Mathematics in India: From Vedic Period to Modern Times
Prof. M. D. Srinivas
Department of Mathematics
Indian Institute of Technology - Bombay

Lecture - 27
Ganitakaumudi of Narayana Pandita 3

So this is the 3rd lecture on Ganitakaumudi of Narayana Pandita.

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Outline

- Outline of *Gaṇitakaumudī*
- *Vargaprakṛti*
  - Nārāyaṇa's variant of *cakravāla* algorithm.
  - Solutions of *Vargaprakṛti* and approximation of square roots
- *Bhāgādāna*: Nārāyaṇa's method of factorisation of numbers.
- *Aṅkapāśa* (Combinatorics)
  - Enumeration (*prastāra*) of generalised *mātrā-vṛttas* (moric metres with syllabic units such as *pluta* etc., in addition to *laghu* and *guru*).
  - Some sequences (*paṅkti*) and tabular figures (*meru*) used in combinatorics.
  - Enumeration (*prastāra*) of permutations with repetitions.
  - Enumeration (*prastāra*) of combinations.

As you can see the lectures on magic squares are also essentially on the last chapter of Narayana Pandita's Ganitakaumudi.

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Gaṇitakaumudī of Nārāyaṇa Paṇḍita (c.1356)

Gaṇitakaumudī, with about 475 *sūtra* verses (rules) and 395 *udāharaṇa* verses (examples), is a much bigger work than Bhāskarācārya's *Līlāvatī*. It is divided into the following 14 *vyavahāras*:

1. *Prakīrṇaka-vyavahāra* (Weights and measures, logistics) – 63 rules and 82 examples
2. *Miśraka-vyavahāra* (Partnership, sales, interest etc) – 42 rules and 49 examples
3. *Śreḍhī-vyavahāra* (Sequences and series) – 28 rules and 19 examples
4. *Kṣetra-vyavahāra* (Geometry of planar figures) – 149 rules and 94 Examples
5. *Khāta-vyavahāra* (Excavations) – 7 rules and 9 examples
6. *Citi-vyavahāra* (Stacks) – 2 rules and 2 examples
7. *Rāśi-vyavahāra* (Mounds of grain) – 2 rules and 3 examples

So Ganitakaumudi written around in 1356, we do not know where Narayana Pandita lived, majority of his manuscripts are found in western India Gujarat, Rajasthan, etc. but maybe he was even in Bihar or somewhere. Ganitakaumudi has a 475 sutra verses and 395 udaharana verses, so it is about 4 times the size of the Lilavati. And it is divided into 14 chapters, it was outlined in the previous talk also it is good to recollect them.

Weights and measures, partnership, sales, interest etc. sequences and series. Geometry of planar figures, which is one of the major chapters in the Ganitakaumudi 149 rules and 94 examples it is almost one third of the number of rules are there. Excavations, stacks, mounds of grain.

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Gaṇitakaumudī of Nārāyaṇa Paṇḍita (c.1356)

8. *Chāyā-vyavahāra* (Shadow problems) – 7 rules and 6 examples
9. *Kuṭṭaka* (Linear indeterminate equations) – 69 rules and 36 examples
10. *Vargaprakṛti* (Quadratic indeterminate equations) – 17 rules and 10 examples
11. *Bhāgādāna* (Factorisation) – 11 rules and 7 examples
12. *Rūpādyaṃśāvatāra* (Partitioning unity into unit-fractions) – 22 rules and 14 examples
13. *Aṅkapāśa* (Combinatorics) – 97 rules and 45 examples
14. *Bhadragaṇita* (Magic squares) – 60 rules and 17 examples

Nārāyaṇa has written his own commentary, *Vāsanā*, which presents the working and solution of all the examples.

The shadow problems, after that Kuttaka, Vargaprakrti, then factorization Bhagadana. Rupadyamsavatara partitioning of unity into various fractions they are very interesting results most of them go back to (FL) actually. Ankapasa combinatorics, then Bhadraganita magic squares. So combinatorics is the next major chapter after (FL) 97 rules, Bhadraganita has 60 rules. Narayana Pandita has written his own commentary Vasana like Bhaskara did.

But it is merely on solutions of the examples that he deals with. So I will be mainly discussing Vargaprakrti, Bhagadana, Ankapasa in this talk. In the other talk on magic squares Narayana's contribution at this chapter on Bhadraganita will be discussed in detail.

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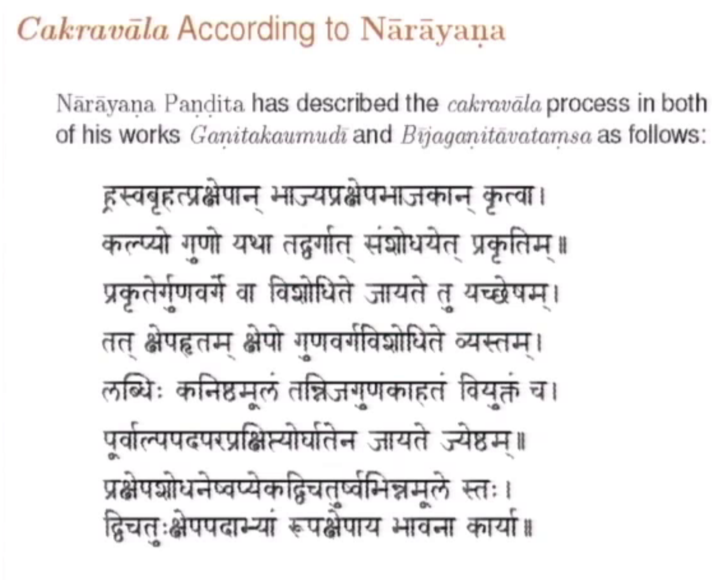

So Narayana brings in cakravala also into Ganitakaumudi, Ganitakaumudi as you know is a pati-ganita text, so it is basically arithmetic and geometry. Lilavati was also pati-ganita text, but Bhaskara brought in kuttaka into Lilavati, kuttaka is the brahmagupta thought of kuttaka was the generic name brahmagupta used for whole of algebra, but Bhaskara brought in kuttaka as a chapter in Lilavati.

And this discussion of kuttaka in Lilavati is same as what he has discussed about kuttaka in his bijaganita also. Then the idea was that the kuttaka problem, one does not need to use algebra symbols, once one knows the kuttaka algorithm one only deals with the numbers and work out the solution of the indeterminate equation, and in the indeterminate equation is also posed as a problem to start with, so one does not need to do algebra.

Similarly, Narayana on the has brought in Vargaprakrti into the pati-ganita text, so he has written a separate book on algebra called bijaganita-vatamsa, unfortunately only the first few chapters of it are available, what is available is only up to cakravala. And a little bit of the (FL) is available, so all the for the discussion on (FL) etc. it would have been perhaps as much of the expansion of Bhaskara's bijaganita, his bijaganita-vatamsa would have been very valuable if the full manuscript where found.

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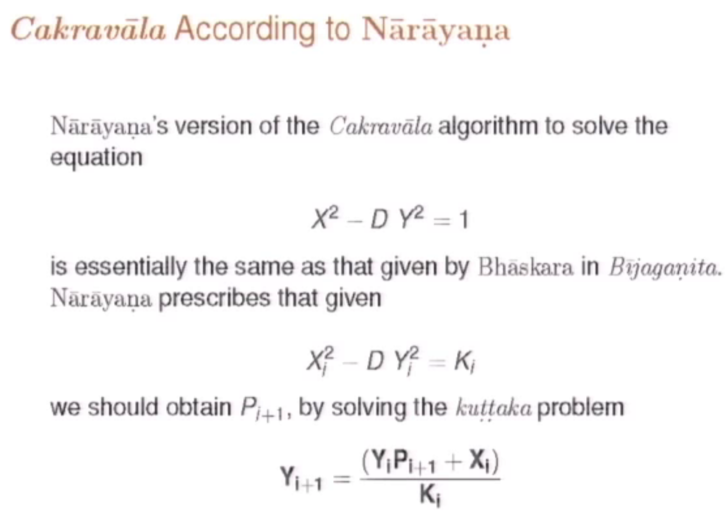

So his description of cakravala is more or less similar to what Bhaskara has said, but he leaves this issue whether this Pi+1 square should be > the, or should be < the in a more ambiguous manner in the statement of cakravala. So again we want to solve the equation X square-D Y square=1, so in the step Xi square-D Yi square =Ki, you want to obtain Pi+1, so you obtain Pi+1 by solving the kuttaka problem Yi+1=Yi Pi+1 +Xi/Ki.

So Pi+1, and Xi+1 are not known, Yi, Xi and Ki are already known integers, so we have 2 unknowns it is a standard kuttaka problem.

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Cakravāla According to Nārāyaṇa

Then Nārāyaṇa states that P_{i+1} is to be so chosen that

$$K_{i+1} = \frac{(P_{i+1}^2 - D)}{K_i} \quad \text{or} \quad K_{i+1} = \frac{(D - P_{i+1}^2)}{K_i}$$

adding that, in the later case, the *kṣepa* (K_{i+1}) should be taken with the opposite sign (*vyasta*). Thus, according to Nārāyaṇa,

P_{i+1}^2 may be chosen to be greater than or lesser than D.

Nārāyaṇa also gives the relation

$$X_{i+1} = P_{i+1} Y_{i+1} - K_{i+1} Y_i$$

Then once Pi+1 is found there are many solutions to kuttaka, so you choose the solution such that P square-D/K or D-P square/K is small, so Ki+1= Pi+1 square-D/Ki or D-Pi+1 square/Ki, and so in the other case ksepa will be negative, so here just said Pi+1 square may be chosen to be > or < D, then of course Xi+1 is related to the known quantities. Now what actually Narayana meant by his condition we can understand by looking at the examples that he has worked out.

So he has worked out as you see the examples are given in Ganitakaumudi, the solution is discussed by in his vasana commentary, so by putting them together you can find out what Narayana intended to do.

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Nārāyaṇa's Example: $X^2 - 103\ Y^2 = 1$

i	P_i	K_i	a_i	ε_i	X_i	Y_1
0	0	1	10	1	1	0
1	10	-3	7	1	10	1
2	11	-6	3	-1	71	7
3	7	9	2	1	203	20
4	11	2	11	-1	477	47
5	11	9	2	-1	5044	497
6	7	-6	3	1	9611	947
7	11	-3	7	-1	33877	3338
8	10	1	20	1	227528	22419

At step 4, we can use *bhāvanā* to obtain the result directly

$$X = \frac{(477^2 + 103.47^2)}{2} = \frac{455056}{2} = 227528$$

$$Y = 2.477.\frac{47}{2} = \frac{44838}{2} = 22419$$

The sequence of steps in this example is the same as would follow from Bhāskara's prescription that P_{i+1} is so chosen that $\left| P_{i+1}^2 - D \right|$ is minimum.

So first example is 103, so in this example again more or less you start with 10 for X and Y, then you put K=1 P=0, and go by the same method which I discussed now about 7-8 times but it does not matter we can do it once more. So 0+10 is divisible by 1, 10 square is closest to 100, and then this 3 is obtained by taking 10 square-103/1 you get-3, immediately the solutions are 10 and 1. And in the next step to 10 a quantity should be added such that the sum is divisible by 3.

So 10+ 11 you could have had 10+ 8, you would have had 10+ 14, so between 8 or 11 or 14 it is 11 whose square is closest to 103 again the same thing, so you choose 11, once you chose 11, 11 square-103/-3 is-6, 11 square-103=121-103 it is 18, 18/-3=-3. So in this stage you have 71 square-103*7 square is-6, so you proceed like that and you can stop at the point where you get K=2 at the step 4, 477 square-103*47 square is = 2, and so you can do bhavana.

And immediately go to the final solution which is to 227528, 22419 of course if you went through the cakravala you would have obtain the same thing. So in this example there is no distinction between the way Bhaskara would have done and in the way Narayana has done the example, so this procedure seems to be the same, but it is not so simple. Narayana has worked out another example.

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Nārāyaṇa's Example: $X^2 - 97\ Y^2 = 1$

i	P_i	K_i	a_i	ε_i	X_i	Y_1
0	0	1	10	1	1	0
1	10	3	7	-1	10	1
2	11	8	3	-1	69	7
3	13	9	2	-1	197	20
4	14	11	3	-1	522	53
5	8	-3	2	1	847	86
6	10	-1	6	-1	5604	569

Note that, in step 3, following Bhāskara's prescription, P_3 can be either 5 or 13.

And this example is X square-97 Y square=1, this example throws up many, many interesting features, so instead of explaining the starting step which is known second step which is also similar. In this step from 11 to 13 is a very interesting feature to start with, so in this step your P2 is 11 and so you are at the stage where 69 square-97*7 square is =8, so you are in this stage, so now to 11 you add a number such that 11 +that number is divisible by 8, so 11+ 5 is possible 11+ 13 is possible.

Then 11+ 21 is possible, 5 squared is 25 97-25 is 72, 13 square is 169 169-97 is also 72, so it is the first time where you are finding an ambiguity Pi+1 square when it is taken above D or below D is equidistant from the D, so what you do Bhaskara is silent on it, now in this example Narayana just given that you take the larger one, in fact Krishnaswami Iyengar in his analysis also has given an algorithm where you are choosing the larger one.

So both 5 and 13 the squares of both these numbers are equidistant from 97 you could have gone ahead and chose and made a rule that you choose the lower also you will immediately you will learn in the eventually you will lead to the proper solution of the Vargaprakrti equation, so this was one thing which was interesting. Now the second thing is even more interesting the step from 3 to 4, so to 13 you add a number such that the sum is divisible by 9.

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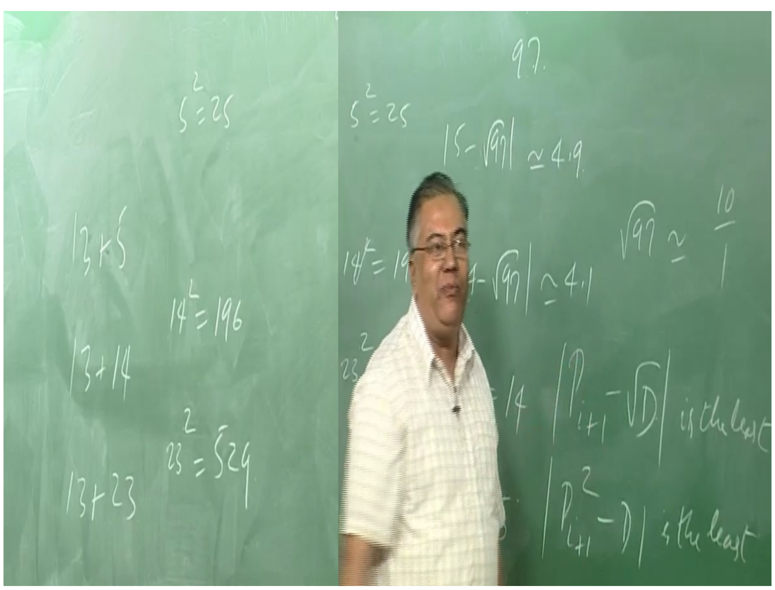

So to 13 what are all the numbers you can add 13+ 5 is divisible, 13+ 14 and 13+ 23, now 5 square is 25, 14 square is 196, 23 square is 529, now your number is 97, so if you proceed the Bhaskara way you can see that 25 is closer to 97 than 196 right, so if you had gone the Bhaskara at this stage you would have to put 5 nothing would have been lost I mean eventually you will get this solution, it is just the specification of the algorithm.

But Narayana saying no we are going to take 14 here, now is there something interesting about 14, now what is the solution at this stage till now we have the solution is 69/7, 69/7 is telling you various steps are what 10/1, then the next approximation is 69/7 these are all better and better approximations to square root of 97, 197/20 right, so this is actually of the order of 9.85 right root 97 is of the order of 9.85, why I am saying this you understand in a minute.

You try to find out or you try to find out, so I will just take the modulus here in this situation if would take 5 okay root 97 is of the order of you have seen it is 9.85, so modulus 5-root D is of the order of 4.9, modulus 14-root D let us write it exactly why write root D, what is this order of 4.1 **"Professor - student conversation starts"** (()) (12:11) you can take 69 upon 7 how different or how much is it different 9.8 anyway you need to know that is more than 9.5 or less than 9.5.

If it is more than 9.5 this is closer to 14 is closer to root D, if it is more than 9.5 this is further away from root D, so if your condition is this is the least then you pick this, if your condition is so you will get P4 step you are doing is P4=5=14 in this case P4=5 minutes case, do you understand what I am saying? **"Professor - student conversation ends."** So if you had this is a Bhaskara condition.

Narayana in this step is choosing this condition which if you implement continuously it leads to a continued fraction known as the nearest integer continued fraction, this is the nearest integer continued fraction algorithm. So what Narayana seems to be doing in this example of 97, this is the only place very differs otherwise, because these 2 conditions are really very close to each other, only rarely you will find them deviating from each other the solution being different from each other.

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Nārāyaṇa's Example: $X^2 - 97\,Y^2 = 1$

In step 4, if we had used Bhāskara's prescription that $\left| P_4^2 - 97 \right|$ is minimum, along with the condition that $13 + P_4$ be divisible by 9, we would obtain $P_4 = 5$ and not 14 as found above.

While
$$|5^2 - 97| < |14^2 - 97|,$$
we also have
$$|14 - \sqrt{97}| < |5 - \sqrt{97}|.$$

In the above example, Nārāyaṇa seems to be using the prescription

(II") $\left| P_{i+1} - \sqrt{D} \right|$ is minimum.

Which leads to the so called Nearest Integer Continued Fraction expansion for $\sqrt{D}$.

Thus, by giving these two examples, Nārāyaṇa seems to be indicating the possibility of there being different variants of the *cakravāla* process.

And therefore in this example Narayana is hinting that you can use the nearest integer algorithm, you can use this as the condition for cakravala instead of this, so by this Narayana just opening up that the cakravala process is indeed is a process where you can discover several kinds of algorithms, and Bhaskara's algorithm is one such which very specific of course, he is hinting that another kind of by giving this example he is not making much more statement than that.

But the definition of cakravala was also in such a way that he did not say this condition unambiguously, he just set that this Pi+1 should be so chosen that its square over around D, and so that is the way he has worked it out.

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Rational Approximation of Square-Roots

We had noted that if X, Y are integers such that $X^2 - D\,Y^2 = 1$, then,
$$\left| \sqrt{D} - \frac{x}{y} \right| \leq \frac{1}{2xy},$$

Nārāyaṇa Paṇḍita gives an instance of how better and better approximations are obtained by successive applications of *bhāvanā*.

Now, $x^2 - 10y^2 = 1$ has solutions $x = 19$ $y = 6$.

By doing *bhāvanā* of this solution with itself

$x_1 = 19^2 + 10.6^2 = 721$ and $y_1 = 2.19.6 = 228$

By doing *bhāvanā* of these two sets of solutions, we get

$x_2 = 19.721 + 10.6.228 = 27379$ and $y_2 = 19.228 + 6.721 = 8658$

Thus, we have successive approximations

$\sqrt{10} \approx \frac{19}{6}, \frac{721}{228}, \frac{27379}{8658}$

The other thing this is well known we have talked about it several times that the solution of cakravala gives good approximation to square root of D, and given one approximation you can work out better and better ones by bhavana. Narayana is the first person who was sort of stated it all this very explicitly, and he has taken the example of square root of 10, it so happens for all his geometry argument Narayana is using the Jain value of square root of 10 for pi.

And so square root of 10 happens to be a important parameter in his geometry, so he has just worked out 19 and 6 is 1 solution, if you do bhavana of that with itself you will get 721/228. If you do the bhavana of these 2 of this with the other one, you get 277/8658 and so on, so he says you can work out better, better square root, he just made this all this very explicit.

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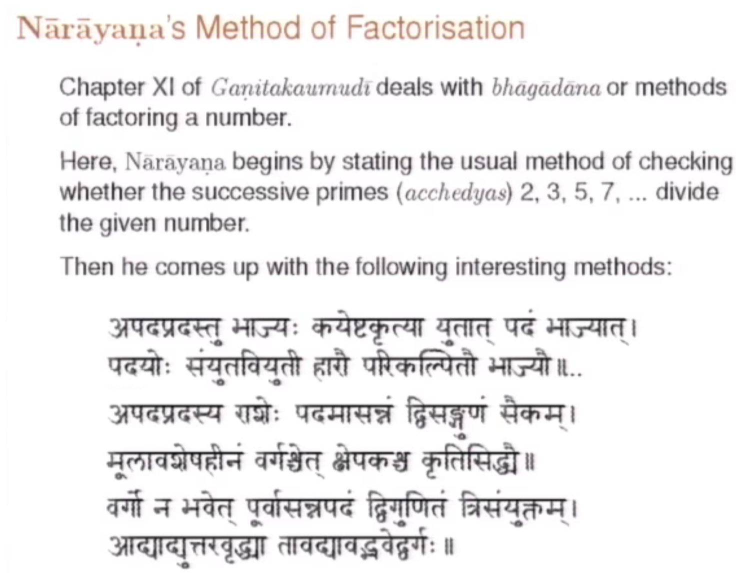

Now we come to Narayana's method of factorization, there is something interesting that he does, so to start with in fact this is the first book in Indian mathematics which seems to talk about the issue of factorization. In Greek mathematics Euclid's book itself starts by talking about factors and primes and things like that, all the discussion is generally missing in Indian Mathematics book. The first time it occurs but Narayana does bring in something very special when you start the subject.

So what he says? The usual procedure that when you want to factorize the number you divide it try dividing it by successive acchedyas this is the name he calls to primes numbers which are not divisible, acchedyas 2, 3, 5, 7 divide this is a well-known sieve of Eratosthenes or whatever standard method for trying to find the factors of a number. Then he comes with the following interesting methods, I will skip the Sanskrit.

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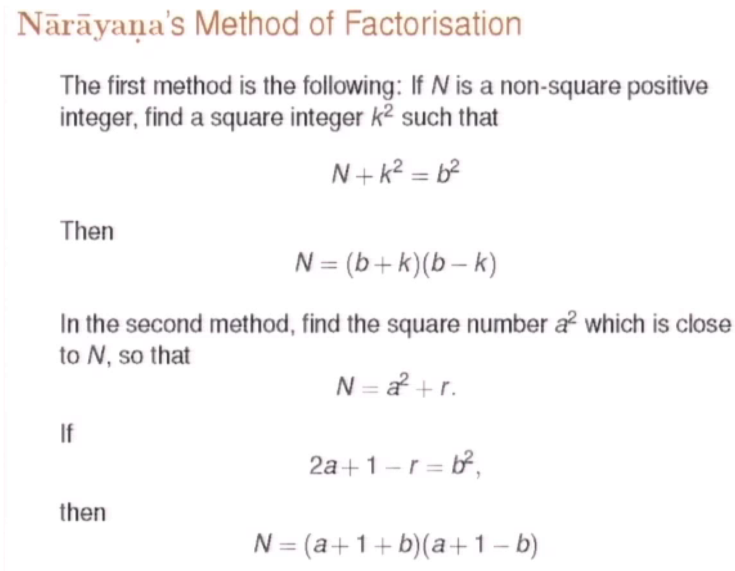

First method he saying is, if N is an non square positive integer, find the square integer such that N+K square is also square, then N=b+k*b-k of course if you succeed then it is well and good you do not succeed this is somewhat simplistic. Then second method N is the number you want to factorize, so the first starting method what you start with 2, 3, 5, 7 go on dividing, so happens that when your first factor itself is very high how high it can be the lowest factor number N which is not prime is of the order of root n, so it can become fairly high.

So you are thinking of factors which are fairly high that is when these methods come important. So the next one is think of N as a square +r, so take out the nearest square from it, suppose it happens that 2a+1-r=a square suppose it happens, then of course n =a+1+b*a+1-b again it is a lucky chance it may happen it may not happen. Now Narayana comes with the exhort of an algorithm, what you can try and do of course it is not a simple thing.

Factorization as all of you know is indeed a difficult problem, but Narayana is giving some systematic method when the factors of an N are fairly large.

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Nārāyaṇa's Method of Factorisation

The above method is then generalised as follows:

Keep adding the arithmetic sequence of numbers $2a+1, 2a+3, \ldots$, with common difference 2, such that their sum minus r becomes a perfect square, b^2. That is

$$(2a+1)+(2a+3)+\ldots+(2a+2m+1)-r=b^2$$

Then we can easily see that

$$N=(a+m+1+b)(a+m+1-b)$$

The above method, which is useful when the factors of N are large, was rediscovered by Fermat in 1640.

So he says keep adding the arithmetic sequence of numbers 2a+1, 2a+3 with common difference such that their sum-r, r is that number a square +r a is obtained by that such that their sum-r becomes a perfect square, so that is 2a+1+2a+ 3 etc. 2a+2m+1 so you have gone m terms in this arithmetic sequence -r=b square, then N will be a+m+1+b a+m+1-b, so this method actually was rediscovered by Fermat, Fermat seems to be re-anticipating rediscovery many things that where earlier done in Indian tradition.

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Nārāyaṇa's Method of Factorisation

Nārāyaṇa's Example 1: To Factorise N = 1161

Here $1161 = 34^2 + 5$

Further $2.34 + 1 - 5 = 64 = 8^2$

Hence, 1161 = (34+1+8) (34+1-8) = 43.27 = 3.3.3.43

Nārāyaṇa's Example 2: To Factorise N = 1001

Here $1001 = 31^2 + 40$

We first note, $2.31 + 1 - 40 = 22$ is not a square

We hence calculate

$$63 + 65 + 67 + \ldots + 89 - 40 = 32^2$$

Therefore

$$1001 = (31+13+1+32)(31+13+1-32) = 77.13 = 7.11.13$$

So Narayana is given 2 examples they are not complex, but you can work out more complex examples by what is today known as the Fermat method of the factorization. So 1161, 1161 closest square is 34 square+5 here you are lucky 2*34+1-5 is already a square, so you got factor straight away. Now it is 1001 nearest square is 31 square has taken 31 square+40, now 2*31+1-40 is not a square.

So start this arithmetic sequence cranking it keep on going sum as many terms such that those sum-40 is a square, so when it happens you will have a factorization of 1001, you have gone 13 terms in this sequence, so 31+13+1+32 31+ so finally you actually find that 7 itself was a factor, so here there was no need to going to do this, but it is an illustration of the method. So Narayana is the first Indian mathematician to systematically talked of factorization of a number and he has come up with the interesting way of a attempting it.

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Aṅkapāśa: Combinatorics

In Chapter XIII of *Gaṇitakaumudī* on *Aṅkapāśa*, Nārāyaṇa Paṇḍita gives a general mathematical formulation of most of the combinatorial problems considered in earlier literature.

After listing the various *pratyayas*, Nārāyaṇa defines various *paṅktis* (sequences) and *merus* (tabular figures) that are going to be useful in different combinatorial problems.

Nārāyaṇa then considers different kinds of *prastāras* which generalise those considered in prosody and music.

He formulates the problem as one of enumerating the various possibilities which arise when there are p slots or places (*sthānas*) in which the q digits $1, 2, \ldots q$, are being placed, subject to various conditions.

Now we come to the chapter on combinatorics, it is a very long chapter and actually a very profound chapter. What Narayana Pandita does is he will give a general mathematical formulation to all the combinatorial problems that where earlier considered in the literature starting from **(FL)** to music and in mathematics, he tries to give a general mathematical formulation.

So in the beginning the chapter starts with defining the various pratyayas the combinatorics so that is **(FL)** all that what do they mean, then he starts by giving auxiliary mathematical quantity, so he first defined various sequences, so in the previous talk on Ganitakaumudi there was an exposition of some of this, so we can continue on that, and merus that is some tabular figures that are going to be useful in combinatorial problems.

So he defines various kinds of sequences panktis, then he starts with different problems, the way he tends to pose the problem is following, so in (FL) the objects where syllables and there were laghu and guru, in (FL) the objects where syllables and the objects where laghu and guru. In permutation the (FL) the objects where swara saregama, and (FL) the objects where the (FL), in each case you were putting them in various orders this object.

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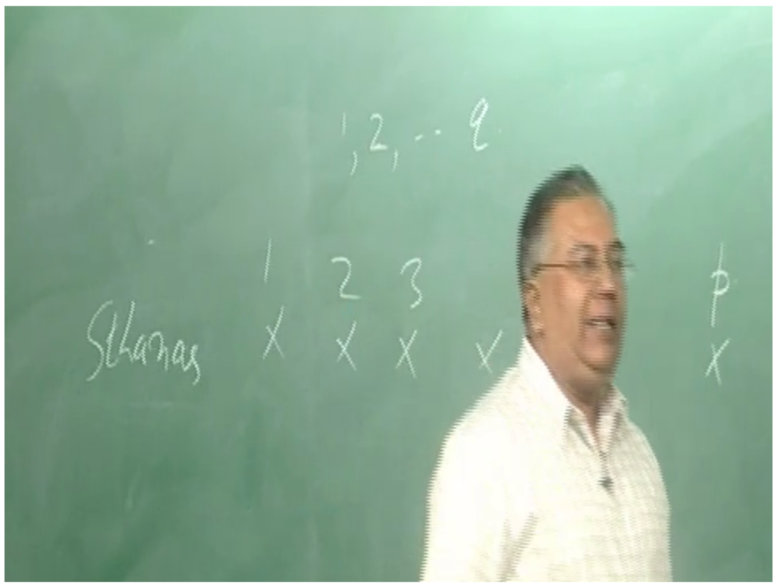

So now he says let me just think of the objects as ankas digits, so we will have digits 1 to q, so these are my objects, so if it is laghu, guru just called them 1 and 2, if it is saregama call them 1, 2, 3, 4 of course in the end of the chapter he says there is a limitation in what I have done, this ankas are only 9, anyway but you can think of larger ways of doing it okay in the end of the chapter. But you think of this, so the objects are and there 1 to q and there are some p places sthanas.

So there are p sthanas in which you place them, and then you put various conditions whether the sum of all the ankas should be constant, or whether all the ankas should come only once, or whether the some of the ankas need not be constant, but the number of paces is fixed, so each of them comes to give you different problems. So in the varuna-vrttas the number of places is fixed in the 3 syllable prastara, 4 syllable prastara, 5 syllable prastara, in each prastara there is a particular P which is fixed.

And the objects varna-vrtta was laghu and guru only, Narayana says we can generalize it you can think of pluta which is the longer syllable than laghu and guru, you can have any way q syllables. So in the varna-vrtta the problem they finally you came up with binary numbers, now the representation that will come is representation of every number to base q or radix q, so that will be the generalization of the varna-vrtta problem.

Now permutation anyway 1 to q are kept in q places, and then they can be permuted, then permutations with repetition also, matra-vrtta problem p is not fixed 1 to q are placed repetitions are allowed sum is a constant that is the value in matra-vrtta total matras (FL) also will come in that only one thing will not come, (FL) generalization of 1, 2, 4, 6 will not come because he is considering all integers ankas sequentially from 1 to q.

So this is the mathematical formulation of all the known prastara problems in terms of digit, so q digits p places, prastara is done under various conditions.

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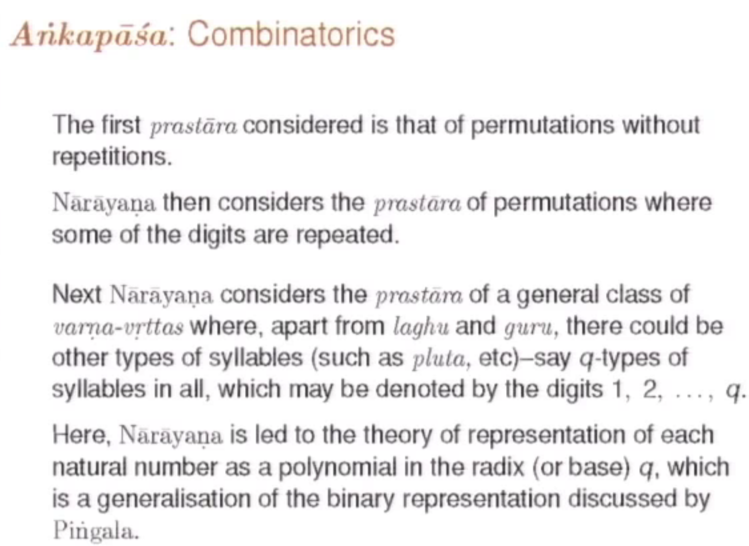

So the first prastara is a permutation without repetition, then he considers prastara of permutations with where digits where repeated. Next, he considers the prastara which is generalization of varna-vrtta, so where instead of laghu and guru there can be other kinds of syllables pluta etc. so in general say q types of syllables 1 to q and this leads into a theory of representation of each natural numbers as a polynomial in base q or radix q, which is a generalization of the binary representation of pingala.

So he is just working on all this as a mathematician giving a complete mathematical framework for all the combinatorial problems.

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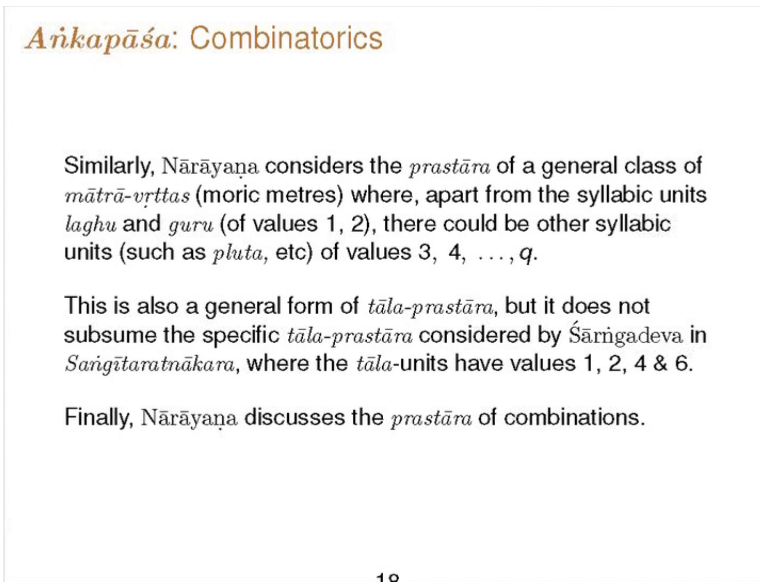

Then he goes to matra-vrttas, so here again instead of laghu and guru which are value 1 and 2 you think of more different kinds of syllables which are values of 3, 4, etc. up to q. Then finally so this is similar to matra-vrtta prastara and tala-prastara also but it does not subsume the Sarngadeva’s tala-prastara. Finally, he discusses prastara of combinations also, I will deal that in the end.

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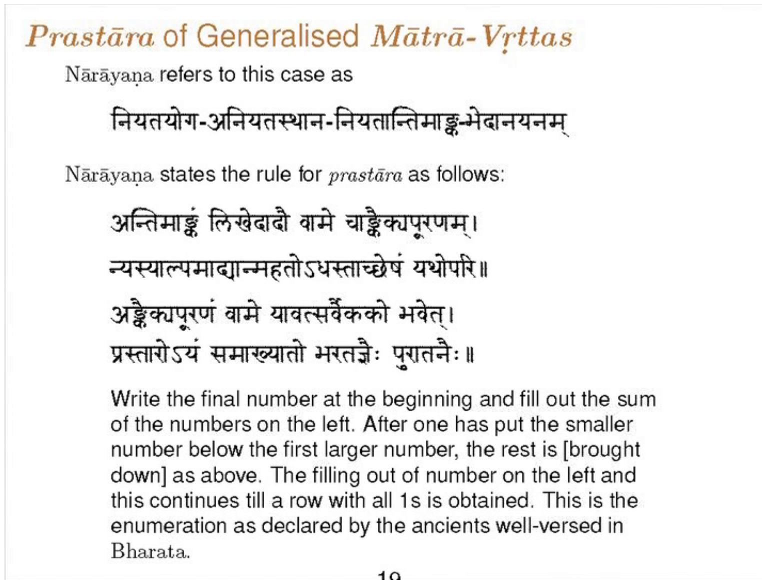

So first I will deal with this generalized matra-vrtta for 5-10 minutes, then we will look at the other things. So he refers to generalize matra-vrttas as (FL) that is the sum of the digits after I place them in these p places that should be fixed. (FL) so the final digit is q some fixed one like that, so you want only laghu and guru the final digit is 2, if you want laghu, guru, pluta final digit is 3 more generalizations you can go to q.

(FL) the number of places can be many because even in prastara of value 3, you can have 3 laghu which can take 3 places or a laghu and guru which take only 2 places right, so sthana is (FL) in his formulation he has put this the matra-vrtta. Then the rule for prastara is standard only he is extols Bharata this is the rule for prastara as declared by the ancient well versed (FL) so the rule is exactly what we had done earlier let us look at this example and understand the rule.

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Prastāra of Generalised Mātrā-Vṛttas

Nārāyaṇa has given the following example of a *prastāra* where the total value $n = 7$ and the highest digit is 3

1	133	12	2122	23	2131	34	3211
2	223	13	11122	24	11131	35	12211
3	1123	14	1312	25	1321	36	21211
4	313	15	2212	26	2221	37	111211
5	1213	16	11212	27	11221	38	13111
6	2113	17	3112	28	3121	39	22111
7	11113	18	12112	29	12121	40	112111
8	232	19	21112	30	21121	41	31111
9	1132	20	111112	31	111131	42	121111
10	322	21	331	32	2311	43	211111
11	1222	22	1231	33	11311	44	1111111

This is nothing but 7-*mātrā-prastāra* with *laghu*, *guru* and *pluta* being the syllabic elements.

So the prastara where the total value is 7 highest digit is 3, so you can think of it as laghu, guru and pluta, 1 is laghu, 2 is guru, 3 is pluta. The total sum is fixed that is 7 now, so you start with maximum of 331, what is the rule for matra-vrtta prastara from the left you start the first place where non one digit occurs, you put the digits just below that below 3 you put 2 bring down this 3 as it is, now fill it up 3+ 2=5 is already done so you can only put 2 there.

So next step below this 2 you will have 1, 2 3 is brought down you can only put 1 there. Next step below this 2 a 1 will come 3 can be brought down, only 3 can be put in the left, like that you go on you will see that you have about 7 such rows which have 3 as the last entry that is followed by about 13 rows, which have 2 as the last entry that is followed by the rest of the rows which is about 22+2=24 rows which have 1 as the last entry.

So this is the prastara this actually 7 matra-vrtta prastara with laghu, guru, pluta, the same rule for matra-vrttas we did earlier only we have instead of laghu and group I have also a pluta there is no change in that.

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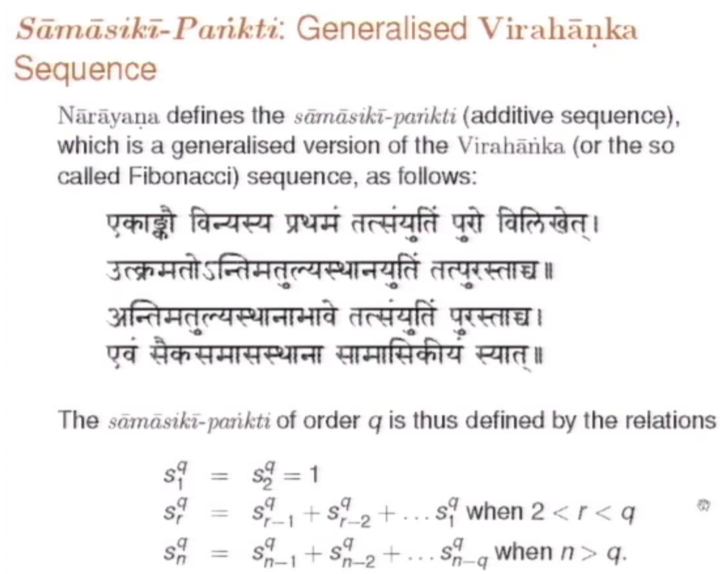

Now to understand the (FL) of this we go back in the beginning of the chapter itself Narayana has put a various sequence, so we will go back to this this samasiki-pankti, this was discussed in the last class, so let us look at it once again. (FL) so first 2 samasiki-pankti comes of 1 and 2 elements of this sequence are just 1, then up to the value q the Srq will be some of whatever are the previous elements which have come in the sequence.

Beyond q each element of the sequence is the sum of the previous q entries, so this is the qth order generalization of the Virahanka sequence, or we should call as what is generally known as the Fibonacci sequence S1q and S2q is 1, Srq is sum from S1 to Sr-1 r<q Snq, so when you put q=2 you will get virahanka the sequence the Fibonacci sequence.

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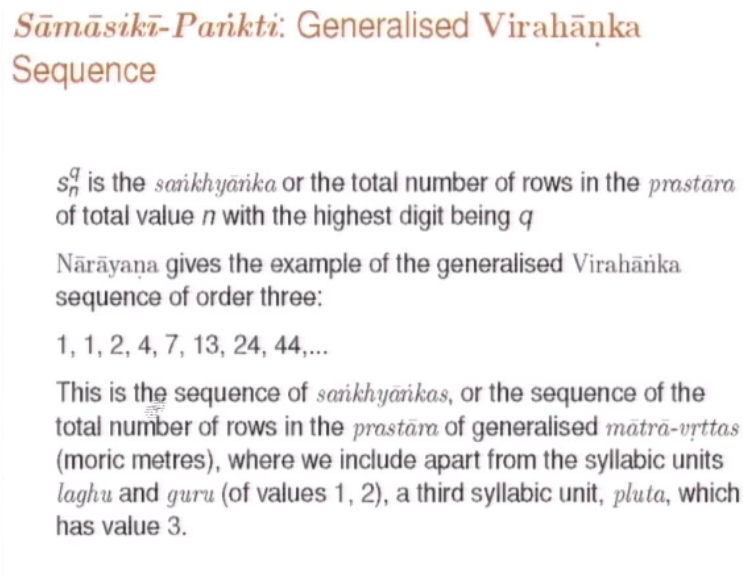

When you put q=3 this is what you will get, when you put q=3 you will get 1, 1, 2, 4, 7, 13, 24, 44 you can see that they were there in that prastara, there were 7 rows with value 6, there were 13 rows with value 5 and I am sorry there were 7 rows with value 4, 13 rows with value 5, 24 rows with value 6, in the right to the right of the first 7 rows you put a 3, to the right of the next 13 rows you put a 2, to the right of the last 24 rows you put 1, that way you generated that 47 row prastara (FL) 1 2 3.

So these are the generalized Virahanka numbers of all the 3, this is called the samasiki-pankti he defines it right in the beginning of the chapter where it has nothing to do with the problem of combinations or prastara, he just defines it generally, this is the general definition which is each number is that some of the previous q numbers after certain point.

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Sūcī-Paṅkti: Needle Sequence

Nārāyaṇa defines the *sūcī-paṅkti* (needle sequence) or the *nārācikā-paṅkti* (arrow-head sequence) as follows:

अन्तिममितवैश्लेषस्थानाङ्कमिताश्च ताः पृथक् स्थाप्याः ।
तासां घातः सूचीपङ्क्तिर्नाराचिका वा स्यात् ॥

If p is the number of places and q is the final digit, then the sequence is defined by the *vaiśleṣikī-paṅkti*, 1, 1, 1, ...1 (repeated q times), multiplied by itself p times, by the *kapāṭa-sandhi* (door-junction) method, or the algebraic method of multiplying keeping in mind the different place-values.

The $(r+1)$-th element of the sequence is a sum of multinomial co-efficients:

$U_{p,q}(r)$ = Coefficient of x^r in $(1+x+x^2+\ldots+x^{q-1})^p$

Now this is the another pankti which was also referred to in the previous talks, this is called suci-pankti this is called the needle sequence. (FL) it is also called the arrow sequence or the naracika-pankti, so 2 names for that. So how is it is to be done you first do the vaislesiki-pankti which is defined earlier that you put just 1 next to itself q times 1 is put next to itself q times that is called vaislesiki-pankti, then you multiply that by itself p times the elements of the naracika-pankti will come.

And the method of multiplication is the algebraic method of multiplication or what in India is called the door junction kapata-sandhi.

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Sūcī-Paṅkti

Nārāyaṇa gives the example of the needle-sequence when $p = q = 3$, which is worked out as follows:

				1	1	1
				1	1	1
				1	1	1
			1	1	1	
		1	1	1		
		1	2	3	2	1
				1	1	1
		1	2	3	2	1
	1	2	3	2	1	
1	2	3	2	1		
1	3	6	7	6	3	1

These are nothing but the coefficients of different powers of x in the expansion of $(1+x+x^2)^3$.

So let us first see that so p=3 q=3 the vaislesiki-pankti is 111, so first you multiplied by itself when you multiply it by itself you get this kind 111, 111, 111 so you add this you will get 12321, so this is the square of 111, you do it once more you get the cube of 111 that is 1367631. So you can see each term here is a coefficient in 1+x+x square to the power 3 coefficient of different powers of x in 1+x+x square to the power 3.

This 1367631 is the naracika-pankti or the suci-pankti when p=3 q=3, so like that he defines. So in general the needle sequence is given by this slightly different notation than the previous talk U p q r put as arguments of Ui put them as subscripts here, so Up, q of r is the coefficient of x to the power r in 1+x+x square+ x to the power q-1 whole to the power p as was emphasized by professor Sriram.

Narayana Pandita does not define it in this manner, but what he is saying is an algebra definition he says you take this vaislesiki-pankti multiplied by itself like a kapata-sandhi that means treat it as a polynomial and multiply okay.

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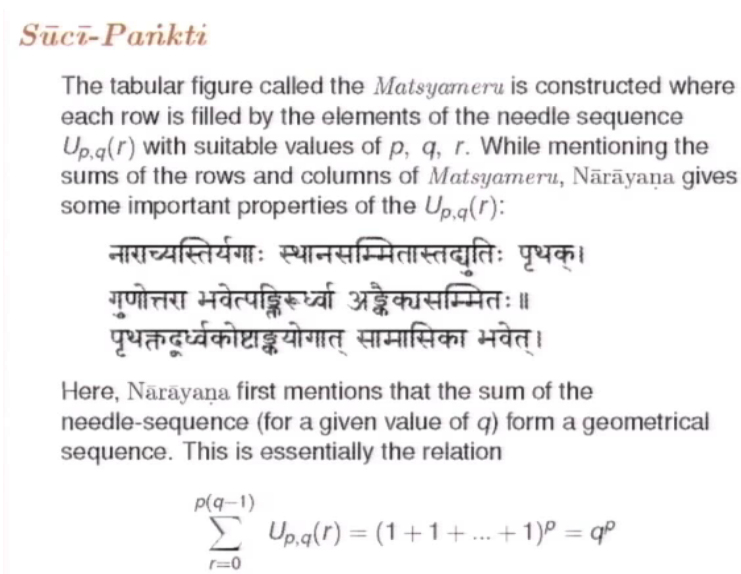

This what is the interesting thing about the suci-pankti is that beautiful relation, which was also defined in the previous talk. So this occurs in context of matra-vrttas I will explain the relation in a minute, before that what Narayana Pandita does he defines a tabular figure called Matsyameru which we will not display somewhat complex. Each row of the Matsyameru is filled by elements of this needle sequence Up, q r.

But anyway the Matsyameru is given by giving value of p and the value of q, p=3 q=3, p=5 q=2 whatever, then Narayan give some of the important properties of Up, q of r in the context of Matsyameru, while summing the rows of the Matsyameru, while summing the columns of the Matsyameru he gets 2 important relations. (FL) so then the rows of the Matsyameru are summed you will get (FL) means geometric sequence, you get the geometric sequence q to the power p q is fixed.

So when you sum the first row you will just get q, when you some the next row you will get q square, when you some the third row you get q cube, it is something like a multinomial theorem for q=2 this will be like the binomial theorem. Then (FL) when you sum the columns of the Matsyameru you will get the samasiki-pankti which are the generalized virahanka numbers that we talked about.

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Sūcī-Paṅkti and *Sāmāsikī-Paṅkti*

Nārāyaṇa then gives the following important relation between the coefficients $U_{p,q}(r)$ and the *sāmāsikī-paṅkti* or the generalised Virahāṅka sequence:

$$\sum_{r=0}^{n-t} U_{n-r,q}(r) = S_n^q,$$

where t is such that $(t-1)q < n \leq tq$.

This relation has the following interpretation in the *prastāra* of the generalised *mātrā-vṛttas*:

We can show that $U_{n-r,q}(r)$ is the total number of generalised *mātrā-vṛttas* of value n, and of length $(n-r)$, i.e., have $(n-r)$ syllables in all.

Therefore the sum of $U_{n-r,q}(r)$ for various possible lengths $(n-r)$, should be equal to S_n^q, which is the *saṅkhyāṅka* or the total number of the generalised *mātrā-vṛttas* of value n.

So the relation is something like this, sum of Un-r,q r r=0 to n-t is the generalized virahanka number Snq where t is such that t-1 of q<n<tq, so this is the relation between the samasiki-pankti and the generalized virahanka sequence which is the I am sorry this is the relation between suci-

pankti and samasiki-pankti that Narayana is the thinking putting everything and the general framework also and it is fairly abstract and complex, but it is very interesting.

Now what is his interpretation for matra-vrtta, Snq as we saw is the sankhya that is the number of rows in the prastara of a matra-vrttas which are of value n, where the syllabic elements you can choose have values from 1 to q or in ordinary language it is the number of rows when you list all the partitions ordered partitions of number n in terms of 1 to q, the repetitions are also allowed, so that is your Snq that is the generalized virahanka sequence or the generalized Fibonacci sequence.

Now Un-r, q r you can show by just simple algebra that it is the total number of generalized matra-vrttas of value n, where the vrttas of stipulated to have just length n-r when you sum the number of syllables of vrttas which are allowed from 0 to that n-t, if you will get the total number of matra-vrttas of value n. So this is by when vrttas are fixed length when you stipulate that the vrttas are of fixed length, this will be the number of matra-vrttas of length n-r of total value n where 1 to q are the syllabic values that are allowed okay.

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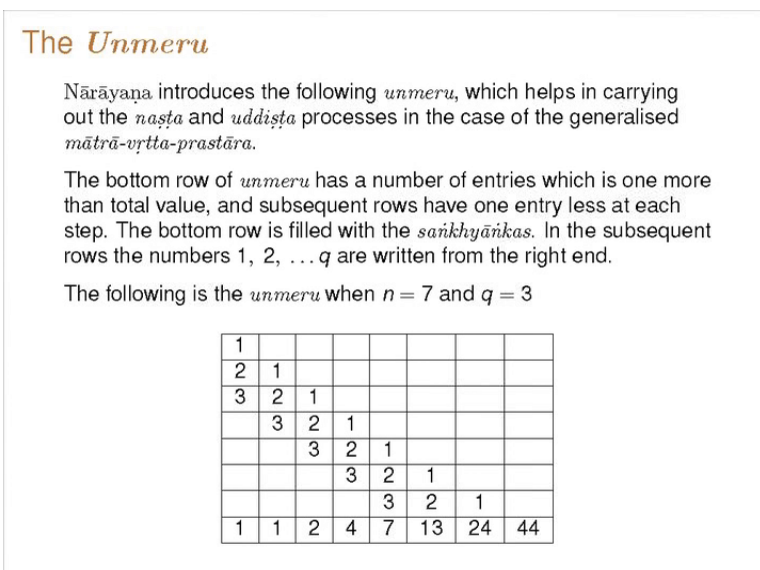

Now nasta and uddista also has to be done, and there also Narayana comes up with a very interesting tabular figure, and this tabular figure is again defined at the beginning of the chapter, this tabular figure is called unmeru the lofty meru. So what is the construction, the bottom row you put the sankhyankas the generalized virahanka sequence 1, 1, 2, 4, 7, 13, 24, 44 when n=7 and q=3, when q=3 you will get the sequence.

Now you are looking at the situation where n=7 put the 8 elements in the bottom row, to consider the problem of the nasta uddista problem, because what is the nasta uddista problem, we have this prastara I want to know where does 11212 appear, or I want to know what is 39th row, this problem I want to solve and Narayana solve this with the unmeru. So you put this sankhyankas 1, 1, 2, 4, 7, 13, 24, 44 brought up and from the top right 1 2 3 to the left, you are considering q=3 so you write 1 2 3 to the left.

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Naṣṭa Process with *Unmeru*

Nārāyaṇa's Example: To Find the 36th row of the 7-*mātrā-prastāra* with highest digit 3

44 − 36 = 8, 8 − 7 = 1, and 1 − 1 = 0

Thus the second 1 and 7 are *patita* and the rest are *apatita-saṅkhyāṅkas*.

1							
2	1						
3	2	**1**					
	3	2	1				
		3	**2**	1			
			3	2	**1**		
				3	2	**1**	
1	**1**	2	4	**7**	13	24	44

Note the entry (1) in the first row above the last *apatita* 24. In the row above the topmost entry of the column of that *apatita*, move left till you reach the column of the next *apatita* 13. Note the corresponding entry (1). And so on. Thus the metric form is 21211.

Now this is the meru it has been constructed now how do we use it to solve the nasta problem. So what is the example of Narayana find the 36th row of the 7 matra-prastara with highest digit 3, so you write this whole thing we want to calculate the 36th row, so what is the process start with 44 subtract 36 from it 8 is left, you can only subtract this 7 from the 8 1 is left, you can only subtract this 1 from this 1 0, so only 7 and 1 are the (FL) only those 2 are the fallen numbers.

All other numbers are apatita-sankhyankas okay. Now you start with first apatita-sankhyankas go up mark the first entry that comes there, then go to the top of that row go left reach the next apatita-sankhyankas mark that number there, go to the top of that go to the left reach the next apatita-sankhyankas mark the number there, go to the top of that go left reach the next apatita-sankhyankas row mark there, go to the top of that go left mark whatever you get in the next apatita-sankhyankas.

So the apatita-sankhyankas columns whatever comes they are all marked, so your final form is 21211, 21211 should be the 36th row of the seven matra-prastara. So 36th throw is used to identify the patita apatita-sankhyankas so only 2 sankhyas are patita 7 and 1, so let us see the 36th row is 21211 okay, so this is nasta with the unmeru.

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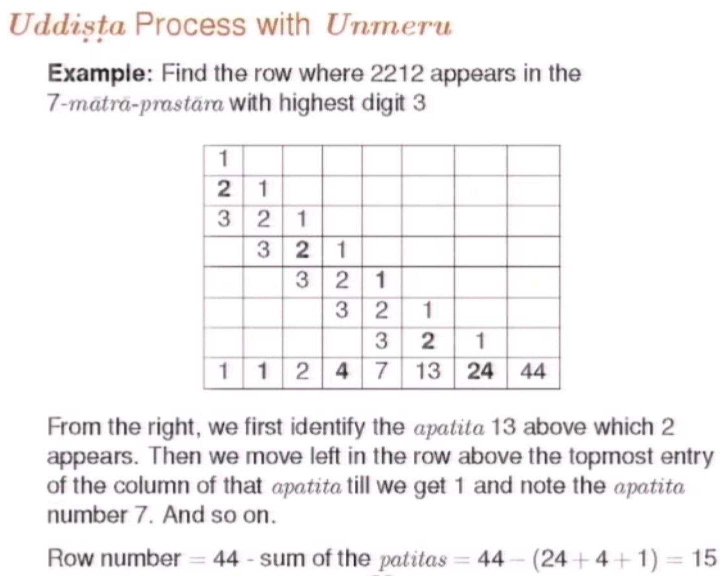

Now uddista, what is uddista given a forum, we have to find out the row in which it appears in the prastara, so again use the unmeru put down 1 to 44 here do the unmeru. Now what you do? You have to identify patita and the apatita-sankhyankas now using your pattern 2212. So start from find out the column in which 2 appears just above the sankhyanka, so that is this column, so this will be apatita-sankhyankas mark that 2 go to the top of that go to the left till you find the next 1 you want to 2122 and mark this, this is another apatita.

Then go to the top of that go to the left till you find the next 2 that is here this is another apatita, then go to the top of that go to the end of that till you find the last 2, so you have found your 2212 there arising in the columns 13, 7, 2 and 1 those are the apatita-sankhyankas, the patita-sankhyankas are the others 13, 7, 2 and I am sorry the patita-sankhyankas are the other those are

24, 4 and 1, so 44-sum of the patitas is 15, so 2212 should appear in the 15th throw of our prastara. So we can go back and check that 15th row is 2212.

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Uddiṣṭa Process: Alternative Method

We can devise an alternative method for *uddiṣṭa*, which is similar to the one we discussed for the usual *mātrā-vṛttas* (with only *laghus* and *gurus*).

Write the sequence of *saṅkhyāṅkas* such that three are written above 3, two above 2 and one above 1. Sum the second and third entries above each 3, the second entry above each 2, and subtract the sum from the *saṅkhyāṅka* of the *prastāra*, to obtain the row number.

Example 1: Find the row where 2212 appears in the 7-*mātrā-prastāra* with highest digit 3

1	**1**	2	**4**	7	13	**24**	44
2		2		1	2		

Row-number = 44 − (1 + 4 + 24) = 15

So there is much simpler way of doing this uddista following what we did in the case of the matra-vrttas, we also did the same thing in the prastara also, so I will describe the simpler method, so what you do write the sequence of sankhyankas, but write them in such a way that you write 3 sankhyankas above 3, 2 sankhyankas above 2, and 1 sankhyankas above 1, this is what we have done in the case of (FL).

We put 4 sankhyankas above the pluta, 6 sankhyankas above the pluta, we put 4 above the guru, 2 above the laghu and 1 above the (FL) or in matra-vrtta prastara we put 2 sankhyankas 1 above and 1 below the guru, and 1 sankhyankas only with the laghu, those is just a generalization of that, write 2 sankhyankas above 2, 3 above 3, 1 above 1, sum the second and third entries above each 3, second entry above each 2, and subtract the sum from the sankhyankas of the prastara which is 44, to obtain the row number.

So here are 1, 1, 2, 4, 7, 13, 24, 44, 2 are written above 2, another 2 are written above 2, 1 is written above 1, 2 are written above 2, so only the second entry above 2 is significant, and those are the patita-sankhyankas 24, 4 and 1, so 44-, so this example we had done just before 2212 we had identified with the 22nd row.

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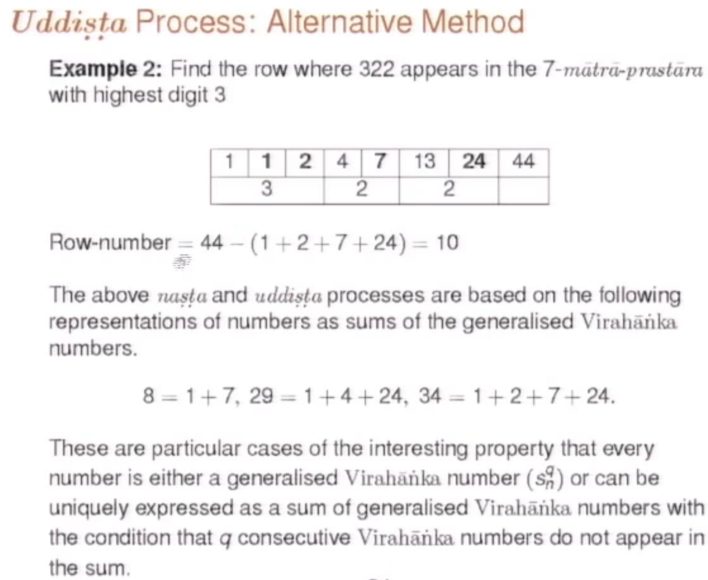

So let us do another example find the row in which 322 appears, in the 7 matra-prastara, so write 3 sankhyankas above 3, 2 sankhyankas above 2, 2 above 2, the second and third above 3, the second above 2 are the patita-sankhyankas they are 24, 7, 2 and 1, so you have so this should be the 10th row 322 should be the 10th row of the prastara. So we will again check the 10th row of the prastara is 322 okay. Now this again gives us another representation.

So what is happening 8 in our examples in this example this number 34 is being written as 1, 2, 7 and 24 it was written as a sum of the generalized virahanka numbers, in the earlier example we had written 8 as sum of 1 9 7, 29 as sum of 1, 4 and 24. So you can see another general theorem is behind the theorem is something like this that every number is either a generalized virahanka number or it can be written uniquely as a sum of the generalized virahanka numbers with the condition that q consecutive virahanka numbers will not appear in this sum.

So when q=2 2 consecutive Fibonacci numbers will not appear, then q=3 3 consecutive will not appear so like that so this is the general result which is behind this unmeru and nasta and uddista process.

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Prastāra of Permutations with Repetitions

Nārāyaṇa Paṇḍita explains that the same rule which is used to generate the enumeration (*prastāra*) of all the permutations of q different digits can be used to generate the enumeration (*prastāra*) of permutations where all the digits are not different. This is the same rule which we found in *Saṅgītaratnākara* for *tāna-prastāra*.

Nārāyaṇa gives the following example of the *prastāra* of permutations of 1, 1, 2, 4:

1	1124	4	1142	7	1241	10	4121
2	1214	5	1412	8	2141	11	2411
3	2114	6	4112	9	1421	12	4211

Of course, the total number of permutations is given by $\frac{4!}{2!} = 12$.

Now next thing is interesting thing is Narayana does this prastara, where the swaras can be repeated, so this is like (FL), so what is the rule for prastara you just follow Sarngadeva's rule you will get prastara of this also. So I will start with 1, 1, 2, 4 wherever there is an ascent 1, 2 you put below 2 the next 1, that is 1 bring down the right things as they are whatever remains you put in the ascending order so it is 1214.

Now 12 is in ascending order below that 2 you put 1 bring down the 14 as it is whatever is left is 2, now 2114 only 14 is in ascending order, so before below this 4 you have to bring down here 2, now the rest of the numbers have to be put in ascending order, so wherever repetition occurs repetition will come, so this is the way you generate all possible permutations of q quantities where they are repeated also.

Of course the 4 sankhyankas here the total number of permutations are the total number of rows in this prastara will be this multinomial or 4 factorial/2 factorial and the method of Sarngadeva's just goes through Narayana Pandita is the first person to have listed the permutations with repetitions, they are called multiset this theory itself very interesting.

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Prastāra of Combinations

Towards the end of the chapter on *Aṅkapāśa*, Nārāyaṇa discusses the *prastāra* of combinations (*mūlakramabheda-prastāra*)

न्यस्याल्पमाद्यान् महतोऽधस्ताच्छेषं यथोपरि।
ऊने तदुत्क्रमादङ्कानेकैकोनान्समालिखेत्॥
चयपङ्क्तिर्भवेद्यावत्तावत् प्रस्तारजो विधिः।

[Starting from] the beginning (i.e, from the left), place the [next] lesser digit below the greater one; the remaining digits [to the right] are to be placed as in the row above. If there are gaps [to the left], then place in reverse order, digits which are successively one less than the previous. This method of generating *prastāra* is to be followed till the [smallest] arithmetic sequence is obtained.

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Prastāra of Combinations

Nārāyaṇa gives the following example of the *prastāra* of the C(8,3) combinations obtained when we select three digits out of the eight digits 1, 2, . . . , 8.

1	**678**	15	**158**	29	**257**	43	**146**
2	**578**	16	**348**	30	**157**	44	**236**
3	**478**	17	**248**	31	**347**	45	**136**
4	**378**	18	**148**	32	**247**	46	**126**
5	**278**	19	**238**	33	**147**	47	**345**
6	**178**	20	**138**	34	**237**	48	**245**
7	**568**	21	**128**	35	**137**	49	**145**
8	**468**	22	**567**	36	**127**	50	**235**
9	**368**	20	**467**	37	**456**	51	**135**
10	**268**	24	**367**	38	**356**	52	**125**
11	**168**	25	**267**	39	**256**	53	**234**
12	**458**	26	**167**	40	**156**	54	**134**
13	**358**	27	**457**	41	**346**	55	**124**
14	**258**	28	**357**	42	**246**	56	**123**

Finally, prastara of combinations, so what is the prastara of combinations basically we will take an example first and then go back to the theory. So you have 8 digits choose 3 of them, how many ways you can do C 8,3 that is 8 factorial/5 factorial*3 factorial that is 56, so C 8,3 stands for number of combinations that is the usual I am using this notations instead of mCr I am using this notation C m,r okay simpler to type.

So here you see all such combinations are all dealt in a sequence, so you are having a rule for prastara and once you have rule for prastara you have nasta, uddista and you also have ultimately representation of every row number in terms of certain sankhyankas, so all that theory will come, we will not go into details of this. **"Professor - student conversation starts"** (()) (48:40) no, no there the sum is constant, here you are taking combinations.

Here, you are taking you are listing all possible combinations which arise when you choose 3 objects from 8 objects, so here you see each row is just 3 entries, their sum is anything their sum is not at all fixed, (FL) is the total (FL) amount is fixed, this is a very different kind of this is permutation this is prastara of combinations, this is generating all possible combinations in a sequence **"Professor - student conversation ends."**

So the rule is it is written in a fairly slick fashion (FL) so the last row should become 123 that is the simplest arithmetic sequence start with the highest arithmetic sequence, so the rule is the following from the left whenever you find something below that can be written you write that and bring down the right things as they are, so this will go on till this point, now below 7 you have to write the number just below that if it is not found to the left of this that is the rule.

So you can put your 6 here 8 is brought down, now with the 6 you write only the sequence of numbers next to it to the left,

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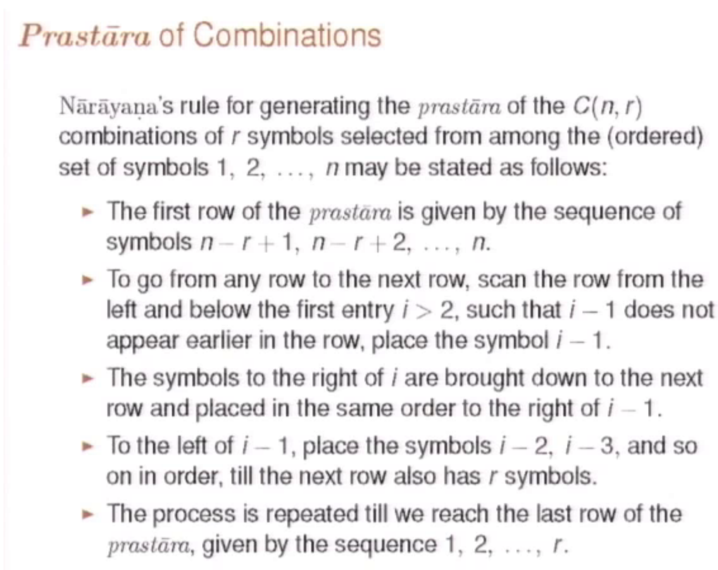

So this rule I have written down in a complex manner here, the first row of the prastara is given by the sequence of symbols n-r+1, n-r+2 etc. n to go from any row to the next row scan it from the left, and the first entry i>0 such that i-1 does not appear earlier in the row place the symbol i-1 there bring down the right things as they are, this in any prastara that is one of the standard referring to the left place the symbols i-2, i-3 in order till the next row has also r symbols by this process you will generate all the combinations.

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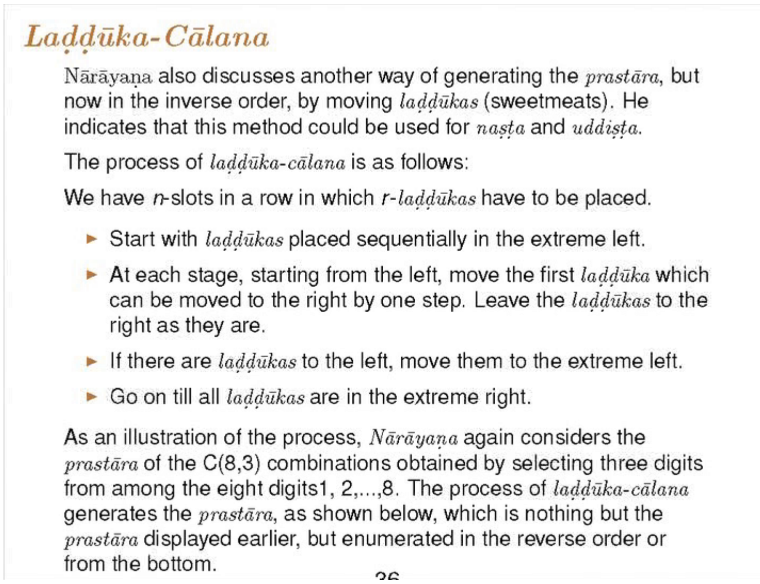

And there is another interesting physical way in which Narayana says this prastara can be done, it is called ladduka-Calana or (FL), this was briefly hinted by (FL) in his half a verse that I mentioned in the case of the 16 perfumes, so the rule is like this.

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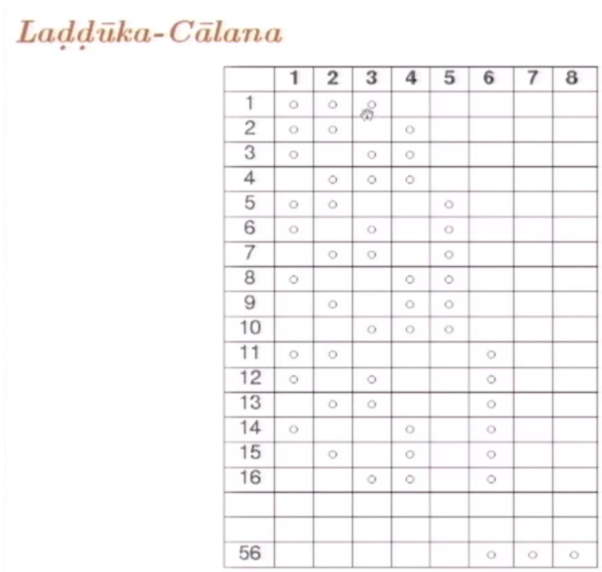

	1	2	3	4	5	6	7	8
1	○	○	○					
2	○	○		○				
3	○		○	○				
4		○	○	○				
5	○	○			○			
6	○		○		○			
7		○	○		○			
8	○			○	○			
9		○		○	○			
10			○	○	○			
11	○	○				○		
12	○		○			○		
13		○	○			○		
14	○			○		○		
15		○		○		○		
16			○	○		○		
56						○	○	○

Let us look at this diagram, keep this laddus here rule is from the left the first laddu which can be moved to the right move that to the right, leave whatever is else is to the right as it is, if there is something else to the left bring them down back to the beginning. So ladduka-calana let us begin we put laddus here, first this will go there, next so from the left this can be move right so I do that, right should be left as it is, left cannot be moved any further left so it is simply stays there.

So in this row from the left this laddu can be moved so move this, and right nothing else is to be done so you are there, now we are here from the left the first laddu can be moved is what is there here so that can be moved to the right so that is brought here, these 2 laddus take it to the left extreme take it to the left extreme, so like that go on and on ladduka-calana, and in the end you will come with this.

Now only thing is this is starting with 123 and ending with 5678, so it is prastara in the reverse so he also mentioned that you can use this ladduka-calana method to do nasta and uddista, it is just a brilliant in the way they are thinking about the problems in multi-various ways in putting them together.

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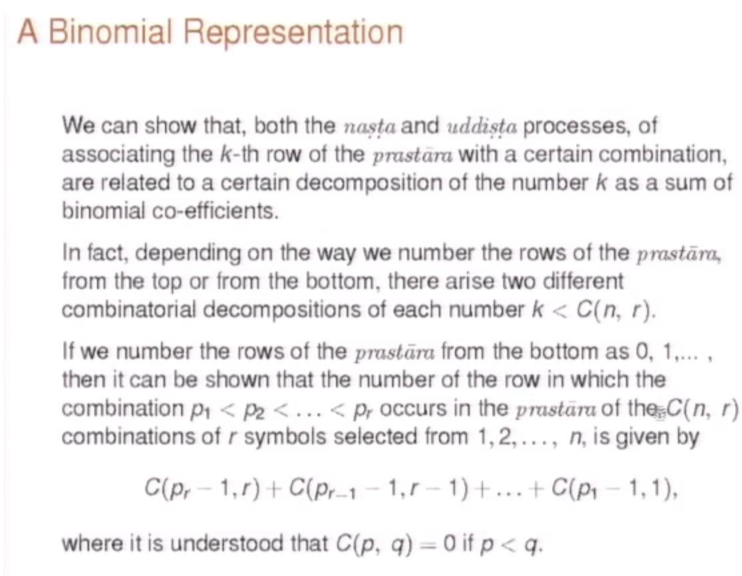

And from this we can generate something called as a representation of every integer in terms of the binomial coefficient, we will not go into the theory of this so I will just write down.

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A Binomal Representation

Example : Binomial Representation for numbers $k < C(8,3)$

No	Combination	Binomial Representation
0	123	0
1	124	C(3,3)
2	134	C(3,3)+C(2,2)
3	234	C(3,3)+C(2,2)+C(1,1)
4	125	C(4,3)
5	135	C(4,3)+C(2,2)
6	235	C(4,3)+C(2,2)+C(1,1)
7	145	C(4,3)+C(3,2)
8	245	C(4,3)+C(3,2)+C(1,1)
9	345	C(5,3)+C(3,2)+C(2,1)
10	126	C(5,3)
11	136	C(5,3)+C(2,2)
12	236	C(3,3)+C(2,2)+C(1,1)

So each number 5 can be written as a combination of the binomial coefficients in a unique manner, for any given prastara such a representation is possible, so as I said the theory of combinatorics always has this representation associated with this. Theory of combinatorics means writing things in an array as a prastara, then associating with the row numbers, the patterns which occurred in the prastara in that row associating the patterns with the row number.

So ultimately you will end up with a representation of numbers in terms of other interesting numbers, which are associating associated with the prastara do the sankhya, so this is a very general method which goes back to pingala and Narayana almost has got it to a beautiful mathematical perfection. So in the end he does say that my theory is limited by the fact that the number of that ankas that are there is has to be <=9. But intelligent people can devise ways by which they can do this problem.

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So Ganitakaumudi was edited by Padmakara Dvivedi in 1936 and 1942 in 2 volumes. This Combinatorics and Magic-Squares chapters are extensively studied by Kusuba in a thesis from Brown University under Professor Pingiri, it has been translated by Professor Paramanand Singh in issues of Ganita Bharati, it is a good introduction to understanding.

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Then there are some articles on particular topics, thank you very much.