it is not necessary condition that f (Refer Time: 23:52) point c if this is third derivative is not 0, then if the function has a point of inflection at c, then the third derivative should not be equal to 0. So, for example, you can take x to the power 5, and look at this. So, this will have a point of inflection at 0, but the third derivative at that point is equal to 0. That is a simple exercise you can just compute and check (Refer Time: 24:19).

So, be careful when applying the conditions for local maxima local minima points of inflection convexity and concavity of functions; that ensure what is the condition whether it is a necessary condition or it is a sufficient condition. So, to possible candidates for local maxima minima and points of inflection are given by the conditions namely first derivative equal to 0 and correspondingly second derivative equal to 0. But not all possible values will give you local maxima minima or inflection points, you have to check by definition or by the test that indeed that is the case.

So, that is how convexity and concavity and calculus helps. Convexity and concavity, are I said are final points of calculus. Namely, it tells you whether the graph of a function is bending towards or away from exists. And in economics and commerce scenario, it tells you the portion where the mathematical it is a second it is the first derivative is increasing or decreasing. Since in economics scenario the first derivative is the marginal of whatever quantity you are looking at. So, whether the marginal is increasing or marginal is decreasing, that is the consequence of convexity and concavity in the economics problems. So, we will continue our study in the next lecture.

Thank you.