

Mathematics in India: From Vedic Period to Modern Times
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Lecture-18
Development of Combinatorics 1

So, as we saw earlier in this course we discuss the ideas of combinatorics which were first systematized under the notion of (FL) in (FL), so these ideas did develop in various other disciplines and in the next almost (FL) these ideas became much more complex but more interesting in different subjects.

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Outline

- ▶ **Combinatorics in *Āyurveda***
- ▶ ***Gandhayukti* of Varāhamihira**
- ▶ ***Mātrā-vṛttas* or moric metres**
 - ▶ *Prastāra*: Enumeration of metres of n -*mātrās* in the form of an array
 - ▶ *Saṅkhyā*: The total number of metrical forms of given number of *mātrās* – the Virahāṅka sequence (so called Fibonacci sequence)
 - ▶ *Naṣṭa* and *Uddiṣṭa* processes: Finding the metrical form given the row-number and vice versa in a *prastāra*.
 - ▶ *Mātrā-meru*: To determine the number of metrical forms with a given number of *gurus*.
 - ▶ Representation of any number as a sum of Virahāṅka numbers.

And we are going to study this in couple of lectures. So, in this lecture we will study combinatorics in the earliest ayurvedic types some ideas then something about combinatorics in the text of (FL) then we will discuss in some greater detail another aspect of (FL) these are called (FL) what in (FL) mostly considers were called the (FL) the syllabic meters we shall discuss the combinatorics or the idea (FL) in (FL) and that would be a fairly detailed discussion very interesting mathematical ideas do come up in that.

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Combinatorics in Āyurveda

The ancient Indian medical treatises of Caraka and Suśruta (prior to 500 BCE) deal with certain combinatoric questions in relation to the six *rasas* and the three *doṣas*. For instance, Caraka (*Sūtrasthāna* Ch. 26) discusses the 63 combinations that are possible from the 6 *rasas*:

भेदशैषां त्रिषष्टिविधविकल्पो द्रव्यदेशकालप्रभवाद्भवति तम्पदेक्ष्यामः ।

स्वादुरस्रादिभिर्योगं शेषैरस्रादयः पृथक् ।

यान्ति पञ्चदशैतानि द्रव्याणि द्विरसानि हि ॥

पृथग्गन्धादियुक्तस्य योगः शेषैः पृथग्भवेत् ।

मधुरस्य तथास्य लवणस्य कटोस्तथा ॥

त्रिरसानि यथा सङ्घां द्रव्याण्युक्तानि विंशतिः ।

वक्ष्यन्ते च चतुष्केण द्रव्याणि दशपञ्च च ॥

So, the ancient is that a treatises of caraka and susruta they deal with combinatory questions in relation to various (FL) and (FL) the one of the most well known quotation from caraka is the following which discusses the 63 possible combinations that can be done from the 5 rasas. So, what are the 6 rasas the (FL) these are the 6 and different words are used here in caraka for each of them in various places. (FL) like that, so (FL) there are 63 (FL) is the different possible combinations of this rasas and that is what we are going to discuss in this.

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Combinatorics in Āyurveda

स्वादुरसौ सहितौ योगं लवणादौः पृथग्गतम् ।

योगं शेषैः पृथग्यातस्तुत्करससंख्यया ॥

सहितौ स्वादुलवणौ तद्वत्कट्टादिभिः पृथक् ।

युक्तौ शेषैः पृथग्योगं यातः स्वादुपणौ तथा ॥

कट्टादौरसलवणौ संयुक्तौ सहितौ पृथक् ।

यातः शेषैः पृथग्योगं शेषैरसकट्ट तथा ॥

युज्यते तु कषायेण सतिक्तौ लवणोपणौ ।

पट् तु पञ्चरसान्याहुरेकैकस्यापवर्जनात् ॥

पट् चैवेकरसानि स्युरेकं षड्रसमेव च ।

इति त्रिषष्टिर्द्रव्याणां निर्दिष्टा रससंख्यया ॥

So, choosing 2 rasas out of 6 how many will come choosing 3 rasas out of 6 how many will come choosing 4 rasas out of 6 how many will come and finally choosing all the 6 rasas how

many possibilities then choosing only one rasa also how many possibilities. So, these are what is discussed in those set of verses of caraka.

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Combinatorics in Āyurveda

The number of *bheda* or combinations that can be obtained by combining different number of *rasas* is 63 as seen below:

The number of combinations of 2 *rasas* selected from the 6 is 15
The number of combinations of 3 *rasas* selected from the 6 is 20
The number of combinations of 4 *rasas* selected from the 6 is 15
The number of combinations of 5 *rasas* selected from the 6 is 6
The number of combinations of 1 *rasa* selected from the 6 is 6
The number of combinations of 6 *rasas* selected from the 6 is 1
Hence the total number of *bhedas* is 63
This is a particular case of the relation

$${}^n C_1 + {}^n C_2 + \dots + {}^n C_n = 2^n - 1$$

The *Suśruta-saṃhitā* (Uttarasthāna Ch. 63) actually lists each of these possibilities in a sequential enumeration akin to a *prastāra*.

So, this 63 combination, so obviously 2 rasas can be selected from a group of 6 that is 6 is 2 that and so to be 15, 3 rasas can be selected in 63 base that is 20, 4 rasas is 6 is 4 which is 62, 5 rasas is in 61 or 6 is 5 that is 6, 1 rasa is selected in 6 ways and when you select all the 6 rasas that is possible only in one way, so the total is 63 and this just one particular case of the usual sum of all binomial coefficients nC_1, nC_2 etc., nC_n this 2 to the power $n-1, 1+1$ to the power n will have all this terms.

And also the term nC_0 , so without that this is, so (FL) is also less this 63 possibilities but it does something more it gives a sequential enumeration of this 63 possibilities that is something akin to a (FL) for each set of combinations they are listed in a particular way anyway we will discuss such (FL) later.

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Gandhayukti of Varāhamihira

Chapter 76 of the great compilation *Bṛhatsaṃhitā* of Varāhamihira (c.550) is devoted to a discussion of perfumery. In verse 20, Varāha mentions that there are 1,820 combinations which can be formed by choosing 4 perfumes from a set of 16 basic perfumes ($^{16}C_4 = 1820$).

षोडशके द्रव्यगणे चतुर्विकल्पेन भिद्यमानानाम्।
अष्टादश जायन्ते श्रतानि सहितानि विंशत्या ॥

In verse 22, Varāha gives the method of construction of a *meru* (or a tabular figure) which may be used to calculate the number of combinations. This verse also very briefly indicates a way of arranging these combinations in an array or a *prastāra*.

पूर्वेण पूर्वेण गतेन युक्तं स्थानं विनान्त्यं प्रवदन्ति सङ्ग्राम्।
दृच्छाविकल्पैः क्रमशोऽभिनीय नीते निवृत्तिः पुनरन्यनीतिः ॥

Bhaṭṭotpala (c.950) in his commentary has explained both the construction of the *meru* and the method of *loṣṭa-prastāra* of the combinations.

Now going to an entirely different subject this is hinted in the introductory overview the discussion in that (FL) is in the 6th century he wrote the 3 very important verse (FL) on the (FL) on the and (FL) on the (FL) these are the 3 different parts of the (FL) the (FL) called (FL). So, in the chapter 76 of (FL) he is talking of the (FL) the the various ways in which perfumes can be made. In this chapter there is a discussion of combinatorics there is also a discussion of certain interesting discussion of certain magic's plates.

So, the magic's plates topic will be taken up and that will be discussed in this course, so he is talking of 16 basic perfumes (FL) and then as a first thing that if you want to make perfumes while choosing 4 out of this 16, so how many possibilities are there and that is of course the number of combinations that can be obtained by selecting core out of 16 and this and that $^{16}C_4$.

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$${}_{16}C_4 = \frac{16!}{12! \cdot 4!} = \frac{13 \cdot 14 \cdot 15 \cdot 16}{1 \cdot 2 \cdot 3 \cdot 4} = 1820$$

P_1, P_2, \dots, P_{16}

$P_2, P_7, P_{12}, P_{13} \downarrow$

So, today we will call this by doing this right 16 factorial by 12 factorial, 4 factorial, so this is 13 into 14 into 15 into 16 by 1 into 2 into 3 into 4, so you will finally get that same number 1820. (FL) so when we have this 16 basic perfumes the number of possibilities that we are choosing 4 they will be 1820. Now a little later (FL) tells you how to how he arrived at this 1820 was that he is going to give you a (FL) a (FL) very similar to what (FL) but still is a different (FL).

The (FL) is stated in a half verse and then in the other half verse (FL) is discussing some other fairly complex topic it is how to list all this possibilities. So, suppose you are calling this as just let me call them as perfume 1, perfume2 etc., perfume 16 these are your perfumes now you are choosing some 4, so p_2, p_7, p_{12}, p_{13} is one such possibility but now there are 1820 possibilities and as I said once you have combinatory problem you should give a rule to list all of them.

So, a rule to list all this combinations and that is what is given, so a 1820 row array and that is what is given in just this half verse of (FL) the first half is discussing the meru that we see. So, as you can (FL) is not much more (FL) than (FL) that he gave the meru in just the sutra (FL) so he is giving you the meru in just (FL) so one of the major commentaries on (FL) is by (FL) is commented on the (FL) also.

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Gandhayukti of Varāhamihira

16			
15	120		
14	105	560	
13	91	455	1820
12	78	364	1365
11	66	286	1001
10	55	220	715
9	45	165	495
8	36	120	330
7	28	84	210
6	21	56	126
5	15	35	70
4	10	20	35
3	6	10	15
2	3	4	5
1	1	1	1

So, (FL) has explained in detail the construction of this meru, so in the first column you just write the numbers 1 to 16, so in the second column you write sum of these numbers, so 1 is 1, 1+2 is 3, 1+2+3 is 6, 1+2+3+4 is 10, 1+2+3+4+5 is 15, 1+2+3+4+5+6 is 21 obviously you do not need to do it that way you have already summed up to 5 this is 15, so 15+6 is 21, 21+7 is 28, 28+8 is 36, so like that you go on and so (FL) is that you start from the beginning and keep on adding (FL).

So, you do not reach the last row just finish before the, so (FL) now the next column is sum of this sums, so 1 is 1, 1+3 is 4, 1+3+6 is 10, 1+3+6+10 is 20 so this column is done by sum of sum this is $16C_1$, this is $16C_2$ this is $16C_3$ and finally this is $16C_4$ which is again sum of the sum of the sum of the sums. So, that is the way of the doing this calculating the number of combinations, so again now this thing is based upon at different reconciliation. the reconciliation of (FL) is this, nCr is sum of $n-1 C_{r-1}, n-2 C_{r-2}$ etc., $r-1 C_{r-1}$.

And you can just if you examine this you can immediately see from this relation (FL) relation follows from (FL) relation if you just expand $n-1C_r$ once again using the same rule this relation for also both these are equivalent reconciliations but (FL) is different (FL) both are simple ways of calculating the nCr but this formula that nCr is equal to various ways in which it is said right n into $n-1$ etc., up to $n-r+1$ by 1 into r .

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$${}^nC_r = \frac{n(n-1)\dots(n-r+1)}{1 \cdot 2 \dots r}$$

So, this is stated in mathematics books, this is stated in the (FL) this is perhaps first stated in (FL). So, this format of a formula for the nCr is given in the text of mathematics they also deal with some combinatorics but in other they are just giving you this fabulous method and the table is equivalent into the formula is the table is based upon of it is certain recurrence relation and (FL) was based upon this (FL) is based upon this.

And this sum of sums this in this formula is a very very important formula later on you will discover Narayanabantitha has discovered this formula for this repeated summation.

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Gandhayukti of Varāhamihira

- ▶ In the first column the natural numbers are written.
- ▶ In the second column, their sums, in the third the sums of sums, and so on. One row is reduced at each step.
- ▶ The top entry in each column gives the number of combinations.

The above *meru* is based on the relation

$${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-2}C_{r-1} + \dots + {}^{r-1}C_{r-1}$$

which is equivalent to Pingala's relation

$${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$$

And that is at the basis of much of the initiation into calculus arises out of this formula, so the entire all his ideas are implicit already in (FL) in that meru.

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Mātrā-ṛtta

- ▶ By assigning the values 1 for *laghu* and 2 for *guru*, we can obtain the total value or *mātrās*, associated with each metrical form.
- ▶ This leads to the notion of *mātrā-ṛttas* or moric metres, where the metrical patterns are classified by their total value or *mātrā*.
- ▶ Piṅgala has only briefly touched upon *mātrā-ṛttas* in Chapter IV of *Chandaḥ-śāstra* while discussing the various forms of *Āryā* and *Vaitālīya ṛttas*.

Now we go on to the study of matra (FL), so (FL) are again formed out of syllables only, now instead of counting the number of syllables we will start counting the total value or in a loose the total time taken to pronounce the syllable. So, a (FL) we will assign a value 1 and to (FL) we will assign a value 2, so if you just takes something like this, just take this (FL)

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रामचन्द्र
4
रामचन्द्र
2 1 2 1

So, this has 4 syllables and each of them is a (FL) or if you take something like ramachnadra, so this is (FL) this is 2, this is 1, this is (FL) this is (FL), (FL) is a (FL), (FL) is a guru because it is

followed by a cogent consonant, so in all there are 6 (FL) in it, there are 4 (FL) in it. So, this way of doing poetry not very common in Sanskrit it become very very common in all the regional languages in kanada, in telugu, in hindi, in ahh (()) (12:45) to start with a many many regional languages.

The meters the (FL) is based upon the matra (FL) the number of matras that each (FL) is having and then you have a combination where you have different matra units in a single (FL) metres. So, Sanskrit also has recline the (FL) meter and the most famous example is Arya and the most famous example for mathematicians of a work which has verses in Arya is the (FL) or the Aryabhattiya.

Arya meter is a complex combination of matra units put together it fixes the total number of matras in total number of units in each of the (FL) composed of various matra units there something called (FL) meters also. So, all these are discuss by (FL) but does not give a detailed discussion of just (FL) alone (FL) alone means you just fix the total number of matra say each (FL) of your verse has this number of matras.

So, that is a much simpler sort of a problem (FL) is discussing a more complex problem of (FL) and things like that in Arya in (FL) ammitters.

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Mātrā-vṛtta

- ▶ *Mātrā-vṛttas* are more commonly met with in *Prākṛta* and regional languages.
- ▶ The *Prākṛta* work *Vṛttajāti-samuccaya* of Virahāṅka (c.600) discusses the *pratyayas* of *prastāra* and *saṅkhyā* for *mātrā-vṛttas*.
- ▶ A more detailed discussion of *mātrā-vṛttas* is available in *Chandonuśāsana* of Hemacandra (c.1200), *Prākṛta-Paiṅgala*, *Vāṅībhūṣaṇā* of Dāmodara (c.1500) and the commentary of Nārāyaṇabhaṭṭa (c.1550) on *Vṛttaratnākara* of Kedāra (c.1000). ☺

He calls some of them (FL) let us come back to the definition of a matra meter I have already told you.

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Mātrā-Prastāra

येष एव प्रस्तारो मात्रावृत्तानां साधितः किन्तु।
मात्रा यत्र न पूर्यते प्रथमं स्पर्शं तत्र देहि॥

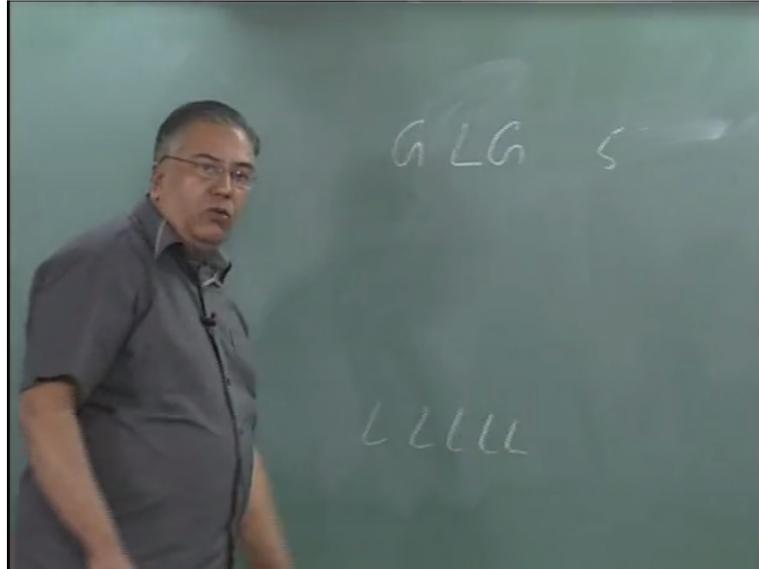
(वृत्तजातिसमुच्चयः ६.२०)

The *prastāra* in the case of a *mātrā-ṛtta* of n -*mātrās* is to be generated following the same procedure as in the case of a *varṇa-ṛtta* except for the following:

- ▶ The first row consists of all Gs if n is even and an L followed by all Gs if n is odd.
- ▶ [Given any row in the *prastāra*, to generate the next, scan from the left to identify the first G. Place an L below that. The elements to the right are brought down as they are.]
- ▶ The remaining *mātrās* to the left are filled in by all Gs, and by placing an L at the beginning, if need be, to keep the total number of *mātrās* the same.

We will repeat it once again a matra meter is (FL) is one were you assign value 1 to a (FL) you assign a value 2 to a (FL) and you look for all possible metrical forms of a total number of matras.

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So, suppose you had GLG as a version this has 5 matras now how many such metrical forums are there which have 5 matras alone of course the ultimate thing approach is L all (FL) that is

also 5 matra. Now you have to list all the so the first question is to start with the (FL) all such matras, (FL) of a given matra.

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Mātrā-Prastāra

येष एव प्रस्तारो मात्रावृत्तानां साधितः किन्तु।
मात्रा यत्र न पूर्यते प्रथमं स्पर्शं तत्र देहि ॥

(वृत्तजातिसमुच्चयः ६.२०)

The *prastāra* in the case of a *mātrā-ṛtta* of n -*mātrās* is to be generated following the same procedure as in the case of a *varṇa-ṛtta* except for the following:

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- ▶ [Given any row in the *prastāra*, to generate the next, scan from the left to identify the first G. Place an L below that. The elements to the right are brought down as they are.]
- ▶ The remaining *mātrās* to the left are filled in by all Gs, and by placing an L at the beginning, if need be, to keep the total number of *mātrās* the same.

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And then start doing (FL) and all the usual stuff that (FL) has told us that once we have a (FL) what we should do. So, one of the most important earliest verse (FL) is a work in (FL) it is (FL) it is by (FL) whose name you will have remember. We will soon see (FL) is the originator of the set of numbers which we commonly think are called by the name of (FL) therefore (FL) is going to appear very often in our discussion you wrote this work along 6th century.

There is a commentary by 1 gopala in Sanskrit then more detailed discussion of matra (FL) is appear in Hemacandra is one of the great scholars in Sanskrit and (FL) so he has written (FL) in 12th century then I mention (FL) et., we will quote some verses of these also for particular issue. So, first is the (FL), so before going into the rule of the (FL).

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Mātrā-Prastāra

1-Mātrā Prastāra

1	G
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2-Mātrā Prastāra

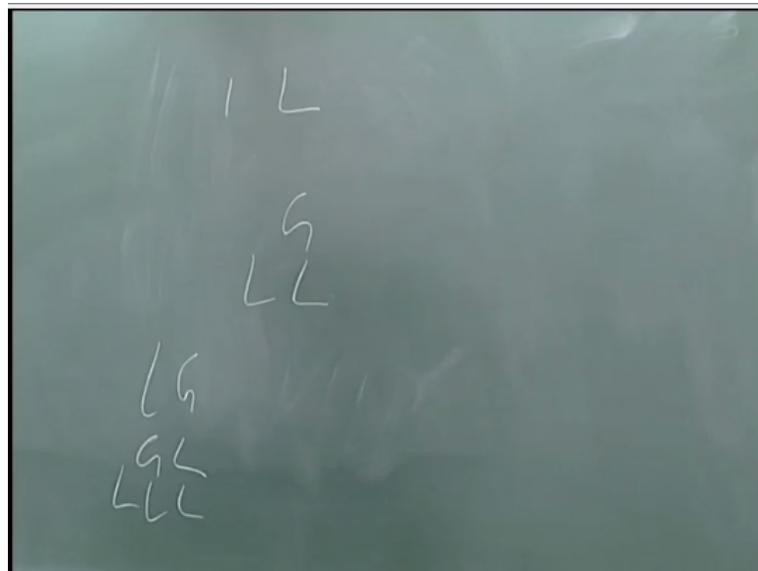
1		G
2	L	L

3-Mātrā Prastāra

1		L	G
2		G	L
3	L	L	L

Let us just see some example, so 1 matra (FL) is just a (FL).

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It cannot be a guru, guru is 2 matras, so the 2 matra (FL) there is only 2 possible (FL) a single guru or 2 lagus, 3 matras LG, GL, LLL, so what is the general rule for having a (FL) of all such (FL). So, this is the simple verse of (FL) means lagu it is is verse of lagu. So, the rule for (FL) is the same as the rule for (FL) and the in the end if the matra is not completing you just put a lagu I will just explain it by considering say the 4 (FL).

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Mātrā-Prastāra

4-Mātrā Prastāra

1			G	G
2		L	L	G
3		L	G	L
4		G	L	L
5	L	L	L	L

5-Mātrā Prastāra

1			L	G	G
2			G	L	G
3		L	L	L	G
4			G	G	L
5		L	L	G	L
6		L	G	L	L
7		G	L	L	L
8	L	L	L	L	L

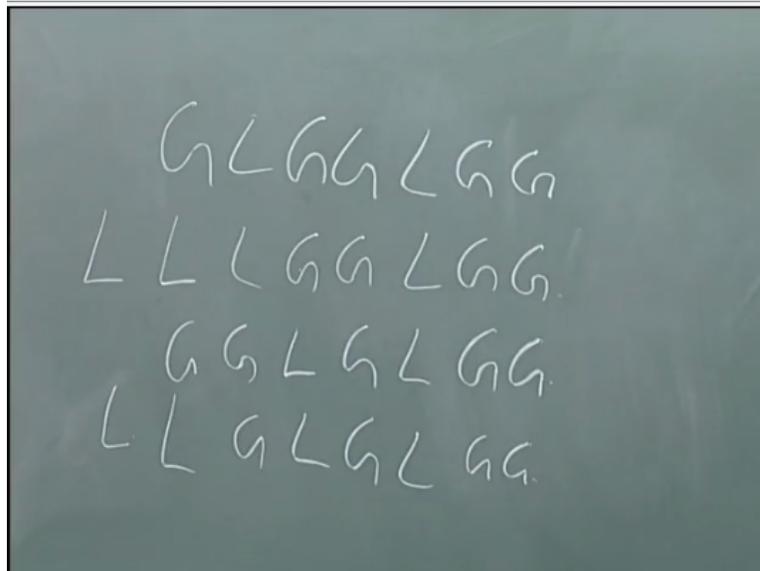
So, you start with G's of course if it is 5 matras you cannot have all G's you have to have a L like the end. So, you start with all G's from the right and in the end if necessary put an L that has to be the starting low. Now what was the earlier rule from the left you starts can be whenever you find the G put an L below carry whatever is there to the right below earlier the rule was in the left fill it up with as many G's.

Then what you are filling up the number of syllables were fixed, so you are filling up those many syllables now the syllables are not numbers syllables is not fixed as you can see a 4 matra (FL) can have 2 syllables, 3 syllables it can have 4 syllables also what is fix this the total number of matras, the total value G is 2, L is 1 you add the number of G's and number of L's by that rule that number should be the same.

So, in some sense it is what can be called as an all dirt partition of the total matra by just 1 and 0 rather 1 and 2. So, at this stage you have to add an L you have a total of 4 matras now again starts scanning from the left you have a G here. So, next row you put an L here in the start putting G's and you cannot put many G's in the end you have to reach 4, so you have to put an L, so as I said if necessary put an L at the end.

Next row below this G put an L bring down the L as it is put a G in the left it is over 4 matras are finished again come down put an L below this put L's in the right all that you can do is put an L in the end. So, 5 (FL) similarly, so we can take something suppose you had some row like this.

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GLGGLGG something like this how many matras are there 2+1, 3, 5, 7, 8, 10, 12 this a in the (FL) of the 12 (FL). So, what you do below the G you put an L bring down whatever you had in the right as it is but now the matras are not hiding up G has been reduced to an L, so you have to put an L in the left that will be the next row what will be the next row start scanning from the left where G occurs put an L below bring down the things to the right as they are.

Now how many in occurs in (FL) 2+2, 4, +1, 5 +2, 7+1, 8, so we can put 2 G's to the left because you are considering the (FL) 12 matra (FL). Now again start from the left this will become LGLGLGG and you have to put an L to complete the total number of matra. So, like this you have to work out the (FL). So, the rule is the same as in the case (FL) starts scanning from the left when you encounter a G put an L below bring down the a syllables to the right as they are.

In the left put as many G's as possible and if necessary add an L such that the total number of matras is fixed at which your (FL) for which the number of matras for which doing (FL). So, the 5 matra (FL) is like this, it has 8 rows, 4 matra (FL) had 5 rows.

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Mātrā-Prastāra

1-Mātrā Prastāra

1	G
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2-Mātrā Prastāra

1		G
2	L	L

3-Mātrā Prastāra

1		L	G
2		G	L
3	L	L	L

The 3 (FL) had 3 rows, 2 (FL) had 2 rows okay.

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Mātrā-Prastāra

6-Mātrā Prastāra

1				G	G	G
2			L	L	G	G
3			L	G	L	G
4			G	L	L	G
5		L	L	L	L	G
6			L	G	G	L
7			G	L	G	L
8		L	L	L	G	L
9			G	G	L	L
10		L	L	G	L	L
11		L	G	L	L	L
12		G	L	L	L	L
13	L	L	L	L	L	L

The 6 (FL) start with GGG end up with LLLLLL, and this has 133 rows, so immediately you can see there is pattern you have this G's here in the first 5 rows you have these L's in the next 8 rows. If you go behind this had 5 rows this had 8 rows, so all that you need to do to obtain the 6 matra (FL) is right the 4 (FL) and write the 5 (FL) below that write 5 G's here write 8 L's here obviously because to the 4 (FL) of the (FL) if you put G on the right it will become all 6.

To the 5 matra of the (FL) you had L to the right it will all again become 6, so the way to generate 6 is the generate all the 4, generate all the 5 to the 4 you put a G on the right to the 5

you put a L on the right that will be (FL) of (FL) of 6 matras. And so, you can see that the 5 is constructed out of 3 and 4 in a very similar way there are 3 rows in the 3 matra of the (FL) those are the same 3 here, those are the same 3 here in the beginning those are the same 3 here.

You put a G to the right, there are 5 rows in the 4 matra of the (FL) there the same here to that you put a L on the right, so this is the rule.

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Saṅkhyā

द्वौ द्वौ पूर्वविकल्पौ या मेलयित्वा जायते सङ्ख्या।
सा उत्तरमात्राणां सङ्ख्याया एष निर्देशः ॥

(वृत्तजातिसमुच्चयः ६.४१)

The number of metrical forms S_n in the n -mātrā *prastāra* is the sum of the number of metrical forms S_{n-1} , S_{n-2} , in the *prastāras* of $(n-1)$ and $(n-2)$ mātrās respectively:

$$S_n = S_{n-1} + S_{n-2}$$

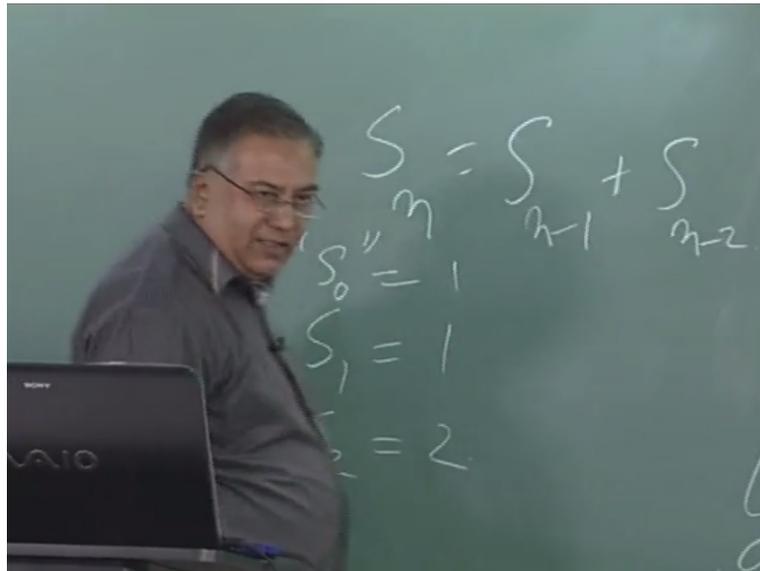
The above rule follows from the fact that the n -mātrā *prastāra* is generated as follows:

The first S_{n-2} rows are obtained by adding a G to the right of each row of the *prastāra* of $(n-2)$ -mātrās.

The next S_{n-1} rows are obtained by adding an L to the right of each row of the *prastāra* of $(n-1)$ -mātrās.

So, this is what our (FL) is telling us (FL), so the number of rows in the n matra (FL) is given by the sum of the number of rows.

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In the n-1 mata (FL) and the number of rows in the n-2 mata (FL) of course we know that there is only (FL) of length you hypothetical think of an S0 is equal to 1 and now you know what this recurrence relation means this is an recurrence relation by which what is call the (FL) sequence of numbers is generated and so this (FL) just that

(Refer Slide Time: 23:59)

Saṅkhyā

The *prastāra* of one *mātrā* has only one metrical form and that of 2-*mātrās* has just two metrical forms (G and LL).

$S_1 = 1$ and $S_2 = 2$

Therefore we get the following Virahāṅka sequence of *saṅkhyāṅkas* —the so called Fibonacci (c.1200) sequence.

n	1	2	3	4	5	6	7	8	9	10
S_n	1	2	3	5	8	13	21	34	55	89

The S_n is sum of $S_{n-1}+S_{n-2}$ why the n mata (FL) is generated by first writing down the m-2 mata (FL) below that the n-1 (FL) write n-(()) (24:13) the S_{n-2} number of G's there write S_{n-1} number of L's below and therefore the number of possibilities of metrical forums of total value n is S_n and those possibilities are $S_{n-1}+S_{n-2}$ and so the (FL) are the number of rows in the (FL) of n matras are nothing but the numbers 1, 2, 3,5, 8, 13, 21, 24, 55, 89 the numbers you are familiar in mathematics this have been called (FL) sequence. Because (FL) wrote down a problem concerning rats in his book.

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Uddiṣṭa

दत्त्वा पूर्वयुगाङ्कं गुरु शीर्षाङ्कं विलुप्य शेषाङ्के।
अङ्कैरितोऽवशिष्टैः शिष्टैरुद्दिष्टमुद्दिष्टम् ॥

(वाणीभूषणम् १.३१)

Given a moric metric form, write down from the left the *saṅkhyāṅkas* sequentially in the following way:

- ▶ For each G write two *saṅkhyāṅkas*, the first above and the next below.
- ▶ For each L, write one *saṅkhyāṅka* above.
- ▶ Add all the *saṅkhyāṅkas* above the Gs. This sum subtracted from the total *saṅkhyā* associated with the *mātrā-ṽṛtta* gives the row number.

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So, now how to do (FL) and (FL), so we are going to call this numbers as (FL) numbers and essentially you will find that this numbers play the major role in fact in understanding (FL) and (FL) for the syllabic meters for the (FL) the number of possible metrical forums was given by 2 to the power n and every number was represented as a sum of the powers of 2 and that is nothing but the what we now know as the binary representation of a number.

Therefore in doing going from the (FL) to the row and from the row number back to the metrical forum the binary representation was the thing that came of hand. So, to do the same kind of problem with (FL) you will see that every number ultimately will be related to the (FL) numbers what are so called (FL) numbers and they will play a role. So, what is the (FL) problem, so given a metrical forum how do you find out the row in which it occurs in the (FL).

So, the rule is for each G write 2 (FL), the first one above and the second one below and for each lagu just write 1 (FL) above add all the (FL) above the (FL) this sum subtracted from the total (FL) gives you the row number (FL) okay, this is from a very small book called (FL) the first chapter in about 20 verses discusses the rule of (FL) both the (FL) and (FL) this written around 15 (()) (27:10) AD thing by Damodhara.

(Refer Slide Time: 27:13)

Uddiṣṭa

Example: To find the row-number of the metrical form GLLG in the 6-*Mātrā-prastāra*:

1	3	5	8
G	L	L	G
2			13

The row number is $13 - (1 + 8) = 4$

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So, an example will make it very clear, so we can here we will do another example on the board let us see this example.

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Saṅkhyā

The *prastāra* of one *mātrā* has only one metrical form and that of 2-*mātrās* has just two metrical forms (G and LL).

$S_1 = 1$ and $S_2 = 2$

Therefore we get the following Virahāṅka sequence of *saṅkhyāṅkas* —the so called Fibonacci (c.1200) sequence.

n	1	2	3	4	5	6	7	8	9	10
S_n	1	2	3	5	8	13	21	34	55	89

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So, the (FL) are (FL) that is these are the total numbers of the (FL) corresponding to each matra value, so for 1 matra is 1, 2 is 2, 3 is 3, 4 is 5, 5 is 8, 6 is 13, so this sequence the (FL) sequence is to be written down.

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Uddiṣṭa

Example: To find the row-number of the metrical form GLLG in the 6-Mātrā-prastāra:

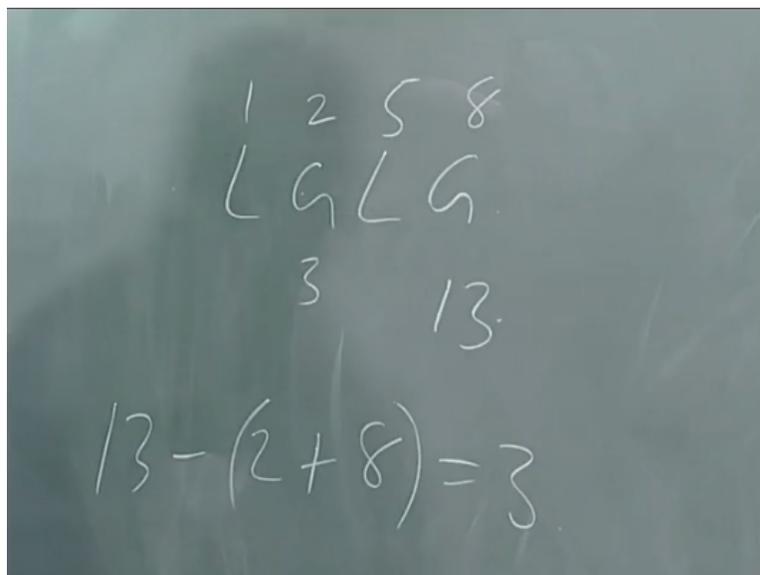
1	3	5	8
G	L	L	G
2			13

The row number is $13 - (1 + 8) = 4$

Now your GLLG is 6 matras, so this is a part of the 6 matra (FL) so write 6 (FL) 2 for each guru, 1 for each (FL) the 1 that you are writing for guru write 1 above and 1 below write 1 above and 1 below add the numbers above the guru subtract it from the total number of metrical forums of 6 matras which is 13 what you what remains is the row number of GLLG in the (FL) 6 matras. So, let us go back to 6 matra (FL), so GLLG is 4, that is the row that you have.

So, you can take anything else, so let us some other example of 6 matras only LGLG let us find out which row it is in the 6 matra.

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So, again 1, 2, 3, 5 I am sorry 8, 13 right which how the (FL) have to written some the (FL) above the gurus subtract it from 13, so that is equal to 3, so actually this would be the row just LGLG. Now what is the logic here will become clear a little bit once we do the other process also we read (FL) because the algorithm is very simple just write this (FL) and add write 2 for each guru add the numbers above the guru and subtract it from the total (FL) you will (()) (29:37) fine.

So, this is now doing given to you like almost like a a puzzle kind of a thing how this is occurring let us see the other way that is the (FL) problem.

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Nasta

नष्टे कृत्वा कलाः सर्वाः पूर्वयुग्माङ्कयोजिताः ।
 पृष्ठाङ्कहीनशेषाङ्कं येन येनैव पूर्यते ॥
 परां कलामुपादाय तत्र तत्र गुरुर्भवित् ।
 मात्राया नष्टमेतत्तु फणिराजेन भाषितम् ॥ (वाणीभूषणम् १.३२-३३)

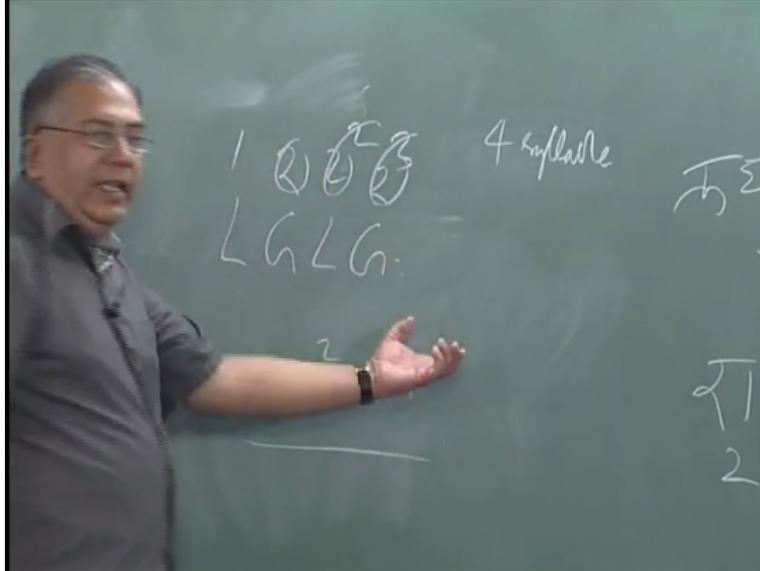
To find the moric metric form associated with a given row-number in the n -*mātrā-prastāra*

- ▶ Write down n Ls with the sequence of *saṅkhyāṅkas* above them.
- ▶ Subtract the given row number from the *saṅkhyāṅka* S_n .
- ▶ From the result, subtract S_{n-1} if possible. Otherwise, subtract S_{n-2} and so on till the end.
- ▶ The moric metric form is obtained by converting each L below a *saṅkhyāṅka* which has been subtracted, together with the L to the right of it, into a G.

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And then we will be able to see some understanding of how this kind of an algorithm is arising. So, what is the nasta problem given a number we have to find out what is the (FL) pattern in the (FL). So, take any number if you want to compare it with what we did in the case of (FL) what were we doing were (FL) if something like LGLG we were talking about.

(Refer Slide Time: 30:28)



So, this will be a 4 syllable (FL) and what you did you wrote 1, 2, 2 square, 2 cube you added the number above L's and added 1 to it that is where that occur. So, the (FL) for (FL) played a crucial role in analyzing the (FL) problem for (FL), now we are doing it in a different way because the nature of (FL) is that syllables are not fixed it is sum of the values of that, that is fixed, so the mathematics of it becomes entirely different.

Now if you want to find out the moric metric form associated with a given row number in the n (FL). First you write down n L's with the sequence of (FL) above them subtract the given row number from the last (FL) S_n from the result subtract S_{n-1} if possible. Otherwise subtract S_{n-2} and so on till the end the moric form is obtained by looking at all those (FL) which have been subtracted, so as a statement of an algorithm it is somewhat complicated by if you look at an example it will become very clear, it is also fairly simple algorithm.

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Naṣṭa

Example: To find the seventh metrical form in the
6-mātrā-prastāra

1 2 3 5 8 13
L L L L L L

- ▶ $13 - 7 = 6$.
- ▶ 8 cannot be subtracted from 6
- ▶ $6 - 5 = 1$
- ▶ 3, 2 cannot be subtracted from 1
- ▶ $1 - 1 = 0$

Thus 5 and 1 are the *saṅkhyāṅkas* which have been subtracted.

Hence the metric form is GLGL

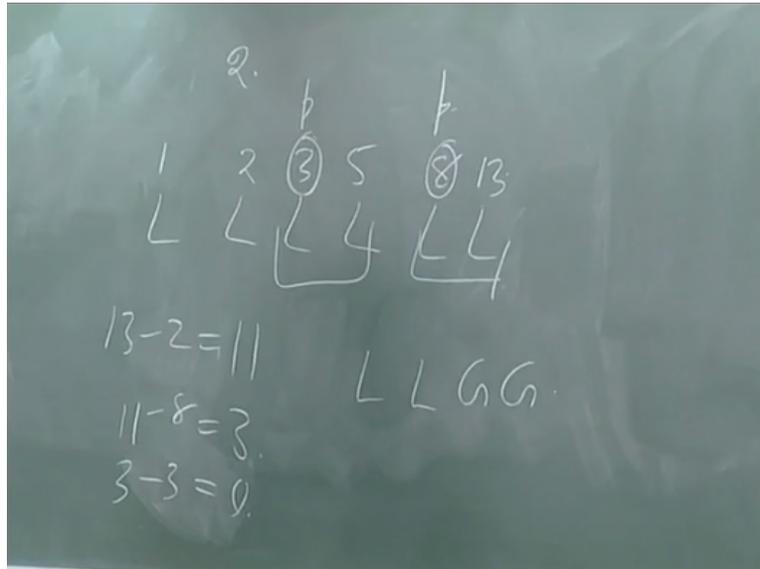
So, seventh metrical form, so first let me explain that looking if the slide then we will do another example on the board. So, the seventh metrical form in this 6 (FL) again so write 7 1's write the (FL) above, 1, 2, 3, 5, 8, 13 okay. Now we want to find the seventh therefore subtract this 7 from 13 what you are left with this 6, now with this 6 start looking at this from the right what is the first number which can be subtracted from this 6 it is 5.

So, this number 5 to use a word which will be using later on in the combinatorics in music you call that 5 as a (FL) a number which can be subtracted which is subtractable, so 5 is the first number which can be subtracted 8 cannot be subtracted 13 (()) (32:55) there is no question because the number of rows are much less than less than or equal to 13, so 8 cannot be subtracted, 5 is subtractable, so $6 - 1$, 5 is 1.

So, you highlight this 5 then the number on hand is 1, only this last 1 can be subtracted from that 1, so there are only 2 numbers which are subtractable in that sequence of operations. So, there are 2 (FL) there are 1, 2, 3, 4 (FL) now with each (FL) the L below you mix it with a L to the next make it a G, the L below this other (FL) you mix it with an next L and make it a G rest of the things you leave them as they are, so you will have a G here, you will have G here.

So, the metrical forum is GLGL, so GLGL is the seventh metrical forum when you are making the (FL) 6 matra, so let us take do some other examples.

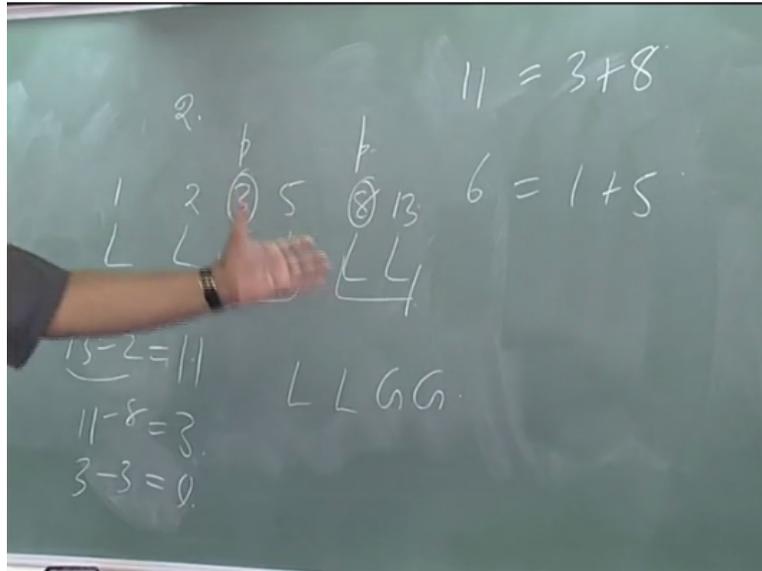
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So, let us do the ninth metrical form so again we have to do the same thing, so 7 (FL) we write 1, 2, 3, 5, 8 of course the 6 (FL) only 13. Let us revise smaller number, so that you will have more maybe, so again let us do the second row of this, so to start with you do $13 - 2$ is equal to 11, now 8 can be subtracted and now 5 cannot be subtracted but 3 can be subtracted. So, this is a (FL) this is a (FL) the rest are all (FL).

So, each (FL) you combine the 2 L's in make it a guru leave the rest of the (FL) as they are, so this form is LLGG, so LLGG is the second row in the (FL) yes LLGG is the second row in the (FL).

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So, finally what is happening this number 13-2 that is 11 you are writing it as 3+8 ultimately this process is occurring by this decomposition of the number 11, this 13-2 this number as replace 8. And similarly in that example that I had shown what was happening the number 6 we wrote it as 1+5, then this is something here what is on the right hand side or the (FL) numbers. So, every number below 13 can be written as a sum of the (FL) numbers in a unique way if I put a certain condition. And that fixes the link between the (FL) and the metrical forum the row number and the metrical forum that link is uniquely fix by right kind of a decomposition of the number.

So, even the (FL) problem was very similar the number 9 was represented as 1+8 that was the way that (FL) problem was solved, so the in the (FL) to link the row number with the metrical forum the mathematical principle that is operating is a decomposition of each of this numbers 1 to 13 as a sum of the (FL) numbers 1, 2, 3, 5, 8. And that way of writing it is unique and that is what I am going to say it here.

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Virahāṅka Representation of Numbers

The above examples of *naṣṭa* and *uddiṣṭa* are based upon the following representations of the numbers 9,6 as sums of Virahāṅka numbers.

$$9 = 1 + 8$$

$$6 = 1 + 5$$

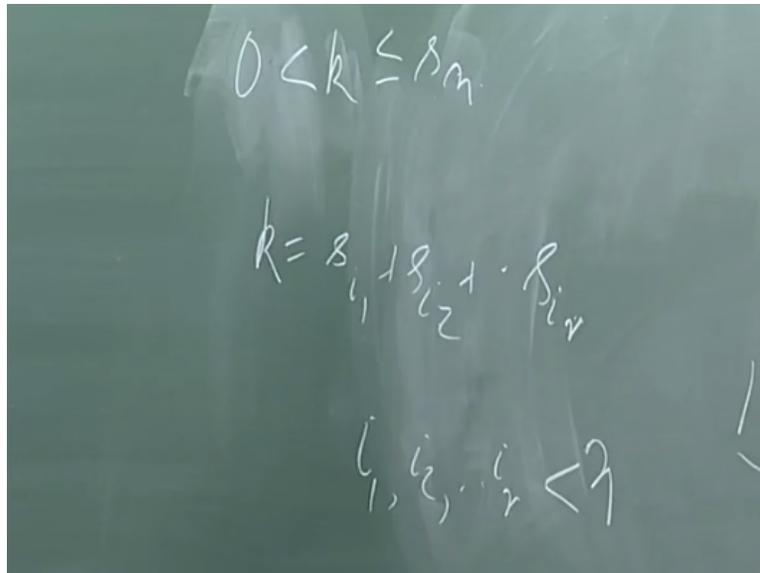
In fact it can be shown the *naṣṭa* and *uddiṣṭa* processes are based on the following very interesting property:

Every integer is either a Virahāṅka number or can be expressed uniquely as a sum of non-consecutive Virahāṅka (or the so called Fibonacci) numbers.

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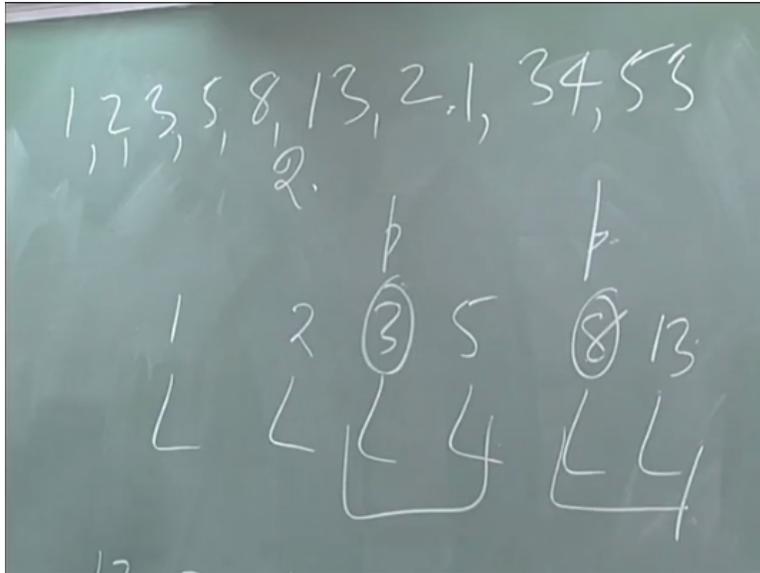
Every integer, so for any $0 \leq k \leq n$.

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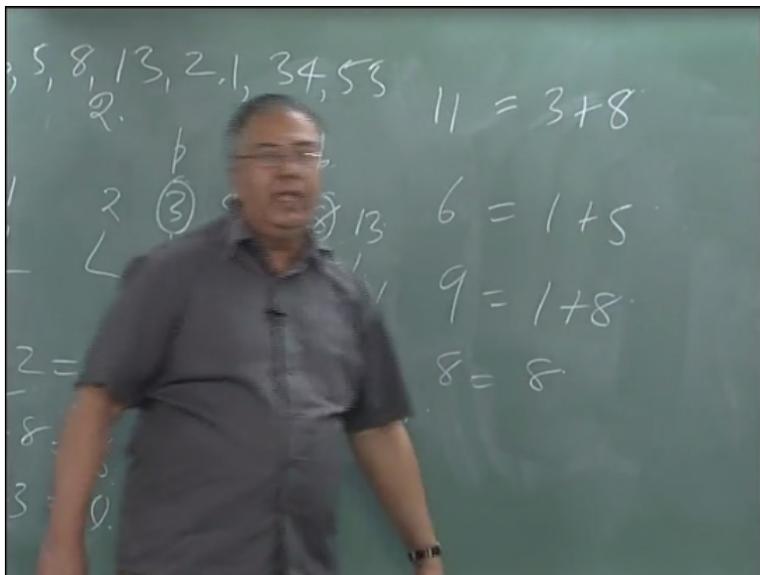
We can see all integers $0 \leq k \leq n$, if you have this less here. So, each number k is equal to sum $S_{i_1} + S_{i_2} + \dots + S_{i_r}$ where i_1, i_2, i_r are all less than n . So, take any (FL) number any integer less than that can be written as a sum of (FL) numbers below that in a unique form as long as you put the condition that the (FL) number should be non consequent.

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So, non consecutively means what 1,2,3,5,8,13,21, 34, 55 like these are the (FL) numbers, these 2 are consecutive these 2 are consecutive, these 2 consecutive, so ultimately the entire mathematics of the (FL) depends on a decomposition like this where every number is writable as a sum of the (FL) numbers were the numbers are non consecutive.

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So, every number is either a (FL) number 8 will have only representation as 8 or if it is not a (FL) number it can be uniquely written as a sum and that is what fixes you the row number or it gives you the metrical pattern the way to go from one to the other and that is at the halt of the calculation.

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Mātrā-Meru

The number of metric forms with different number of *laghus* (*gurus*) can be found from the following table known as the *Mātrā-meru*:

द्वयं द्वयं समं कोष्ठं कृत्वान्त्येष्वेकमर्पयेत्।

एकद्विकत्रिकचतुःक्रमेण प्रथमेष्वपि ॥

श्रीर्षाङ्कानपराङ्काभ्यां शेषकोष्ठान् प्रपूरयेत्।

मात्रामेरुरयं दुर्गः सर्वेषामतिदुर्गमः ॥ (वाणीभूषणम् १.३७-३८)

The successive rows of the *mātrā-meru* have 1, 2, 2, 3, 3, etc cells.

Place 1 in the top row and in the end of each row.

Place 1 in the beginning of all the even rows and 2, 3, 4 etc in the beginning of odd rows.

The other cells are filled by the sum of the number in the row above which is above and the number in the row further above which is to the right.

Now last is (FL) and (FL) is saying (FL), so there is nothing (FL) about it this also very simple only slightly more complex than (FL) but what is the question that in n matra (FL) how many metrical forums are there with any particular number of gurus, 3 gurus or 5 gurus or any particular number of (FL). So, to answer that question that was called the (FL) question, so for (FL) case it was very simple, the total syllables were fixed each syllable can be guru or (FL).

So, you are just picking up, so how many eighth syllable (FL) syllable (FL) are there with 3 (FL) that is 8C3 then nothing more to be done but this is a more complex thing right let us see the (FL) there you will see let how the, so look at this (FL) how many matra (FL) are there with 3 gurus there is only one, how many (FL) are there with 2 gurus this here, it is here, it is here, so it is 3 and there is also 1 below here there is one power 5, 6 the 6 how many matra (FL) are there with 1 guru.

So, 1 here, 1 here, 1 here, 1 here I think that is all there are 4 how many (FL) are there with 0 gurus there is just 1. So, what about the (FL) how many 6 (FL) are there with 1 (FL) what is the answer 0 because there cannot be a 6 (FL) with 1 (FL) or 3 (FL) or 5 (FL). Because guru will give you a 2, there is to be at least 2 (FL) to make 6 in r, so whenever you have an even (FL) the number of (FL) in that has to be even when you have an odd (FL) the number of (FL) there have to be odd.

That condition does not apply to guru, so like that this problem is more complicate it is not just picking, so many slots out of the total number of slots but this was also solved where there and this matra meru was written down.

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Mātrā-Meru

						1	$S_1 = 1$
					1	1	$S_2 = 2$
					2	1	$S_3 = 3$
				1	3	1	$S_4 = 5$
				3	4	1	$S_5 = 8$
			1	6	5	1	$S_6 = 13$
			4	10	6	1	$S_7 = 21$
		1	10	15	7	1	$S_8 = 34$
		5	20	21	8	1	$S_9 = 55$
	1	15	35	28	9	1	$S_{10} = 89$
	6	35	56	36	10	1	$S_{11} = 144$
1	21	70	84	45	11	1	$S_{12} = 233$
7	56	126	120	55	12	1	$S_{13} = 377$

From the above *meru* we can see that, in the 6-*mātrā-prastāra*, there is 1 metrical form with 0L, 6 with 2L, 5 with 4L and 1 with 6L. We can also infer that in the same *prastāra* there is one form with 3Gs, 6 with 2Gs, 5 with 1G and 1 with 0G.

So, this is the (FL), so what this (FL) is doing this is just the total number of meters, total number of meters with 1 matra is 1, total number of meters with 2 matras is 2, total number of meters with 3 matras is 3, total number of meters with 5 matras is 8 you saw all that total number of meters with 6 matras is 13, so these are all the (FL) numbers. So, any row of this meru, so first thing this meru has unlike the simple meru of (FL) which had 1 row first 1 entry in the first row 2 entries in the second row, 3 entries in the third row, 4 entries in the fourth row.

This goes in a more complex way, first row is 1 entry, second and third rows have 2 entries, 4 and fifth rows have 2 entries, 3 entries, 6 and seventh row have 4 entries like that, so to 2 rows have the same number of entries then what other pattern you can see it has 1 in the end and if these 2 entries that you are having the top one is always 1, that is also simple to see, the bottom one is 2, 3, 4, 5, 6 like that.

That also we are able to see if you want to see something more than you have to take it from me 10 is given by 4+6, 15 is given by 10+5, 6 is given by 5+1, 21 is given by 15+6 you go up 1 step then you go across by 1 step the sum of those 2, so that is the recurrence relation. Now what are

these numbers, so let us take the 6 (FL), 6 (FL) so there is one metrical forum with 0L, 6 metrical forums with 2L, 5 metrical forums with 4L and 1 metrical forum with 6L.

Other way of talking about it there is one metrical forum with 3G, 6 metrical forum with 2G, 5 metrical forum with 1G and 1 metrical forum with 0G. So, each row of this (FL) is giving you how many different meters are there with a given number (FL) which also giving you how many meters are there with a given number of gurus. So, how did they arrive it, now first how did they describe this meru.

(Refer Slide Time: 44:26)

Mātrā-Meru

The number of metric forms with different number of *laghus* (*gurus*) can be found from the following table known as the *Mātrā-meru*:

द्वयं द्वयं समं कोष्ठं कृत्वान्त्येष्वेकमर्पयेत्।
एकद्विकत्रिकचतुःक्रमेण प्रथमेष्वपि ॥
शीर्षाङ्कानपराङ्काभ्यां शेषकोष्ठान् प्रपूरयेत्।
मात्रामेरुरयं दुर्गः सर्वेषामतिदुर्गमः ॥ (वाणीभूषणम् १.३७-३८)

The successive rows of the *mātrā-meru* have 1, 2, 2, 3, 3, etc cells.
Place 1 in the top row and in the end of each row.
Place 1 in the beginning of all the even rows and 2, 3, 4 etc in the beginning of odd rows.
The other cells are filled by the sum of the number in the row above which is above and the number in the row further above which is to the right.

(FL) so, this is (FL) so they have 1, 2, 2, 3, 3 etc., number of cells in each row 1 in top of each row and the end of each row out 2, 3,4 and finally (FL) that is how he is explain to you the recurrence relation which we are now going to discuss.

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Mātrā-Meru

In the n -mātrā-ṛtta-prastāra, if a metrical form has i gurus, then it will have $(n - 2i)$ laghus.

Hence the metrical forms which have i gurus will all have a total of $(n - i)$ syllables, of which i will be gurus.

Therefore, it follows that G_n^i the number of metrical forms with i Gs, in the n -mātrā-ṛtta prastāra, is given by:

$$G_n^i = \binom{n-i}{i} C_i = \frac{(n-i)!}{(i)!(n-2i)!} \text{ for } 0 \leq i \leq \frac{n}{2}$$

and

$$G_n^i = 0 \text{ otherwise.}$$

Note that G_n^i satisfy the recurrence relation:

$$G_n^i = G_{n-1}^i + G_{n-2}^{i-1}$$

So, the question if you have an n matra (FL) how many (FL) are there with i gurus this is the question right, so how do we understand this if there are i gurus the total matras are n , so i gurus will have $2i$ value, so what is the value left out $n-2i$ therefore if there are i gurus and we have n (FL), so we have $n-2i$ (FL). So, we have $n-2i$ (FL), so when we have i gurus we will have $n-2i$ (FL) this is fixed, i gurus is fixed total matras n is fixed.

So, if you have i gurus there has to be $n-2i$ (FL), now add these 2 you have $n-i$ syllables, so all those matra (FL) which have i gurus have $n-i$ syllables and i of them are gurus how many possibilities $n-i$ C_i , so that will tell you the number of (FL) of value n which have i guru, so I will repeat if you have n (FL) it has i gurus then it should have $n-2i$ (FL) because gurus have value 2, so the total number of syllables also it is fixed become $n-i$ syllables.

So, you have n -syllable i syllables of which i will be guru how many possibilities are there $n-i$ C_i and this i has to be less than or equal to $n/2$, if you have n (FL) you cannot have more than $n/2$ gurus each guru as value 2 therefore you have this relation, this is the number of matra (FL) of n matras with i gurus and you can straight away see they will satisfy this recurrence relation $G_n^i = G_{n-1}^i + G_{n-2}^{i-1}$, the G_{n-1}^i is the number above +the number diagonally across to the right, 56 is 35+21, 35 is 15+20..

So, that is how this (FL) is constructed and is one of the most beautiful constructions in combinatorics in ancient Indian mathematics and a very nice combinatorial problem solve brilliantly. So, this was the first complexity that arises once you go from (FL) fixed number of syllables and only (FL) guru being there to a situation they fixed amount of total value where guru has value 2, (FL) has value 1.

So, the number of syllables is not fixed the total value is fixed, so you do all the same things you do (FL), so this is the beautiful mathematics of combinatorics developed in connection with (FL) and much of it is derived in literature and hardly discussed in mathematical literature in modern time.

(Refer Slide Time: 48:10)

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I mean the older mathematicians did know what was being discussed. So, with this we will come to an end of today's lecture thank you very much.