

**Mathematics in India: From Vedic Period to Modern Times**  
**Prof. M.D. Srinivas**  
**Centre for Policy Studies, Chennai**

**Lecture-23**  
**Bijaganita of Bhaskaracarya 1**

So, there are 3 works of Bhaskaracharya (FL) and (FL) all the 3 are also called put together in (FL) also, then he wrote the (FL) commentary on (FL) on (FL) on (FL) also. The (FL) and (FL) and (FL) mainly deal with explain in the examples given in his works where the (FL) discusses (FL) explanations, proofs and things like that.

Then Bhaskaracharya also wrote (FL) and also he wrote a commentary on (FL) with a (FL). Anyway we will discuss the (FL) mainly Bhaskara's treatise we will quickly look at the development of algebra.

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**Outline**

- ▶ Development of *Bījaganita* or *Avyaktaḡanita* (Algebra) and Bhāskara's treatise on it.
- ▶ Understanding of negative quantities.
- ▶ Development of algebraic notation.
- ▶ The *Vargaprakṛti* equation  $X^2 - D Y^2 = K$ , and Brahmagupta's *bhāvanā* process
- ▶ The *Cakravāla* method of solution of Jayadeva and Bhāskara.
- ▶ Bhāskara's examples  $X^2 - 61 Y^2 = 1$ ,  $X^2 - 67 Y^2 = 1$
- ▶ The equation  $X^2 - D Y^2 = -1$ .
- ▶ Solution of general quadratic indeterminate equations
- ▶ Bhāskara's solution of a biquadratic equation.

Then look at something's like how they understood negative quantities, how algebra is notation develop we will mainly talk about in both this lecture in the next lecture about the (FL) equation and particularly the so called (FL) method of solution of a (FL) equation. In this talk I will also tell you something about some other equations interesting equation at Bhaskara solved in it (FL). So, the notion of (FL) unknown quantity goes back to (FL) you all saw that (FL) gives you how to construct another square which is n times given square.

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**Development of *Bījagaṇita***

- ▶ The notion of a variable quantity, *yāvat-tāvat* (as many as), goes back to *Śulvasūtras*. The *Kātyāyana-Śulvasūtra* deals with the problem of constructing a square whose area is  $n$ -times that of a given square
- ▶ *Āryabhaṭīya* (c.499) uses the term *gulikā* for the unknown. There, we also find the solution of linear and quadratic equations and also the *kuṭṭaka* process for the solution of linear indeterminate equations.
- ▶ Bhāskara I (c.629) uses the notion of *yāvat-tāvat* in his commentary of *Āryabhaṭīya*.
- ▶ Brahmagupta has given a detailed exposition of *Bījagaṇita* in the Chapter XVIII, *Kuṭṭakādhyāya*, of his *Brāhmasphuṭasiddhānta* (c.628). This work has been commented upon by Pṛthūdakasvāmi (c.860).
- ▶ Śrīpati deals with *avyakta-gaṇita* in Chapter XIV of his *Siddhāntaśekhara* (c.1050)

And that  $n$  was explained expressed in term (FL) the term (FL) occurs in Aryabhattiya for the unknown and Aryabhatta of course give the (FL) method of solving the linear indeterminate equations. Bhaskara uses the idea of (FL) in his commentary, Brahmagupta of course give a detailed exposition of algebra in the (FL) Brahmasphutasiddhanta in 628 AD discussion of algebra appears in the work of (FL) which was written in 1050.

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**Development of *Bījagaṇita***

- ▶ Bhāskarācārya II has written the most detailed available treatise on *Bījagaṇita* (1150), where he states that he has only compiled and abridged from the treatises of Śrīdhara (c.750) and Padmanābha, which are not available to us.
- ▶ *Bījagaṇita* has about 100 verses giving various rules and about 110 verses giving various examples. Bhāskara has written his own commentary *Vāsanā*, which gives details of solutions of the various examples.
- ▶ *Bījagaṇita* has also been commented upon by Sūryadāsa (c.1540) and Kṛṣṇa Daivajña (c.1600). The commentary *Bījapallava* or *Bījānavāṅkurā* of Kṛṣṇa provides detailed proofs (*upapattis*) of various results presented in *Bījagaṇita*.
- ▶ Nārāyaṇa Paṇḍita (c.1350) has also composed a treatise, *Bījagaṇitāvatanṣa*, of which only the first few chapters are available.

But the most important work an algebra that is Indian algebra that available to us today these are (FL) of (FL) 2 as you are know he wrote this work when he was 34 years around 1150 and he says he has compiled and abridged well-known treatises of Sridhara and Padmanaba on (FL)

both of which are not available to us (FL) is also a somewhat smaller 100 verses giving the various sutras are roots and above 110 verses giving examples and as I said this Bhaskara's commentary (FL) on this.

Two important commentaries on (FL) one by (FL) and the one by (FL) the commentary on of (FL) has been partially edited and printed the commentary of (FL) is available in, in published form in full. He discusses proofs of various results contained in (FL), there is another very important work on algebra due to (FL) solved (FL) but only the first few chapters of this book are available to us now up to (FL).

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*Bījagaṇita* of Bhāskarācārya II (c.1150)

The following are the topics dealt with in the *Bījagaṇita* of Bhāskarācārya:

- ▶ Six operations with positive and negative numbers (*dhanarṇa-śaḍvidha*)
- ▶ Arithmetic of zero (*kha-śaḍvidha*)
- ▶ Six operations with unknowns (*avyakta-śaḍvidha*)
- ▶ Arithmetic of surds (*karaṇī-śaḍvidha*)
- ▶ Linear indeterminate equations (*kuṭṭaka*)
- ▶ Second order indeterminate equation  $x^2 - D y^2 = 1$  (*vargaprakṛti*)
- ▶ The cyclic method (*cakravāla*)

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So, quickly the contents on Bhaskara's (FL) the 6 operations with positive and negative numbers, third (FL), 6 operations with 0 (FL) 6 operations with unknown quantities, arithmetic of surds (FL) leaving the linear indeterminate equations (FL), second order indeterminate equation (FL), the cyclic method for solving them (FL).

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## Bījagaṇita of Bhāskarācārya II

- ▶ Equations with single unknown (*ekavarṇa-samikaraṇa*)
- ▶ Elimination of middle term in quadratic equations (*madhyamāharaṇa*)
- ▶ Equations with several unknowns (*anekavarṇa-samikaraṇa*)
- ▶ Elimination of middle term in equations with several unknowns (*anekavarṇa-madhyamāharaṇa*)
- ▶ Equations with products of unknowns (*bhāvita*)

According to Bhāskarācārya, the topics dealt with in the first few chapters, upto *cakravāla*, are said to be “*bījopayogi*” —useful for the consideration of equations that are to follow.

Then these portions Bhaskara calls them (FL) they are useful for algebra can else he saw other equations and the main portion (FL) is actually deals with (FL) equation with single unknown, (FL) elimination of the middle term in a quadratic equation, equations with several unknowns (FL) then elimination of middle term in equations with several unknowns and finally the (FL) form of equation which this classification is standard it has come down from Brahmagupta as you can see.

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## Invocation to *Avyakta-Gaṇita*

*Bījagaṇita* or *avyakta-gaṇita*, is computation with seeds, or computation with unmanifest or unknown quantities, which are usually denoted by *varṇas*, colours or symbols.

The following invocatory verse of *Bījagaṇita* of Bhāskarācārya has been interpreted in three different ways by Kṛṣṇa Daivajña:

उत्पादकं यत्प्रवदन्ति बुद्धेरधिष्ठितं सत्पुरुषेण साङ्गः।

व्यक्तस्य कृत्स्नस्य तदेकबीजमव्यक्तिमौशं गणितं च वन्दे ॥

तदव्यक्तम् ईशं गणितं च वन्दे।...अव्यक्तं प्रधानम्। साङ्गशास्त्रे जगत्कारणतया प्रसिद्धम्। ईशं सच्चिदानन्दरूपं वेदान्तवेदात्। गणितमव्यक्तमेव।

**Meaning I:** I salute that *avyakta* (*prakṛti* or primordial nature), which the philosophers of the *Sāṅkhya* School declare to be the producer of *buddhi* (the intellectual principle *maha*), while it is being directed by the immanent *Puruṣa* (the Being). It is the sole *bīja* (seed or the cause) of all that is manifest.

So, in the first verse of (FL) does an invocation to (FL) and it is a very interesting beautiful verse which has been given 3 different meanings where the (FL) can be either thought of as the (FL) primordial nature as understood in the (FL) philosophy it can be thought of as (FL) in his

commentary says (FL), so there are 3 invocations made in a single verse (FL) in the (FL) school of philosophy at the primordial nature (FL) second meaning you think of it has the (FL) as gone in (FL).

And in the third meaning (FL) is (FL) of the mathematics done with un manifest quantity or algebra. So, the 3 meanings are here you can read it at you leisure.

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### Kṛṣṇa on the Notion of Negative Quantities

The verse of *Bījagaṇita*

संशोध्यमानं स्वमृणत्वमेति स्वत्वं क्षयः तद्गुतिरुक्तवच्च ॥

gives the rule of signs that, in subtraction, a negative quantity becomes positive and vice versa.

Kṛṣṇa Daivajña, in his commentary explains how negativity is to be understood in different contexts. He then goes on to show that this physical interpretation of negativity can be used to demonstrate the rule of signs in algebra in different situations.

The idea of negative quantities where host a great problem in history of mathematics in many cultures but in India from the beginning it had a very nice and good interpretation there was no fear of the negative numbers or no uncertainty are confusion about the way they have to used. Now while commenting on the verse of (FL) that a negative number when you subtract become positive (FL) try to explain what is a negative number.

So, he says a negativity can be understood in 3 different ways depending on the physical situation and each of them what negativity means he tries to explain.

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## Kṛṣṇa on the Notion of Negative Quantities

ऋणत्वमिह त्रिधा तावदस्ति देशतः कालतः वस्तुतश्चेति । तच्च  
वैपरीत्यमेव । यत् उक्तमाचार्यैर्लीलावत्यां क्षेत्रव्यवहारे  
“दशसप्तदशप्रमौ भुजौ ...” इत्यस्मिन्नुदाहरणे । ऋणगता आवाधा  
दिग्वैपरीत्येनेत्यर्थ इति । तत्रैकरेखास्थिता द्वितीया दिक् विपरीता  
दिगित्युच्यते । यथा पूर्वविपरीता पश्चिमा दिक् । यथा वोत्तरदिग्विपरीता  
दक्षिणा दिगित्यादि । तथा च पूर्वापरदेशयोर्मध्ये एकतरस्य धनत्वे  
कल्पिते तं प्रति तदितरस्य ऋणत्वम् ।...

एवं पूर्वोत्तरकालयोरन्योन्यम् ऋणत्वं वारप्रवृत्त्यादिषु प्रसिद्धम् । एवं  
यस्मिन् वस्तुनि यस्य स्वस्वामिभावः सम्बन्धः तस्य तद्धनमिति  
व्यवह्रियते । तस्मिन् वैपरीत्यं तु परस्य स्वस्वामिभावः सम्बन्धः ।

So, (FL), so the negativity can be of 3 kinds one in spacial direction, one in temporality and one in the terms of objects and he goes and explaining what it is. So, in spacial direction if 1 direction means thought of to be positive, the opposite direction is to be thought of to be negative a very clear physical explanation of, similarly temporarily if one way of your use of time is to be thought of it is positive, the other plane can be thought of to be negative.

Similarly if I possess a set of object and then it can be called my well and if I call some of it to you then it becomes negative for me and positive for you. So, this (FL) and (FL) which arouse normally in commercial mathematics, so negativity has all this different ways of understanding.

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## Kṛṣṇa on the Notion of Negative Quantities

“Negativity (*ṛṇatva*) here is of three types: Spatial, temporal and that pertaining to objects. In each case, [negativity] is indeed the *vaiparītya* or the oppositeness. As has been clearly stated by the Ācārya in *Līlavatī* in the example ‘The *bhujās* are ten and seventeen etc.’ The negative base intercept (*ābādhā*) is to be understood to be in the opposite direction. There, the other direction in the same line is called the opposite direction (*viparītā dik*); just as west is the opposite of east... Further, between two stations, if one way of traversing is considered positive, then the other is negative...”

In the same way past and future time intervals will be mutually negative of each other... Similarly, when one possesses the said objects they would be called his *dhana* (wealth). The opposite would be the case when another owns the same objects...”

And what (FL) is doing he is trying to prove this result, that – of – is + by using a special way of understanding negatively.

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**Bhāskara's Example of Negative Intercept**

The example discussed by Bhāskara in *Līlāvātī* has to do with the calculation of the base-intercepts (*ābādhas*) in a triangle. There, Bhāskara explains that if the calculation leads to a negative intercept, it should be interpreted as going in the opposite direction. This happens when the foot of the altitude falls outside the base.

दशसप्तदशप्रमौ भुजौ त्रिभुजे यत्र नवप्रमा मही।  
अवधे वद लम्बकं तथा गणितं गणितिकाऽऽशु तत्र मे ॥

"In a triangle, which has sides 10, 17 and 9, tell me quickly, Oh mathematician, the base intercepts and the area."

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Then he also brought attention to the beautiful verse (FL) and (FL) which was explained in the morning class by professor Sriram this is to calculation of the perpendicular intersect in a triangle where the base the perpendicular falls outside the base (FL).

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**Bhāskara's Example of Negative Intercept**

In his *Vāsanā*, Bhāskara gives the solution of this problem as follows

अत्र त्रिभुजे भुजयोर्योग इत्यादिना लब्धम् २१।  
अनेन भूजना न स्यात्। तस्मादेव भूजपनीता शेषार्धं  
ऋणगताऽऽबाधा दिग्वैपरीत्येनेत्यर्थः। तथा जाते आबाधे ६\*,  
९५।

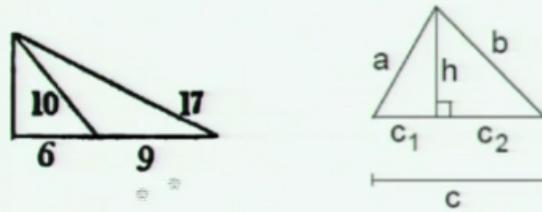
Here, using the rule, "The sum of the sides...", we get (the difference of the intercepts to be) 21. We cannot subtract this from the base (9). Hence, the base has to be subtracted from this (difference) only and the half of the result is the intercept which is negative, and it is to be understood to be in the opposite direction. Thus the intercepts are (-6, 15).

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So, Bhaskara his this is Bhaskara's (FL) commentary we will come to in the moment.

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### Bhāskara's Example of Negative Intercept



In a triangle with sides  $a$ ,  $b$ , and base  $c$ , let the base intercepts be  $c_1$ ,  $c_2$ . Now  $(c_1 + c_2) = c$  and  $(b^2 - a^2) = (c_2^2 - c_1^2)$ . Therefore,

$$c_2 - c_1 = \frac{(b^2 - a^2)}{(c_1 + c_2)} = \frac{(b^2 - a^2)}{c}$$

Thus, in our problem,

$$c_1 + c_2 = 9 \text{ and } c_2 - c_1 = \frac{(17^2 - 10^2)}{9} = 21.$$

$$\text{Hence } c_2 = 15 \text{ and } c_1 = -\frac{(21-9)}{2} = -6$$

The triangle that is being thought of is 9 is the base 17 and 10 are the sides and this is the perpendicular and normally when you calculate the base intercepts you think of a triangle with an acute angle at that top and in all the 3, so the usual calculation has been explained to you several times,  $C_1$  and  $C_2$  have to be calculated,  $b^2 - a^2 = C_2^2 - C_1^2$  therefore is  $b^2 - a^2 = c_1 + c_2$  which is  $c$  itself and so you have 2 equations one for  $c_2 - c_1$  one for  $c_2 + c_1$  and by the method (FL) which was explained in the morning you find  $c_1$  and  $c_2$ .

But you plug in 17 and 10 and 9 immediately you will find that the  $c_2 - c_1$  21 is larger than the base 9 and therefore we will one of the intercepts  $c_1$  will have to become negative, so this was Bhaskara explain in here (FL) that is by using the formula for the (FL) you will get that  $c_2 - c_1$  is 21 (FL) we cannot subtract 9 21 from 9 (FL) therefore we subtract 9 from 21 and we get an intercept which is (FL) which is negative which means it goes in the opposite direction.

He is explaining the entire concept of negative number in this beautiful example to which (FL) is drawing attention in his commentary here. (FL) mean (FL), so this is a very beautiful and important way of understanding negative numbers.

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## Algebraic Notation

In the *Āryabhaṭīyabhāṣya* of Bhāskara I (c.629) we find references to the algebraic notation used in Indian mathematics. Various features of the notation are more clearly known from the *Bakhshālī* manuscript (c. 700).

The system of algebraic notation is explained and fully exemplified in the *Bījagaṇita* of Bhāskara II (c.1150), together with his own auto-commentary *Vāsanā*.

In *Vāsanā* on verse 3 of *Bījagaṇita*, Bhāskara says:

अत्र रूपाणामव्यक्तानां चादाक्षराण्युपलक्षणार्थं लेख्यानि ।  
तथा यान्यनगतानि तान्यूर्ध्वविन्दुनि च ।

Here (in algebra), the initial letters of both the known and unknown quantities should be written as their signs. Similarly those (quantities) which are negative, they have (to be shown with) a dot over them.

Now the development of algebra is notation this will also means talks about one of the best indicators for it is of course is in the (FL) manuscript in the Aryabhattiya (FL) we find the references to algebra it is notation. Bhaskara very clearly writes down the notation (FL) in algebra (FL), so the first letter of the unknowns have to be written for denoting this objects (FL), so you have to put a dot above the negative quantities.

So, we saw that Bhaskara was putting a dot here (FL) 6 with the dot there -6 that was the base intercept in, so here in algebra the initial letters of the unknown quantity should be written as their science.

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## Algebraic Notation

- ▶ The symbols used for unknown numbers are the initial syllables *yā* of *yāvat-tāvat* (as much as), *kā* of *kālaka* (black), *nī* of *nīlaka* (blue), *pī* of *pīta* (yellow) etc.
- ▶ The product of two unknowns is denoted by the initial syllable *bhā* of *bhāvita* (product) placed after them. The powers are denoted by the initial letters *va* of *varya* (square), *gha* of *ghana* (cube); *vava* stands for *varyavarya*, the fourth power. Sometimes the initial syllable *ghā* of *ghāta* (product) stands for the sum of powers.
- ▶ A coefficient is placed next to the symbol. The constant term is denoted by the initial symbol *rū* of *rūpa* (form).
- ▶ A dot is placed above the negative integers.
- ▶ The two sides of an equation are placed one below the other. Thus the equation  $x^4 - 2x^2 - 400x = 9999$ , is written as:

यावव १ याव २• या ४००• रु ०  
यावव ० याव ० या ० रु ९९९९

So, the symbols used as (FL) so you use (FL) for the (FL) black colour (FL) for the (FL) for that (FL) p for that product of 2 unknowns will be written with a (FL) following them (FL) the product (FL) sum quantity will be written by writing a (FL) following that quantity standing for (FL) if it is cube write a (FL), so you can see that the equation  $x^2 - 2x - 400$  is equal to 9999 an equation which Bhaskara solved a very famous by quadratic equation that Bhaskara solved in his (FL) , it is to be written in symbols like this.

The 2 sides of the equations are written 1 below each other instead of writing on the left and the right they are written 1 below each other (FL) 1 (FL) 2. (FL) 400. (FL) 0 that is constant term 0, (FL) 0 (FL) 0 (FL) 0 (FL) 9999 stands for this equations, this is the kind of algebra is notation this is clearly there in the commentary (FL), so this kind of notation is standard and was very established by 1150 in Indian algebra, (FL) is a constant term.

So, 9 9999 is (FL) 9999, so it is not an unknown it is a constant (FL) the one that is visible to you straight away as a number (FL) is something that is not known it is (FL) this is (FL).

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**Vargaprakṛti**

In the *Kuṭṭakādhyaḡya* of his *Brāhmasphuṭasiddhānta* (c.628), Brahmagupta considered the problem of solving for integral values of  $X$ ,  $Y$ , the equation

$$X^2 - D Y^2 = K$$

given a non-square integer  $D > 0$ , and an integer  $K$ .

$X$  is called the *jyestha-mūla*,  $Y$  is called the *kanishtha-mūla*

$D$  is the *prakṛti*,  $K$  is the *kṣepa*

One motivation for this problem is that of finding rational approximations to square-root of  $D$ . If  $X$ ,  $Y$  are integers such that  $X^2 - D Y^2 = 1$ , then,

$$\left| \sqrt{D} - \left( \frac{X}{Y} \right) \right| \leq \frac{1}{2XY} < \frac{1}{2Y^2}$$

The *Śulva-sūtra* approximation  $\sqrt{2} \approx 1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34} = \frac{577}{408}$  is an example as  $(577)^2 - 2(408)^2 = 1$ .

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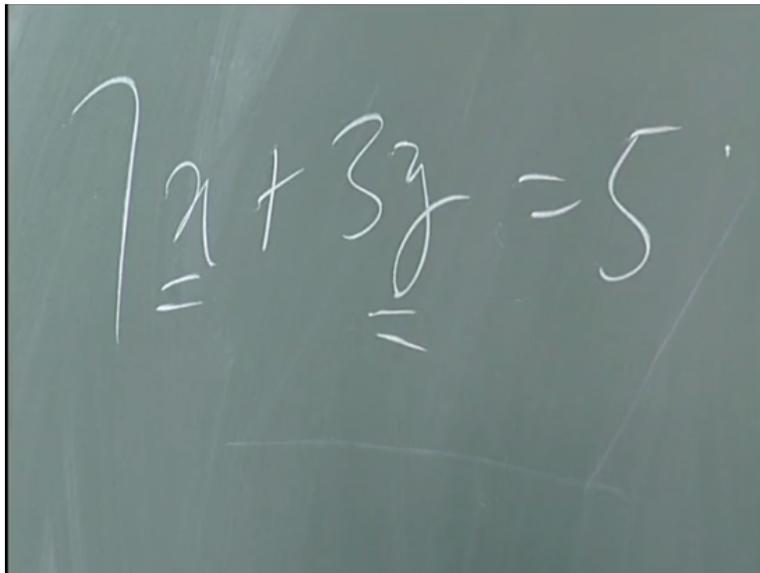
Incidentally there is a very interesting discussion on the second verse of Bhaskara for where he says (FL) that the (FL) is (FL) so this can be interpret in 2 ways (FL) is the (FL) of (FL) it can also interpreted as (FL) is that for which (FL) is the (FL) so both can be use and (FL) tries to

explain it that in order to understand algebra you should first know how to calculate with manifest quantities with numbers etc.,

But in order to prove the rules given in arithmetic like the rule for square root of the rule for cube root you take recourse to algebraic methods. So, in that algebra is the (FL) of arithmetic okay now we the rest of the lecture is on the (FL) equation what is the equation. He had to solve this quadratic indeterminate equation the  $k$  is a given integer  $D$  is a given non-negative non-square integer.

So,  $D$  can be 2, 3 not 4, 5,6,7,8 not 9, so you have to solve the  $x$  and  $y$  in integers, so such class of equations where you want to solve for the equation in integers are called indeterminate equations u they are wrongly called (FL) equations (FL) is consider equations but the solution he look forward rationale so you look for equations solutions of equation like say an equation like this  $7x+3y$  is equal to 5.

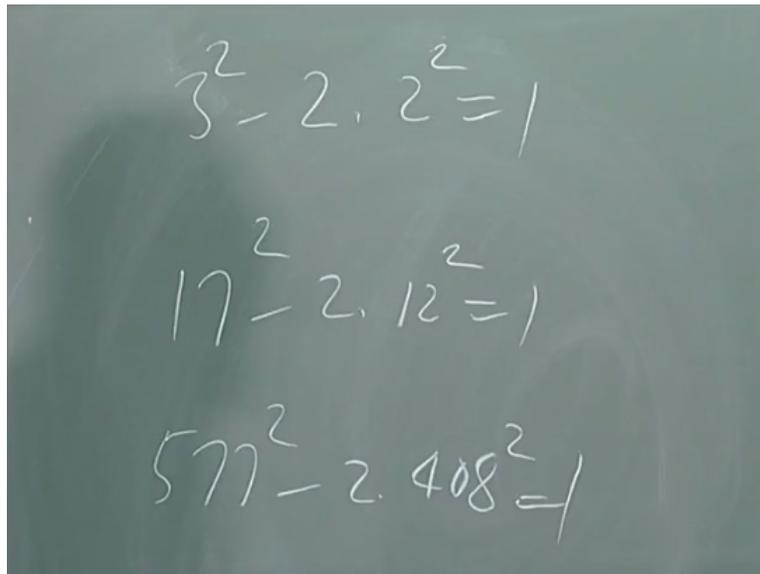
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$$7x + 3y = 5$$

Now if I demand that you solve are  $x$  and  $y$  in integers this becomes a (FL) problem but if I do not demand anything then given any  $x$  you can always find the  $y$  and there find that is something intermediate between that he said we want to solve for  $x$  and  $y$  in rationale numbers, so given any rationale number for  $x$  you obviously obtain a rationale number for  $y$ , so he did not discuss indeterminate equations at all.

In European tradition it was (FL) who started the detailed study of indeterminate equation before that (FL) try to solve the equations like this we will come to that later. So, in this (FL) version x is called (FL) y is (FL) D is the (FL) K is the (FL) we saw this equation also that if x square-Dy square is equal to 1, then x/y is a good approximation to root D, the larger the y here the base term will be the approximation and (FL) approximation is a particular case of (FL) solution of a (FL) equation if the D is equal to 2.

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$$3^2 - 2 \cdot 2^2 = 1$$
$$17^2 - 2 \cdot 12^2 = 1$$
$$577^2 - 2 \cdot 408^2 = 1$$

So, we can start with 3 square-2 into 2 square is equal to 1, next is 17 square-2 into 12 square is equal to 1, next is 577 square-2 into 408 square is equal to 1 these are 3 steps see that each of them come from the previous (FL).

**(Refer Slide Time: 16:47)**

### Use of *Bhāvanā* when $K = -1, \pm 2, \pm 4$

The *bhāvanā* principle can be used to obtain a solution of equation

$$x^2 - D y^2 = 1,$$

if we have a solution of the equation

$$x_1^2 - D y_1^2 = K, \text{ for } K = -1, \pm 2, \pm 4.$$

$$K = -1 : x = x_1^2 + D y_1^2, y = 2x_1 y_1.$$

$$K = \pm 2 : x = \frac{(x_1^2 + D y_1^2)}{2}, y = x_1 y_1.$$

$$K = -4 : x = (x_1^2 + 2) \left[ \frac{1}{2}(x_1^2 + 1)(x_1^2 + 3) - 1 \right],$$

$$y = \frac{x_1 y_1 (x_1^2 + 1)(x_1^2 + 3)}{2}.$$

$$K = 4 : x = \frac{(x_1^2 - 2)}{2}, y = \frac{x_1 y_1}{2}, \text{ if } x_1 \text{ is even.}$$

$$x = \frac{x_1(x_1^2 - 3)}{2}, y = \frac{y_1(x_1^2 - 1)}{2}, \text{ if } x_1 \text{ is odd.}$$

So, Brahmagupta not only discuss this equation he gave this (FL) which has been done in the morning what is the (FL) principle.

(Refer Slide Time: 16:50)

### Brahmagupta's *Bhāvanā*

मूलं द्विधेष्टवर्गाद् गुणकगुणादिष्टयुतविहीनाद्य।

आदावधो गुणकगुणः सहान्त्यघातेन कृतमन्त्यम्॥

वज्रवधैकां प्रथमं प्रक्षेपःक्षेपवधतुल्यः।

प्रक्षेपशोधकहते मूले प्रक्षेपके रूपे॥

If  $X_1^2 - D Y_1^2 = K_1$  and  $X_2^2 - D Y_2^2 = K_2$  then

$$(X_1 X_2 \pm D Y_1 Y_2)^2 - D (X_1 Y_2 \pm X_2 Y_1)^2 = K_1 K_2$$

In particular, given  $X^2 - D Y^2 = K$ , we get the rational solution

$$\left[ \frac{(X^2 + D Y^2)}{K} \right]^2 - D \left[ \frac{(2XY)}{K} \right]^2 = 1$$

Also, if one solution of the Equation  $X^2 - D Y^2 = 1$  is found, an infinite number of solutions can be found, via

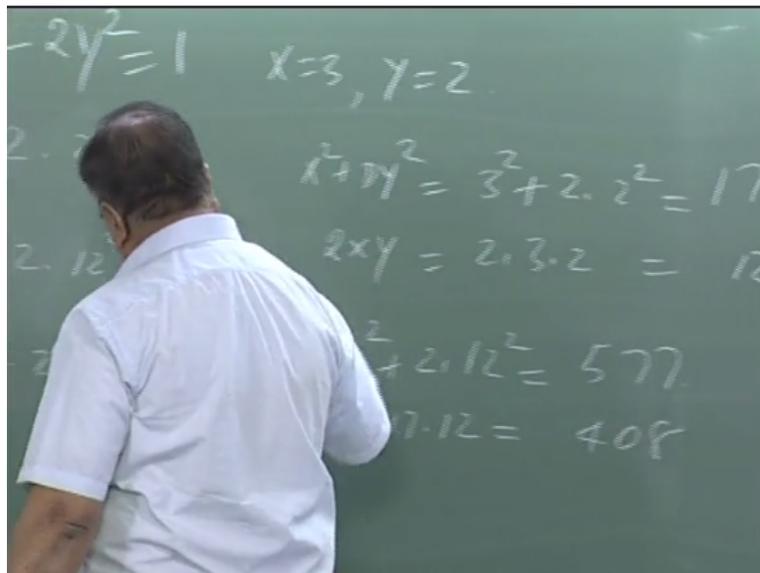
$$(X, Y) \rightarrow (X^2 + D Y^2, 2XY)$$

If you had a solution for the (FL) equation for a given D (FL) for K1 if you had the solution for the (FL) equation for the same D with some other shape of K2 then you can construct the solution for the same (FL) equation with D with shape of K1, K2. So, given equation with a K1 a solution given in equation with K2 another solution you can construct a solution for K1, K2 that can just be done by taking the product of this 2 equations that will immediately it come just an algebraic identity given this and this it immediately follows algebraically that this 2.

But this was the great discovery because it enable Brahmagupta to handle this equation, so first thing he did he said you can construct always rational solutions for the so you can start with an x and a y get whatever k you want and then do this we will get a rational solution. But more importantly if I if k1 is equal to 1 and k2 is equal to 1 then k1k2 is equal to 1. Therefore if I know a solution xy with k is equal to 1 then you know another solution x square+D y square into 2xy which is also had it for k is equal to 1.

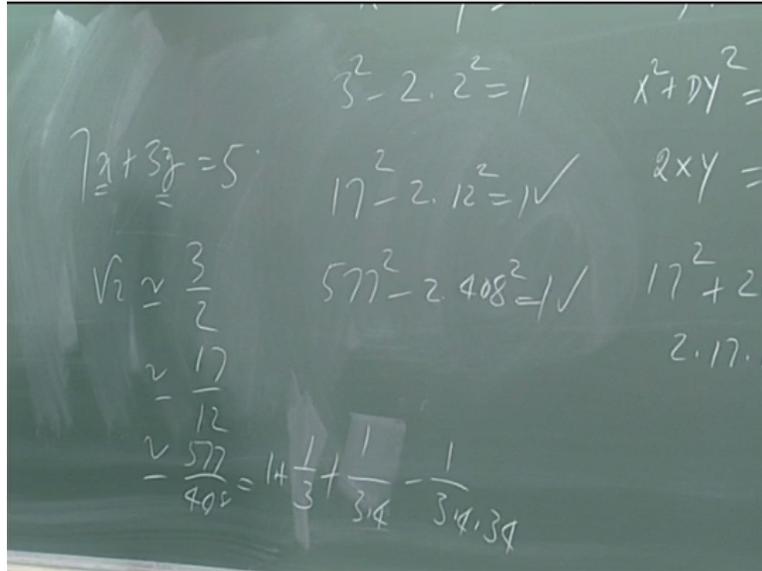
So, if this equation can be solved to get one solution then you can obtain another then you can obtain another, therefore it has infinite number of solution. So, how this are obtain by (FL) you can straight away see.

**(Refer Slide Time: 18:34)**



So, we are solving x square-2 y square is equal to, so x is equal to 3, y is equal to 2 is one solution we can straight away see from inspection. So, we do (FL) on this, so X square+Dy square will give me 3 square+2 into 2 square, so this is 17, 2xy if you mean 2 into 3 into 2 is equal to 12 and you will obtain the other solution 17 square this 289, 12 square\*2 is 288 and again I will do it 17 square+2 into 12 square this gives you 577 and 2\*17\*12 it is 408.

**(Refer Slide Time: 19:31)**



So, you will immediately get the (FL) approximation for root 2. So, you can say root 2 is approximate equal to  $3/2$  is 1, next is  $17/12$  that one is  $577/408$  and this is, so this is the power of (FL) you know 1 solution.

**(Refer Slide Time: 19:48)**

**Brahmagupta's Examples**

राशिकलाशेषकृतिं द्विनवतिगुणितां त्र्यश्रोतिगुणितां वा ।  
 सैकां जदिने वर्गं कुर्वन्नावत्सराद्गणकः ॥

To solve,  $x^2 - D y^2 = 1$ , for  $D = 92, 83$

$10^2 - 92 \cdot 1^2 = 8$

Doing the *bhāvanā* of the above with itself,

$192^2 - 92 \cdot 20^2 = 64$  [ $10^2 + 92 \cdot 1^2 = 192$  and  $2 \cdot 10 \cdot 1 = 20$ ]

Dividing both sides by 64, we get  $24^2 - 92 \cdot (\frac{5}{2})^2 = 1$

Doing the *bhāvanā* of the above with itself,

$1151^2 - 92 \cdot 120^2 = 1$  [ $24^2 + 92 \cdot (\frac{5}{2})^2 = 1151$  and  $2 \cdot 24 \cdot (\frac{5}{2}) = 120$ ]

Similarly,  $9^2 - 83 \cdot 1^2 = -2$

Doing the *bhāvanā* of the above with itself,

$164^2 - 83 \cdot 18^2 = 4$  and hence,  $82^2 - 83 \cdot 9^2 = 1$

And Brahmagupta has was pointed out in the lecture on Brahmasphutasiddhanta solved 2 examples  $x^2 - D y^2 = 1$  for  $D$  is equal to 92 and  $x^2 - D y^2 = 1$  for  $D$  is equal to 83 by near inspection and application of (FL). So, he just started with so 92 look at the nearest square 10,  $10^2 - 92 = 8$ , then just by doing (FL) of it with itself and manipulating both the sides he finally lands up with a solution  $x$  is equal to 1151,  $y$  is equal to 120 which is not at all easy to guess unless you knew the (FL) method.

(Refer Slide Time: 20:40)

### *Cakravāla* : The Cycle Method

The first known reference to *Cakravāla* or the Cyclic Method occurs in a work of Udayadivākara (c.1073), who cites the following verses of Ācārya Jayadeva:

ह्रस्वज्येष्ठक्षेपान् प्रतिराश्या क्षेपभक्तयोः क्षेपात्।  
कुट्टाकारे च कृते कियद्गुणं क्षेपकं क्षित्वा ॥  
तावत्कृतेः प्रकृत्या हीने प्रक्षेपकेण संभक्ते।  
स्वल्पतरावाप्तिः स्यादित्याकलितोऽपरः क्षेपः ॥  
प्रक्षिप्तप्रक्षेपककुट्टाकारे कनिष्ठमूलहते।  
सज्येष्ठपदे प्रक्षेपकेण लब्धं कनिष्ठपदम् ॥  
क्षिप्तक्षेपककुट्टागुणितात् तस्मात्कनिष्ठमूलहतम्।  
पाश्चात्यं प्रक्षेपं विशोधय शेषं महन्मूलम् ॥

So, we now move on to the historic (FL) method which was a systematic method algorithm for solving the equation  $x^2 - Dy^2 = 1$  for any  $D$  which is not a spec till that it 40 as wrote it first thought that this (FL) method occurred in the (FL) of (FL), so people thought that (FL) 2 himself was the originator of that later on in 1957 professor K. Sukra discovered verses about (FL) in a commentary of (FL) is writing a commentary (FL) on the (FL) of (FL) 1, so in that commentary he is sighting (FL) and quoting the (FL) method.

So, (FL) method it is goes back to (FL) who must have lived some kind prior to 1073 now this set of verses of (FL) roughly translate to what I am writing now.

(Refer Slide Time: 21:50)

### *Cakravāla according to Jayadeva*

Given  $X_i, Y_i, K_i$  such that  $X_i^2 - D Y_i^2 = K_i$

First find  $P_{i+1}$  as follows:

(I) Use *kuttaka* process to solve

$$\frac{(Y_i P_{i+1} + X_i)}{K_i} = Y_{i+1}$$

for integral  $P_{i+1}, Y_{i+1}$

(II) Of the solutions of the above, choose  $P_{i+1}$  such that

$$\frac{(P_{i+1}^2 - D)}{K_i} \text{ has the least value}$$

So, what is our procedure at some stage you have a solution  $x_i^2 - D y_i^2 = K_i$  suppose you want to solve  $X^2 - D Y^2 = 1$  for a given  $D$  suppose you have reached this what should you do (FL) says you calculate a quantity  $P_{i+1}$  by these 2 conditions they are somewhat complex but they will keep repeating themselves repeatedly, so first thing you already know  $X_i$  as an integer  $Y_i$  as an integer,  $K_i$  as an integer.

So, here  $X_i$  is known,  $K_i$  is known,  $Y_i$  is known solve this (FL) equation these are known integers  $X_i, K_i$  and  $Y_i$ , so solve this (FL) equation for the unknowns  $P_{i+1}$  and  $Y_{i+1}$  and as you know in the (FL) problem solution is not unique given 1 solution there can be infinite number of others you have choose amongst the  $P_{i+1}$  which satisfy this (FL) equation that  $P_{i+1}$  such that  $P_{i+1}^2 - D$  divided by  $K_i$  as the least value, find them at  $P_{i+1}$  is found  $Y_{i+1}$  is found.

**(Refer Slide Time: 23:15)**

### Cakravāla according to Jayadeva

Then set

$$K_{i+1} = \frac{(P_{i+1}^2 - D)}{K_i} \quad Y_{i+1} = \frac{(Y_i P_{i+1} + X_i)}{K_i}$$

$$X_{i+1} = P_{i+1} Y_{i+1} - K_{i+1} Y_i$$

These satisfy  $X_{i+1}^2 - D Y_{i+1}^2 = K_{i+1}$

Iterate the process till  $K_{i+1} = \pm 1, \pm 2$  or  $\pm 4$ , and then solve the equation using *bhāvanā* is necessary.

Jayadeva's verses do not reveal how condition II is to be interpreted.

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Then you find  $K_{i+1}$  and  $X_{i+1}$  by the following algebraic relations then you have found a new set of quantity  $X_{i+1}, Y_{i+1}$  such that  $X_{i+1}^2 - D Y_{i+1}^2 = K_{i+1}$ . So, we have obtained from starting from this step  $X_i^2 - D Y_i^2 = K_i$  to a new step  $X_{i+1}^2 - D Y_{i+1}^2 = K_{i+1}$  of course this is not the solution but (FL) tells you that do not worry you keep going on you read that obtain at same stage  $K_{i+1}$  is equal to 1 itself then you have solved the problem.

Or you may hit upon  $-1, \pm 2$  or  $\pm 4$  in which case Brahmagupta has told you how to solve the problem by using (FL), this explains this to in the morning I did not spend time on that this is that.

**(Refer Slide Time: 24:18)**

### Use of *Bhāvanā* when $K = -1, \pm 2, \pm 4$

The *bhāvanā* principle can be used to obtain a solution of equation

$$x^2 - D y^2 = 1,$$

if we have a solution of the equation

$$x_1^2 - D y_1^2 = K, \text{ for } K = -1, \pm 2, \pm 4.$$

$$K = -1 : x = x_1^2 + D y_1^2, y = 2 x_1 y_1.$$

$$K = \pm 2 : x = \frac{(x_1^2 + D y_1^2)}{2}, y = x_1 y_1.$$

$$K = -4 : x = (x_1^2 + 2) \left[ \frac{1}{2} (x_1^2 + 1)(x_1^2 + 3) - 1 \right],$$

$$y = \frac{x_1 y_1 (x_1^2 + 1)(x_1^2 + 3)}{2}.$$

$$K = 4 : x = \frac{(x_1^2 - 2)}{2}, y = \frac{x_1 y_1}{2}, \text{ if } x_1 \text{ is even.}$$

$$x = \frac{x_1 (x_1^2 - 3)}{2}, y = \frac{y_1 (x_1^2 - 1)}{2}, \text{ if } x_1 \text{ is odd.}$$

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Brahmagupta have told you that if you know how to solve this equation for  $-1 \pm \sqrt{2}$  and  $\pm \sqrt{4}$  then a simple algebra it formula for the solution for  $K$  is equal to 1 can be obtain. So, for instance if you know  $X_1^2 - D y_1^2$  is equal to -1 then you can say  $X_1^2 + D y_1^2$  and  $2x_1y_1$  will solve this equation, if you know that  $x_1^2 - D y_1^2$  is equal to +2 then you can immediately see that this will solve this equation.

So, like that if you know it  $\pm \sqrt{4}$  and  $\pm \sqrt{4}$  also (FL) has given you the formulae for cal obtain in this solution. So, once you reach  $K$  is equal to  $\pm \sqrt{2}$  or  $\pm \sqrt{4}$  are -1 you do not need to follow this complex procedure you do not need to do this.

**(Refer Slide Time: 25:18)**

**Cakravāla according to Jayadeva**

Given  $X_i, Y_i, K_i$  such that  $X_i^2 - D Y_i^2 = K_i$   
 First find  $P_{i+1}$  as follows:

**(I) Use *kuttaka* process to solve**

$$\frac{(Y_i P_{i+1} + X_i)}{K_i} = Y_{i+1}$$

**for integral  $P_{i+1}, Y_{i+1}$**

**(II) Of the solutions of the above, choose  $P_{i+1}$  such that**

$$\frac{(P_{i+1}^2 - D)}{K_i} \text{ has the least value}$$

You can use (FL) use of, so this was the (FL) verses according to (FL) unfortunately this verses do not tell us how exactly are these conditions to be implement it is see in a many what is the problem is about this conditions, if they are not very clear this statement.

**(Refer Slide Time: 25:40)**

### Cakravālā according to Bhāskara (c.1150)

In his *Bījagaṇita*, BhāskaraĀcārya gives the following description of *cakravālā*:

ह्रस्वज्येष्ठपदक्षेपान् भाज्यप्रक्षेपभाजकान्।  
कृत्वा कल्प्यो गुणस्तत्र तथा प्रकृतितश्च्युते ॥  
गुणवर्गे प्रकृत्योनेऽथवालयं शेषकं यथा।  
तत्तु क्षेपद्वतं क्षेपो व्यस्तः प्रकृतितश्च्युते ॥  
गुणलब्धिः पदं ह्रस्वं ततो ज्येष्ठमतोऽसकृत्।  
त्यक्त्वा पूर्वपदक्षेपांश्चक्रवालमिदं जगुः ॥  
चतुर्द्वौकयुतावेवमभिन्ने भवतः पदे।  
चतुर्द्विक्षेपमूलाभ्यां रूपक्षेपार्थभावना ॥

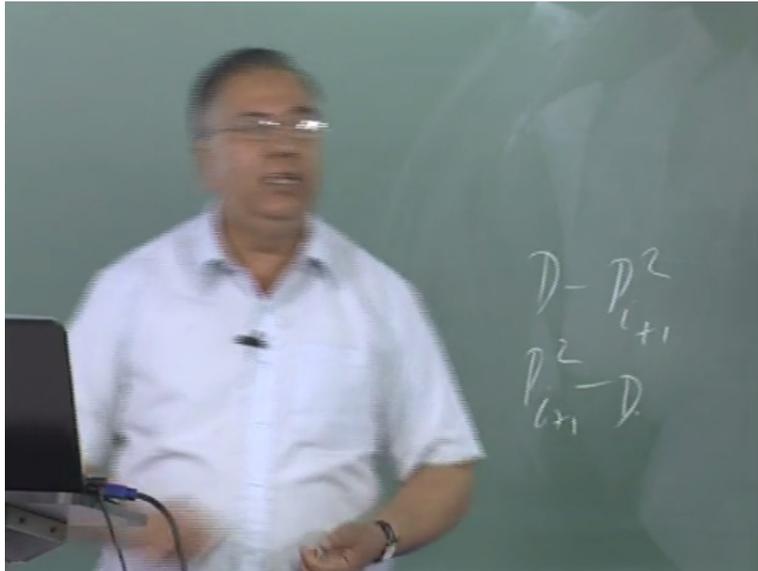
Bhāskara has given the Condition II in the precise form:

(II) Choose  $P_{i+1}$  such that  $|(P_{i+1}^2 - D)|$  has the least value

And there are examples of (FL) also available, now we will come to (FL) statement of (FL) and it is a very simple statement in about 4 verses and he is saying (FL) people call this method as the (FL) method which in fact made many people suspect that (FL) is talking of a method that was already well known it his stand and the mean difference between this condition.

And the condition Bhaskara is giving is that Bhaskara is making a very specific condition choose your  $P_{i+1}$  such that (FL)  $P_{i+1}$  square-D as the least value. So, he has made written very clear condition and where does this appear here (FL) that is that  $P_{i+1}$  (FL) is the  $Y_{i+1}$   $P_{i+1}$  there (FL)  $P_{i+1}$  square (FL) subtracted from D (FL) are  $P_{i+1}$  D subtracted from  $P_{i+1}$  + square (FL).

(Refer Slide Time: 27:05)



So, either way that  $D \cdot P_{i+1}$  square or  $P_{i+1}$  square- $D$  whenever either of them becomes the least you choose that, so the condition of (FL) is very clearly this. So, now let us write down the algorithm in final form.

**(Refer Slide Time: 27:21)**

*Cakravāla* according to Bhāskara

In 1930, Krishnaswami Ayyangar showed that the *cakravāla* procedure always leads to a solution of the *vargaprakṛti* equation with  $K = 1$ . He also showed that condition (I) is equivalent to the simpler condition

**(I')  $P_i + P_{i+1}$  is divisible by  $K_i$**

Thus, we use the *cakravāla* algorithm in the following form:

To solve  $X^2 - D Y^2 = 1$  : Set  $X_0 = 1, Y_0 = 0, K_0 = 1, P_0 = 0$ .

Given  $X_i, Y_i, K_i$  such that  $X_i^2 - D Y_i^2 = K_i$  <sup>⊗</sup>

First find  $P_{i+1} > 0$  so as to satisfy:

**(I')  $P_i + P_{i+1}$  is divisible by  $K_i$**

**(II)  $|P_{i+1}^2 - D|$  is minimum.**

A further small simplification was done in 1930 by Krishnaswami Ayyangar then when you analyse Bhaskara (FL) algorithm and he showed that this complex (FL) condition for determining  $P_{i+1}$  can be replaced by a much simpler condition, that  $P_i + P_{i+1}$  is divisible by  $K_i$ . So, finally the algorithm that we are going to use and workout 1 or 2 examples is the following. We want to solve  $x$  square- $D$   $y$  square is equal to 1 some initial condition let us start with  $X_0$  is equal to 1  $y_0$  is equal to 0,  $K_0$  is equal to 1,  $p_0$  is equal to 0.

At any step in the algorithm you have reached a point  $X_i - D y_i$  square is  $K_i$  find the next  $P_{i+1}$  by solving by using this 2 condition choose  $P_{i+1}$  such that  $P_i + P_{i+1}$  is divisible by  $K_i$  and modulus of  $P_{i+1}$  square- $D$  is minimum.

**(Refer Slide Time: 28:22)**

## Cakravāla according to Bhāskara

Then set

$$K_{i+1} = \frac{(P_{i+1}^2 - D)}{K_i} \quad Y_{i+1} = \frac{(Y_i P_{i+1} + X_i)}{|K_i|} = a_i Y_i + \varepsilon_i Y_{i-1}$$

$$X_{i+1} = \frac{(X_i P_{i+1} + D Y_i)}{|K_i|} = P_{i+1} Y_{i+1} - \text{sign}(K_i) K_{i+1} Y_i = a_i X_i + \varepsilon_i X_{i-1}$$

These satisfy  $X_{i+1}^2 - D Y_{i+1}^2 = K_{i+1}$

Iterate till  $K_{i+1} = \pm 1, \pm 2$  or  $\pm 4$ , and then use *bhāvanā* if necessary.

Note: We also need  $a_i = \frac{(P_i + P_{i+1})}{|K_i|}$  and  $\varepsilon_i = \frac{(D - P_i^2)}{|D - P_i^2|}$ , with  $\varepsilon_0 = 1$

Then you are able to obtain you a  $K_{i+1}$ ,  $Y_{i+1}$ ,  $X_{i+1}$  such that you reach  $X_{i+1}^2 - D Y_{i+1}^2 = K_{i+1}$  square is  $K_{i+1}$  iterate this till either you obtain 1 or -1 or +/-2 or +/-4 in the later case do (FL) there are 2 other quantities which will also use which will be interesting later  $P_i + P_{i+1}$  by mod  $K_i$  we will call it as  $a_i$  and the sign whether  $P_i$  square is greater than  $D$  or lesser than  $D$  we will denote it by writing an epsilon and is 1 or -1.

And this  $a_i$  and epsilon and  $i$  will simplify the calculation of the  $Y_{i+1}$  and  $X_{i+1}$  in terms of  $Y_i$  and  $X_i$  you can straight away do the calculation without divisions and all that.

(Refer Slide Time: 29:17)

## Bhāskara's Examples

का सप्तषष्टिगुणिता कृतिरेकयुक्ता का चैकषष्टिनिहता च सखे सरूपा।  
स्यान्मूलदा यदि कृतिप्रकृतिर्नितान्तं त्वघेतसि प्रवद तात ततालतावत्॥

द्वितीयोदाहरणे न्यासः

प्र ६१ क १ ज्ये ८ क्षे ३ ।

कुट्टार्थं न्यासः भा १ हा ३ क्षे ८ ।

हरतटे घनक्षेपे इति लब्धिगुणौ ३, १। इष्टाहतेति द्वाभ्यामूत्थाप्य जातौ लब्धि  
गुणौ ५, ७। गुणवर्गं ४९। प्रकृतेः शोधिते १२ व्यस्त इति ऋणम् १२• इदं  
क्षेपहृतं जातः क्षेपः ४• अतः प्राग्वज्जाते चतुःक्षेपमूले क ५ ज्ये ३९।

इष्टवर्गहृतः क्षेप स्यादित्युपपन्नरूपशुद्धिमूलयोर्भावनाथं न्यासः

क  $\frac{5}{2}$  ज्ये  $\frac{39}{2}$  क्षे १

क  $\frac{5}{2}$  ज्ये  $\frac{39}{2}$  क्षे १•

So, Bhaskara's first example was 61 and 67 (FL) which number square multiplied by 67 (FL) which square (FL) multiplied by 67 or another example (FL) so which square multiplied by 61 (FL) 1 added to it (FL) square root of that will be an integer, please tell us if you really know the (FL) if you really know how to calculate the (FL) equation tell us. So, now Bhaskara's own calculation goes like this I have just given it to tell you, you can compare with the table that I am going to write down later to see that how Bhaskara is arguing in his (FL) for solution.

**(Refer Slide Time: 30:14)**

**Bhāskara's Example:  $X^2 - 61 Y^2 = 1$**

i	$P_i$	$K_i$	$a_i$	$\varepsilon_i$	$X_i$	$Y_i$
0	0	1	8	1	1	0
1	8	3	5	-1	8	1
2	7	-4	4	1	39	5
3	9	-5	-3	-1	164	21

To find  $P_1$  :  $0 + 7, 0 + 8, 0 + 9 \dots$  divisible by 1. Of them  $8^2$  closest to 61. Hence  $P_1 = 8, K_1 = 3^{\text{rd}}$

To find  $P_2$  :  $8 + 4, 8 + 7, 8 + 10 \dots$  divisible by 3. Of them  $7^2$  closet to 61. Hence  $P_2 = 7, K_1 = -4$

After the second step, we have:  $39^2 - 61.5^2 = -4$

Now, since have reached  $K = -4$ , we use *bhāvanā* principle to get

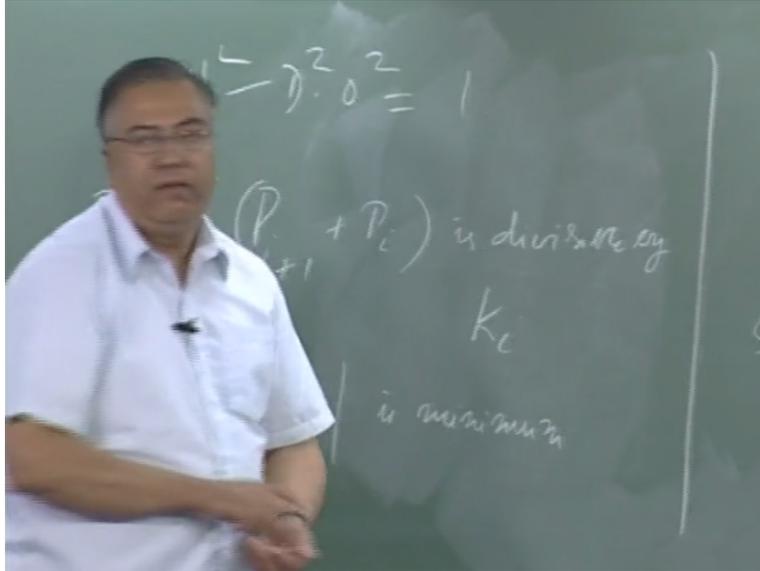
$X = (39^2 + 2) \left[ \left( \frac{1}{2} \right) (39^2 + 1)(39^2 + 3) - 1 \right] = 1,766,319,049$

$Y = \left( \frac{1}{2} \right) (39.5)(39^2 + 1)(39^2 + 3) = 226,153,980$

$1766319049^2 - 61.226153980^2 = 1$

It is not the entire solution a portion of the solution I have just told you till you reach the 39 and 5 okay. So, as I said the first step involved was that you take  $X_0$  is equal to 1,  $y_0$  is equal to 0,  $p_0$  is equal to 0,  $k_0$  is equal to 1,

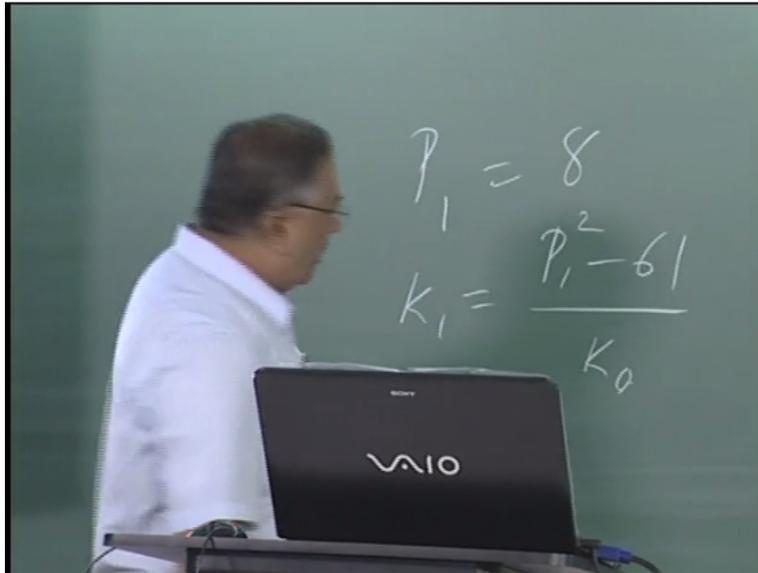
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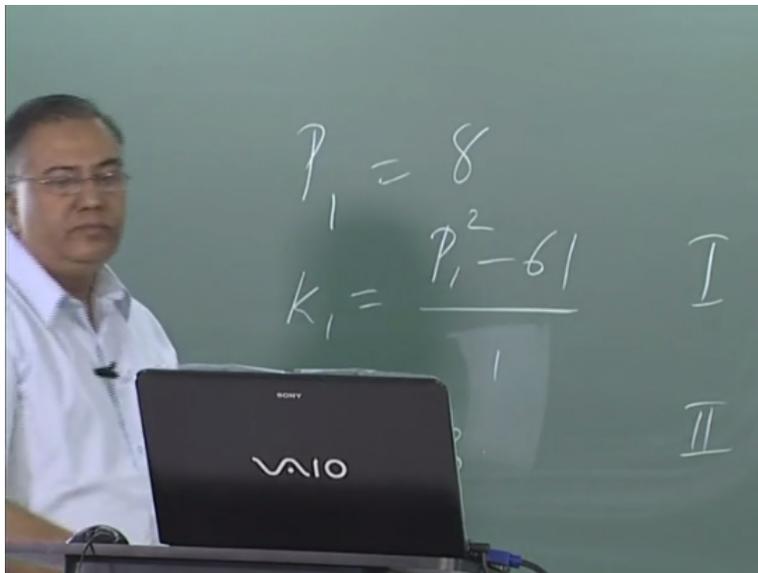
So, this is just the statement that 1 square-D square into 0 square is equal to 1. Now from this  $p_0$  I have to go to next  $p$ , so what are my 2 conditions, so those 2 conditions I will write them on the board  $P_{i+1}+P_i$  is divisible by  $k_i$ , so this our condition 1, condition 2  $P_{i+1}$  square-D is minimum, so keep this in mind when we understand this table, so my  $P_1$  should be such that,  $P_0+P_1$  is divisible by 1, know every integer is are  $0+7$ ,  $0+8$ ,  $0+9$  are all divisible by 1, square of  $P_1$  square should be closest to 61.

So, amongst all this squares it is 8 square which is closes to 61, so immediately we will choose our  $P_1$  to be 8. Once  $P_1$  has been chosen to be 8 what you need to do you have to calculate your  $K_1$ , so  $p_1$  we have already obtain to be 8.

**(Refer Slide Time: 32:02)**



So, your  $K_1$  is equal to  $P_1$  square-61 you are doing the 61 example right by  $K_0$ ,  $k_0$  is already 1.  
**(Refer Slide Time: 32:14)**



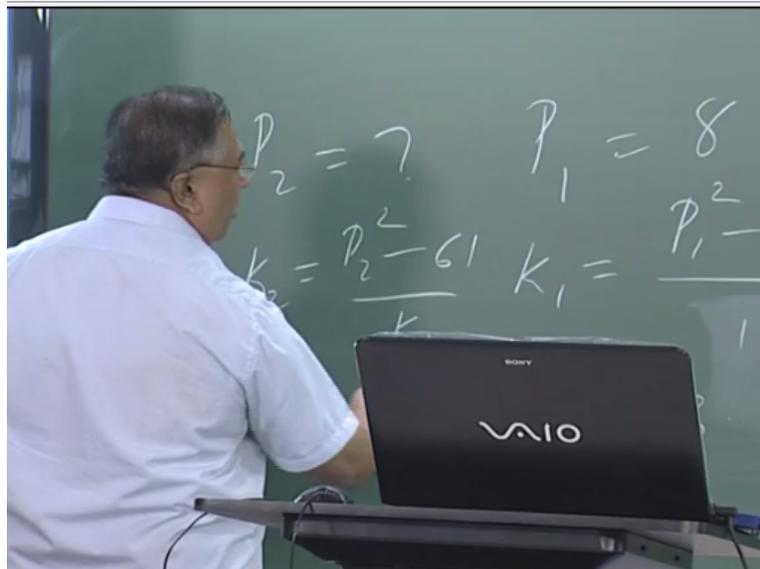
And so we will get this term yeah you can write sign, so this equal to 3, so 8 and 3 have come you really do not calculate all this at this stage you can keep on going till you find your  $K$  to be 1 you can fill up all this columns later, but still for the sake of convenience let us remember what  $x_1$  is what  $y_1$  is to do that we need to go back to this formula  $x_1$  is  $a_1$  into  $x_0$ ,  $x_2$  is  $a_1$  into  $x_1 + \epsilon_1$  into  $x_0$ .

Similarly  $y_2$  is  $a_1$  into  $y_1 + \epsilon_1$  into  $y_0$ , so you have to just apply that and remember your  $a_1$  is  $P_0 + P_1$  divided by  $k$ ,  $0 + 8$  divided by  $k$  that is 8 here. In this  $K$  can take positive and

negative values, a can take positive and negative values, P always takes positive values epsilon can take positive negative values, x and y always take positive value. So, what is the next step you have already obtained your P1 to be.

And so you have arrived at a stage which is very trivial 8 square-61 into 1 square is equal to 3 which is something which could have written just without doing all this, now what is the next step, P1+P2 is divisible by 3, so what are the possibilities 8+4 is divisible by 3, 8+7 is divisible by 3, 8+10 is divisible by 3 of this 4, 7 and 10 of these 3 the square of which is closest to 61, 10 square is 100 it is 39 away, 7 square is 49 it is 12 away, 4 square is 16 it is 35 away 45 away.

**(Refer Slide Time: 34:35)**



So, 7 is the so P2 is 7 once P2 is 7, so K2 is equal to P2 square-61/k1 that is equal to 7 square-61/3 therefore you get -1 and you can go ahead and calculate your x1, x2 and y2 before that what is your a2, a2 is 8+7/3 that is 5 and your epsilon is -1 because here this D square is larger than 61

**(Refer Slide Time: 35:33)**

$$17^2 + 2 \cdot 12^2 = 577$$

$$2 \cdot 17 \cdot 12 = 408$$

$$39^2 - 61 \cdot 5^2 = -4$$

And so at the step 2 you are obtaining  $39^2 - 61 \cdot 5^2$  is equal to  $-4$ , of course you in the normal course you have to go on but (FL) meaning the Bhaskara and (FL) has told us if you obtain  $\pm 2$ ,  $\pm 4$  use (FL) principle, so this is the formula in case  $-1$ , so let us go back and see that formula in case  $k$  is equal to  $-4$ , in case  $x^2 - Dy^2$  is  $-4$  then construct your  $x$  by  $x^2 + 2$  into half of  $x^2 + 1$  into  $x^2 + 3 - 1$  construct your  $y$  this way that will be solution of  $x^2 - Dy^2$  is equal to  $1$ .

So, let us see that is same thing that will do here  $39^2 + 2$  half into  $39^2 + 1$  into  $39^2 + 3 - 1$  and this is your  $x$  and this is your  $y$  and for such a big number this is smallest value of  $x$  and smallest value of  $y$  such that this equation satisfy. So, 1 is of the order of trillion, 2 is of the order of 100 226 million (()) (36:47).

**(Refer Slide Time: 36:51)**

### Bhāskara's Example: $X^2 - 61 Y^2 = 1$

i	P <sub>i</sub>	K <sub>i</sub>	a <sub>i</sub>	ε <sub>i</sub>	X <sub>i</sub>	Y <sub>i</sub>
0	0	1	8	1	1	0
1	8	3	5	-1	8	1
2	7	-4	4	1	39	5
3	9	-5	3	-1	164	21
4	6	5	3	1	453	58
5	9	4	4	-1	1,523	195
6	7	-3	5	1	5,639	722
7	8	-1	16	-1	29,718	3,805
8	8	-3	5	-1	469,849	60,158
9	7	4	4	1	2,319,527	296,985
10	9	5	3	-1	9,747,957	1,248,098
11	6	-5	3	1	26,924,344	3,447,309
12	9	-4	4	-1	90,520,989	11,590,025
13	7	3	5	1	335,159,612	42,912,791
14	8	1	16	-1	<b>1,766,319,049</b>	<b>226,153,980</b>

If you have not stopped at this -4 if you are continued the (FL) you would have obtain the solution you would obtain your k is equal to 1 at the 14<sup>th</sup> step in between you would have gone through various other k values, so 8 square-61 into 1 square is -3, 39 square-61 into 5 square is -4, 164 square-61 into 21 square is -5 like that it will go you would have reached the -1 point also and you can see a certain symmetry in this p and k that is the thing, that is moving in the site that is the (FL) about it.

That there is a you can see a very nice symmetry both in the k and in the p and that is what is the (FL) about it. Let us do the second example of Bhaskara quickly x square-67 y square is equal to 1 again we start with x<sub>0</sub> is equal to 1, y<sub>0</sub> is equal to 1, p<sub>0</sub> is equal to 0, k<sub>0</sub> is equal to 1 we have to find P<sub>1</sub>, p<sub>0</sub>+p<sub>1</sub> is divisible by 1 all integers are at visible of them again 8 is what is closest to 67, x square is to be taken but 8 square-67/1 is now -3, 8 square-61/1 was+ the 8 square-67/1 is -3.

So, the first step you will have k is -3, now again to this 8 again 8+ so this that is also very similar to the 61 example 8+4, 8+7, 8+10 are all divisible by 3 of them 7 square is closest only now 7 square-67/-3 is equal to 6, 7 square-67 is -18 divided by -3 is 6 and at this stage you are having 41 square-67\*5 square is equal to 6. Now next step 7+the next quantity should be divisible by 6, so 7+5, 7+11, 7+17 are the possibilities of them 5 square is what is closest to 67.

So, you choose 5 then 5 square-67 divided by 6 will be the next k that is -7 and at this stage you have 90 square-67\*11 square is -7, now again 5+p4 is divisible by 7, 2 is possible, 9 is possible, 16 is possible between 2, 9 and 16 it is 9 square which is closest to 67, 2 square is 4 it is 63 away, 9 square is 81 it is 14 away, 16 square is 256 it is 189 away therefore 9 is what is closest, so you choose 9 once you choose 9, 9 square-67/-7 it will turn out to be -2, you are still not obtain k is equal to 1.

But you have obtain 1 of these 4 quantities +/-2, +/-4 now we can do (FL) and we can do (FL) and we get 221 square-67\*27 square-2 from that we get 48842 square-67\*5967 square, so this is the final solution.

**(Refer Slide Time: 40:23)**

**Bhāskara's Example:  $X^2 - 67 Y^2 = 1$**

i	P <sub>i</sub>	K <sub>i</sub>	a <sub>i</sub>	ε <sub>i</sub>	X <sub>i</sub>	Y <sub>i</sub>
0	0	1	8	1	1	0
1	8	-3	5	1	8	1
2	7	6	2	1	41	5
3	5	-7	2	1	90	11
4	9	-2	9	-1	221	27
5	9	-7	2	-1	1,899	232
6	5	6	2	1	3,577	437
7	7	-3	5	1	9,053	1,106
8	8	1	16	1	48,842	5,967

Again if you do not stop at -2 you go on doing (FL) you will obtain the 1 value in the end and so this has taken you 8 steps but -2 occurred at the fourth step itself and so you could go on from there. I will not speak more about (FL) in this talk I will speak more about (FL) in the next talk I will speak about other things also done by (FL) in the (FL) work. So, one is solution of x square- D y square is equal to -1.

Obviously this equation is not solvable for all non square integers D, in fact if it solvable and all those cases where this -1 will appear in the sequence that you do. For instance in the case of 61 you have a -1 appearing and in fact you have a solution for x1-61 y square is equal to -1 and that

solution is beside this 297 only and 3805. So, Bhaskara just says this problem cannot be solved unless  $D$  is sum of 2 squares so, 61 is a sum of 2 squares what 2 squares 5 square and 6 square, so  
**(Refer Slide Time: 41:54)**

### The Equation $X^2 - D Y^2 = -1$

Bhāskara states that the equation  $X^2 - D Y^2 = -1$  cannot be solved unless  $D$  is a sum of two squares.

रूपशुद्धौ खिलोद्दिष्टं वर्गयोगो गुणो न चेत्।

Taking  $D = m^2 + n^2$  Bhāskara gives two rational solutions

$$(X, Y) = \left( \frac{n}{m}, \frac{1}{m} \right) \text{ and } (X, Y) = \left( \frac{m}{n}, \frac{1}{n} \right)$$

From these, it is sometimes possible to get integral solutions by ingenious use of *bhāvanā* and *cakravāla*, as Bhāskara shows in the case of  $X^2 - 13Y^2 = -1$ . He obtains  $X = 18$ ,  $Y = 5$ .

Now Bhaskara gives a method for calculating rational solutions for this equations when  $D$  is sum of 2 square if  $D$  is of the form  $m^2 + n^2$  Bhaskara gives you these two values which are rational solutions not integral solution. And he takes this example  $x^2 - 13 y^2$  is equal to  $-1$ , we can take this example and do (FL) on it and see whether you can obtain the solution, so 13 can be thought of as  $3^2 + 2^2$ .

**(Refer Slide Time: 42:39)**

$$\left(\frac{3}{2}\right)^2 - 13\left(\frac{1}{2}\right)^2 = -1$$

$$\frac{9}{4} - \frac{13}{4}$$

$$17^2 + 2.1$$

So,  $3/2$  and  $1/2$  will be a solution, so what is this is  $9/4-13/4$  this indeed correct okay. But there was another example where Bhaskara did not come up in the previous case he did come up with the integral solution.

**(Refer Slide Time: 42:29)**

The image shows a chalkboard with two equations written in white chalk. The first equation is  $18^2 - 13.5^2 = -1$ . The second equation is  $344 - 13.25$ , where the number 13.25 has a horizontal line underneath it.

That  $18^2 - 13.5^2$  is equal to  $-1$ , this is  $18^2$  is  $324 - 13.25$  no it is square is  $324$ .

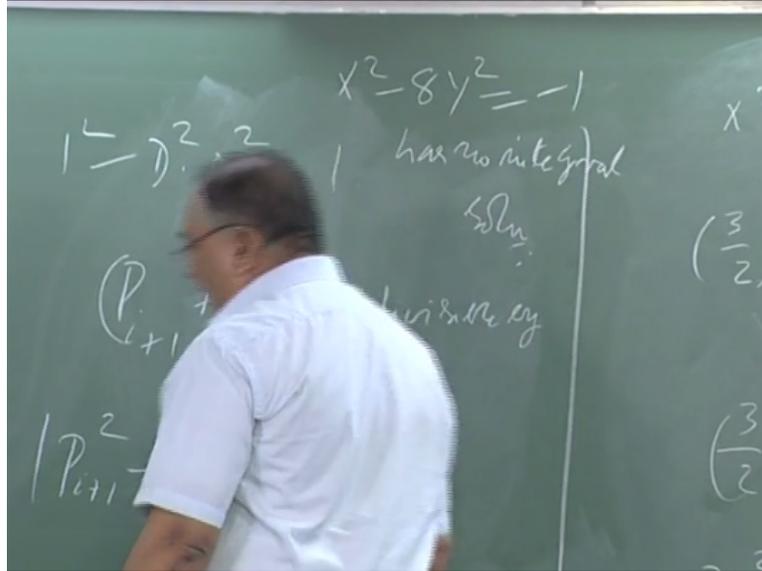
**(Refer Slide Time: 43:36)**

The image shows a chalkboard with three equations written in white chalk. The first equation is  $18^2 - 13.5^2 = -1$ . The second equation is  $324 - 13.25$ , with a horizontal line under 13.25 and an equals sign followed by a minus sign and a 1. The third equation is  $9 \quad 13 \quad 22 \quad 11$ .

This is  $325$  therefore is equal to  $-1$ . In the next example Bhaskara is nearly giving the rational solution for the case  $x^2 - 8y^2$  is equal to  $-1$  he just giving the solution  $x$  is equal to  $1$ ,

y is equal to half but here our (FL) tries to say that by doing (FL) of this solution you can obtain integral solutions by that is in the not 2.

(Refer Slide Time: 44:14)



You can convince yourself by a simple argument that  $x^2 - 8y^2 = -1$  has no integral solutions. One symbol were seeing it is think of  $x$  and  $y$  as quantities which are of the type  $4m+1$ ,  $4m+2$ ,  $4m+3$  and then see that the none of these 3 cases you have a solution for an equation like that. So, here is 1 sort of an error which was pointed out in a recent thesis by (FL) on (FL) commentary.

(Refer Slide Time: 44:49)

### General Quadratic Indeterminate Equations

Bhāskara has shown how the solution of *vargaprakṛti* equation with  $K = 1$  can be used in determining solutions of general quadratic indeterminate equations.

We illustrate the method with a couple of examples.

को राशिर्द्विगुणो राशिवगैः षड्भिः समन्वितः ।  
मूलदो जायते बीजगणितज्ञ वदाशु तम् ॥

अत्र यावत्तावद्भ्राशिर्द्विगुणो वर्गैः षड्भिः समन्वितः याव ६ या २ ।  
एष वर्ग इति कालकवर्गेण समीकरणार्थं न्यासः

याव	६	या	२	काव	०१
याव	०	या	०	काव	२१

(FL) is not use this (FL) in certain other examples these are not very complicated examples this a simple example. So, I will just give the English translation of this (FL).

(Refer Slide Time: 45:03)

**General Quadratic Indeterminate Equations**

अत्र समशोधने जातौ पक्षौ  
याव ६ या २, काव १।  
अथैतौ षड्भिः संगुण्य रूपं प्रक्षिप्य प्राग्वत् प्रथमपक्षमूलम्  
या ६ रू १।  
अथ द्वितीयपक्षस्यास्य काव ६ रू १। वर्गप्रकृत्या मूले क २ ज्ये ५,  
वा क २० ज्ये ४९। ज्येष्ठं प्रथमपक्षपदेनानेन, या ६ रू १ समं  
कृत्वा लब्धं यावत्तावन्मानम् ३ वा ८। ह्रस्वं प्रकृतिवर्णस्य कालकस्य  
मानम् २ वा २०।  
एवं कनिष्ठज्येष्ठवशाद्बहुधा।

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I am giving this (FL) argument just 2 show you the way you are arguing in the commentary on the solution.

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**General Quadratic Indeterminate Equations**

Here the two sides are  $6x^2 + 2x + 0y^2$  and  $0x^2 + 0x + 1y^2$   
Equating and clearing the sides, the equation is

$$6x^2 + 2x = y^2$$

Multiplying both sides by 6 and adding 1, we get

$$(6x + 1)^2 - 6y^2 = 1$$

The solutions of the above equation are, for instance,  
 $y = 2, 20$  and correspondingly  $6x + 1 = 5, 49$  and so on.  
In this way we get the integral solutions  $x = 8, y = 20$ .

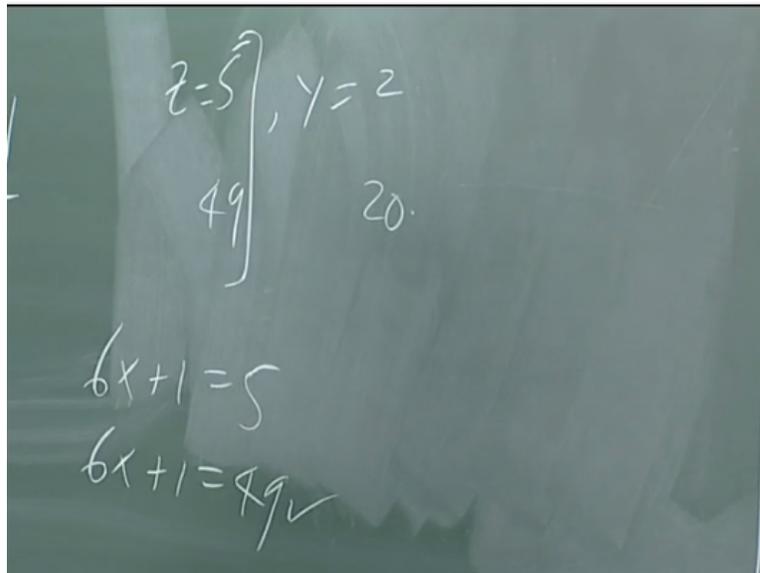
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So, one example he is trying to tell us is try and solve  $6x^2 + 2x$  is equal to  $y^2$ , so what does Bhaskara do of course as it is looking it is not looking like (FL) or any such thing, so you have to complete some squares to make it look like (FL) and so first is multiply both sides by 6

add 1, so if you do that left hand side becomes  $6x+1$  whole square and then this  $y$  square if you bring it, it becomes  $6y$  square is equal to 1.

So, if you call this  $6x+1$  as  $z$  or something else this is a (FL) equation, so now I mean if this is the (FL) equation you solve (FL) in the usual manner then one solution is  $y$  is equal to 2,  $6x+1$  is equal to 5 that is.

**(Refer Slide Time: 46:08)**



The image shows handwritten mathematical work on a chalkboard. At the top, it says  $z=5, y=2$ . Below that, it says  $49, 20$ . At the bottom, it says  $6x+1=5$  and  $6x+1=49$ .

So, you are solving  $z$  square- $6y$  square is equal to 1, so  $z$  is 5 and  $y$  is equal to 2 this one solution,  $z$  is equal to 49 and  $y$  is equal to 20 other this can be done by (FL) once you know one solution. So, your  $z$  is  $6x+1$ , so this part now you have to solve  $6x+1$  is equal to 5, you have to solve  $6x+1$  is equal to 49 this you can solve straight away,  $x$  is equal to 8, so you obtain integral solution  $x$  is equal to 8. So, it is just an application and algebra it transformation of this equation and then application of (FL).

**(Refer Slide Time: 46:59)**

## General Quadratic Indeterminate Equations

त्रिकादिद्वुत्तरश्रेढ्यां गच्छे क्वापि च यत्फलम्।  
तदेव त्रिगुणं कस्मिन्नन्यगच्छे भवेद्दद ॥

Say what is the number of terms of a sequence (in AP) whose first term is 3 the common difference is 2; but whose sum multiplied by three is equal to the sum of a different number of terms (of the same AP).

Let  $x$  and  $y$  be the number of terms. Then

$$3(x^2 + 2x) = y^2 + 2y$$

Multiplying both sides by 3 and adding 9,

$$(3x + 3)^2 = z^2 = 3y^2 + 6y + 9$$

--

Another problem now he is trying to positive and more sort of motivated way say what is the number of terms in an arithmetic progression whose first one is 3, common difference is 2 but whose sum multiplied by 3 is equal to the sum of a different number of terms of the same arithmetic progression recall the (FL) is the number of terms, so (FL) starting with 3 (FL) 2 is the addition at each stage in the arithmetic sequence.

So, you can immediately see if an arithmetic sequence has  $x$  terms 3 it is first term and common difference is 2, then if sum is  $x^2 + 2x$ . Similarly another arithmetic sequence which has wide terms 3 is the starting term and 2 is the common difference if sum is  $y^2 + 2y$  this you can straight away do by simple algebra now 3 times this should be equal to the other,  $x$  is the number of terms in the first case,  $y$  is the number of terms in the second case.

**(Refer Slide Time: 48:24)**

$$\begin{aligned}
 a &= 3 \\
 d &= 2 \\
 n &= x \\
 \frac{1}{2}n [2a + (n-1)d] \\
 &= x^2 + 2x
 \end{aligned}$$

I think you are all staring at it so I have to do this a is equal to 3, d is equal to 2 x is the number of terms, so what is the sum  $3 + \frac{1}{2}x$  into  $3 + x - 1$  into 2 right. So, this is  $2x$ . So, similarly y is the number of terms  $y^2 + 2y$  is the sum of the arithmetic is, so 3 times 1 is the other, so again now you have an quadratic equation in x and y you have to ultimately convert it into a (FL) form by transforming it suitably.

**(Refer Slide Time: 49:36)**

### General Quadratic Indeterminate Equations

Again multiplying by 3 and subtracting 18

$$3z^2 - 18 = t^2 = (3y + 3)^2$$

We find that a solution is  $z = 9$  and  $t = 15$ .

Other solutions can be found by doing *bhāvanā* with solutions (2, 1) of  $t^2 - 3z^2 = 1$ .

Thus we get  $z = 33$  and  $t = 57$ .

Hence we get  $x = 2, y = 4$  and  $x = 10, y = 18$ , and so on.

So, the transformations are given here multiplied by both sides by 3 add 9 you get an equation like this. So all that you have converted 1 into a square. Again multiplied by 3 and subtracting 18 you get an equation like this now you are in the domain that you can handle  $t^2 - 3z^2 = -18$  is a (FL) kind of equation. So, start with a solution of this and obtained the solution.

(Refer Slide Time: 49:55)

**Bhāskara's Solution of a Biquadratic**

Bhāskara II has given an example of the method of solution of a biquadratic equation of the special form

$$x^4 + px^2 + qx + r = 0$$

which involves adding  $ax^2 - qx + b$  to both sides, choosing  $a$  and  $b$  such that both sides are perfect squares.

This can be done in general, but it could involve solving a cubic equation for  $a$ . In his example, Bhāskara seems to have guessed the values of  $a$ ,  $b$ .

Bhāskara's example is in the *madhyamāharaṇī* section of *Bījagaṇita*:

को रशिर्द्विशतोक्षुण्णो रशिवर्गयुतो हतः ।  
द्वभ्यां तेनोनितो रशिवर्गवर्गोऽयुतं भवेत् ।  
रूपोनं वद तं रशिं वेत्सि वीजक्रियां यदि ॥

$$x^4 - 2(x^2 - 200x) = 10000 - 1$$

So, finally I come to the by quadratic I was talking about in general an equation like this  $x^4 + px^2 + qx + r = 0$  (50:05) by adding a term like this  $ax^2 - qx + b$  but what  $a$  to be used in this equation. If you try in  $c$  you will see that you will ultimately and the but the cubic equation. Therefore is not easy to solve this equation Bhaskara is given one example various slight to find out this  $a$  yes this  $a$  very correctly.

So, the equation Bhaskara is giving this much is transformable into this form (FL) is  $10000$  (FL) is coming  $x^4 - 2(x^2 - 200x) = 10000 - 1$  (FL) if you know algebra please answer this. And this the way Bhaskara goes above solving this.

(Refer Slide Time: 51:11)

## Bhāskara's Solution of a Biquadratic

Bhāskara first obtains the two sides of the equation in the form

$$x^4 - 2x^2 - 400x = 9999$$

He then remarks that if we add  $400x + 1$  to the left side we get a complete square, but the same thing added to the right hand side will not produce one, and hence proceeding in this way we cannot accomplish anything.

Hence, says Bhāskara, here one has to apply one's intellect.

If we add  $4x^2 + 400x + 1$  to both sides, we get the roots

$$x^2 + 1 = 2x + 100$$

This can be solved in the usual way to obtain

$$x = 11$$

Bhāskara remarks that this is how the intelligent should attempt such problems.

Krishnaswami Ayyangar has noted that Bhāskara's is indeed the first solution of a non-trivial biquadratic equation.

So, first Bhaskara says if I can add  $400x+1$  to the left hand side it will become a perfect square but eh right hand side will not become a perfect side,  $400x+9999+1$  is not a perfect square. So, in this were proceeding we are not reaching anywhere.

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## Bhāskara's Solution of a Biquadratic

अत्राद्यपक्षे किल यावत्तावच्चतुःशतीं रूपाधिकां प्रक्षिप्य मूलं लभ्यते  
परं तावति क्षिप्ते नान्यपक्षस्य मूलमस्ति एवं क्रिया न निर्वहति।  
अतोऽत्र स्वबुद्धिः।

इह पक्षयोर्यावत्तावद्द्वर्गचतुष्टयं यावच्चतुःशतीं रूपं च प्रक्षिप्य मूले

$$\begin{array}{r} \text{याव } १ \text{ या } ० \text{ रू } १ \text{ ।} \\ \text{याव } ० \text{ या } २ \text{ रू } १०० \text{ ।} \end{array}$$

पुनरनयोः समीकरणेन प्राग्वल्लब्धं यावत्तावन्मानम् ११।

इत्यादि बुद्धिमता ज्ञेयम् ॥

So, he says (FL) therefore we are not able to proceed for them (FL) therefore the mathematician is now pre the device his method to solve this equation, so what is the method add  $4x^2+100x+1$ , so this particular form  $+400x$  is coming from this  $+1$  is also coming but this choosing of this 4 is very special and this what he chooses if we add to left hand side and right hand side both become perfect squares and there square roots are like this.

So, left hand side becomes  $x^2+1$  whole square right hand side will become  $2x+100$  whole square and this can be solve this a ordinary quadratic equation Bhaskara is giving only the solution 11 what is the other solution -9, so that can also be used there is nothing.

(Refer Slide Time: 52:39)

### Bhāskara's Solution of a Biquadratic

अत्राद्यपक्षे किल यावत्तावद्यतुःशतीं रूपाधिकां प्रक्षिप्य मूलं लभ्यते  
परं तावति क्षिप्ते नान्यपक्षस्य मूलमस्ति एवं क्रिया न निर्वहति।  
अतोऽत्र स्वबुद्धिः।

इह पक्षयोर्यावत्तावद्गर्गचतुष्टयं यावद्यतुःशतीं रूपं च प्रक्षिप्य मूले

याव	१	या	०	रू	१।
याव	०	या	२	रू	१००।

पुनरनयोः समीकरणेन प्राग्वल्लब्धं यावत्तावन्मानम् ११।  
इत्यादि बुद्धिमता ज्ञेयम्॥

So in the end Bhaskara is saying (FL) therefore there is no algorithm for solving this equation one has to epsilon once intelligent to find a suitable solution if 1 is a b why do so. So, if that (FL) noted that this in the the first solution of such a non trivial biquadratic equation.

(Refer Slide Time: 52:58)

### References

1. *Bijaganita* of Bhāskara with Comm. of Kṛṣṇa Daivajña, Ed. Radhakrishna Sastri, Saraswati Mahal, Thanjavur 1958.
2. *Bijaganita* of Bhāskara with *Vāsanā* and Hindi Tr. by Devachandra Jha, Chaukhambha, Varanasi 1983.
3. H. T. Colebrooke, *Algebra with Arithmetic and Mensuration from the Sanskrit of Brahmagupta and Bhāskara*, London 1817.
4. *Bijaganita* of Bhāskara with *Vāsanā*, with notes by T. Hayashi, SCIAMVS, 10, December 2009, pp. 3-303.
5. A. A. Krishnaswami Ayyangar, New Light on Bhāskara's *Cakravāla* or Cyclic method, J.Indian Math. Soc. 18, 1929-30, pp.225-248.

So, the paper of (FL) was the paper which first showed that Bhaskara's cyclic methods actually in variably always leads to the solution of the, the quadratic indeterminate equation of the (FL)

equation he also showed that it is more optimal and different than what is called the (FL) method for solving what is called the (FL) equation I will something about that in my talk on (FL) tomorrow.

**(Refer Slide Time: 53:28)**

References

6. A. A. Krishnaswami Ayyangar, The Earliest Solution of the Biquadratic, *Current Sci.* Oct 1938, p.179.
7. B. Datta and A. N. Singh, *History of Hindu Mathematics, Vol II, Algebra*, Lahore 1938; Reprint, Asia Publishing House, Bombay 1962.
8. K. S. Shukla, Acārya Jayadeva the Mathematician, *Gaṇita*, 5, 1954, pp.1-20.

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This is Krishnaswami Ayyangar's paper on the biquadratic in this Shukla's paper on (FL) verses, thank you.