Quantitative Investment Management
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Lecture 46
Random Walks

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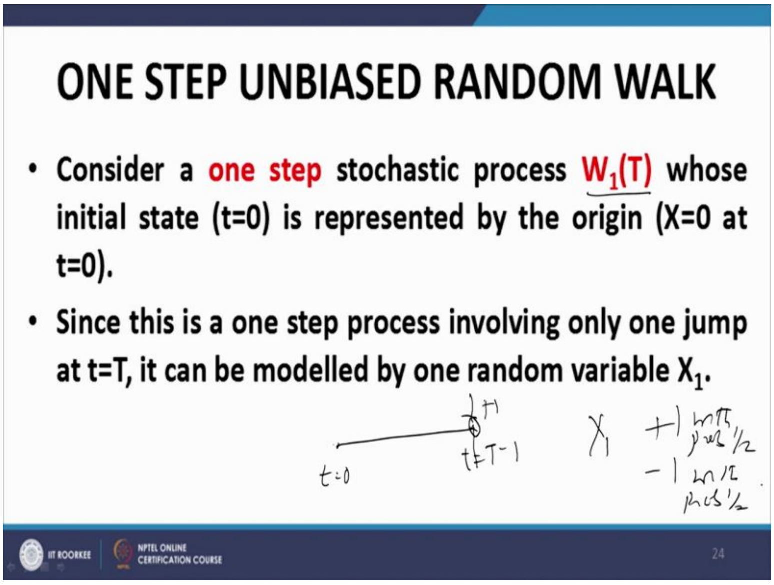

Welcome back. So let us start with the simplest prototype of a stochastic process that is a one step unbiased random walk. Consider a one step stochastic process W1(T). 1 is the number of steps, T is the time length of that step whose initial state t equal to 0 is represented by the origin, X equal to 0 at t equal to 0. Since this is a one step process involving only one jump at t equal to capital T, it will be modelled by one random variable.

So basically, what we are talking about is this is say t equal to 0. This is the starting point of the process and then at t equal to capital T when this clock strikes t equal to capital T, the process makes the transition either upwards or downwards.

Sitting at t equal to 0, we do not know what the transition is going to be, whether it is going to be upwards or downwards. The magnitude of this transition is either upwards, plus 1 or it is downwards, minus 1 and between t equal to 0 and t equal to capital T, the process is dormant, although the time is flowing.

Time is moving on but the process is not undergoing any transition. As soon as the clock strikes capital T, the process makes either an up jump or a down jump of magnitude plus 1 that is up jump, minus 1 a down jump, each of which has a probability of 1 by 2 because sitting at t equal to 0 we do not know whether it is going to jump up or down.

Therefore, the up jump or the down jump sitting at t equal to 0 can be modelled as a random variable. Let us call it X1. So X1 can take the values plus 1 or minus 1 with probabilities 1 by 2 in each case.

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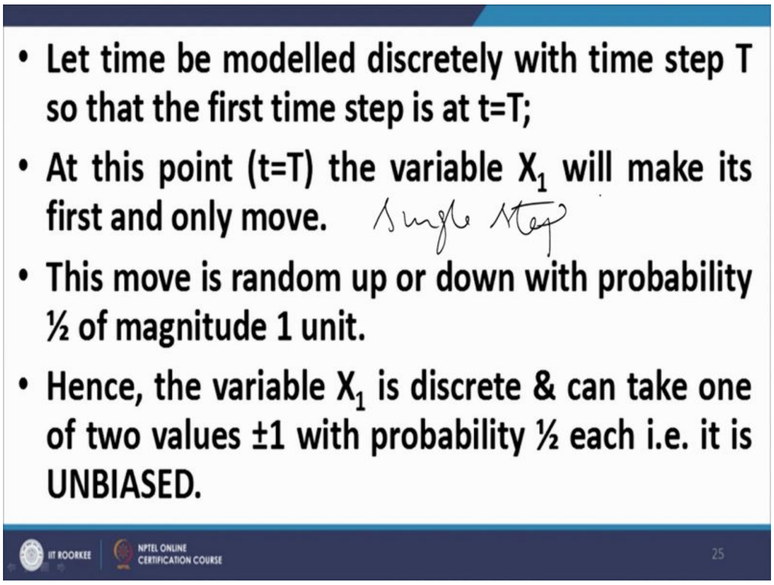

Let time be modelled discreetly with time step capital T so that the first time step is at t equal to capital T, as I explained. At this point t equal to capital T, the variable X1 will make its first and only move because we are having a single step, single step random walk, single step. So there is only one time step here.

And at the end of that time step for the at t equal to capital T that is the time step, the process makes an up jump or a down jump and it is modelled by a random variable X1. This move is random up or random down with a probability of 1 by 2 and the magnitude of the up jump or the down jump is 1 unit. Hence, the variable X1 is discrete, and can take one of two values plus or minus 1 with probability 1 by 2 in each case and that is it is unbiased.

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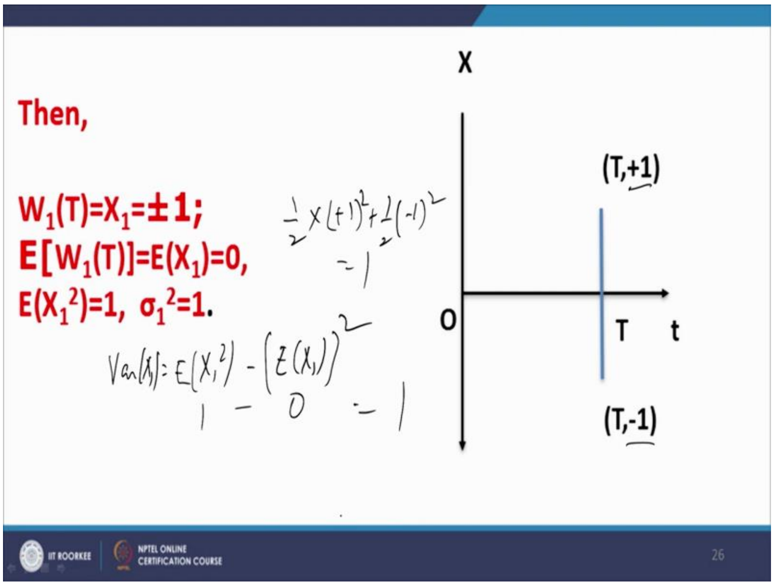

So this is the representation of the process. We are here at t equal to 0. At t equal to capital T, when the process will, when the clock will strike capital T, immediately, simultaneously, the process will make an up jump or a down jump. Sitting at t equal to 0, we do not know whether it is going to be an up jump or a down jump.

The magnitude of the up jump or the down jump is plus 1, is 1 that is it will either be at plus 1 coordinate or it will be at minus 1 coordinate and at capital T and the probability of the up jump and the down jump are identical and equal to 1 by 2.

Now on this basis, what are the cardinals of this process, single step. And please note this is the only transition in this process because you are talking about a single step random walk. There is only one step in it and at the end of that step, the process either makes up jump or down jump and it is represented by the random variable X1.

So W1(T) is equal to X1. W1(T) is a stochastic process and it is represented by the random variable X1 and the random variable X1 can take the values plus 1 with probability 1 by 2 or minus 1 with probability 1 by 2 at t equal to capital T.

So the expected value of W1(T) is equal to 0. It is quite obvious, 1 by 2 into plus 1, plus 1 by 2 into minus 1. The expected value of X1 square is equal to 1. You can work it out, 1 by 2 into plus 1 squared plus 1 by 2 into minus 1 squared. That is equal to 1.

So expected value of X1 squared is equal to 1 and what is the variance of this process? E of variance of X1 is equal to E of X1 squared minus E(X1) whole squared. This is equal to 0. This is equal to 1. So, this variance is equal to 1.

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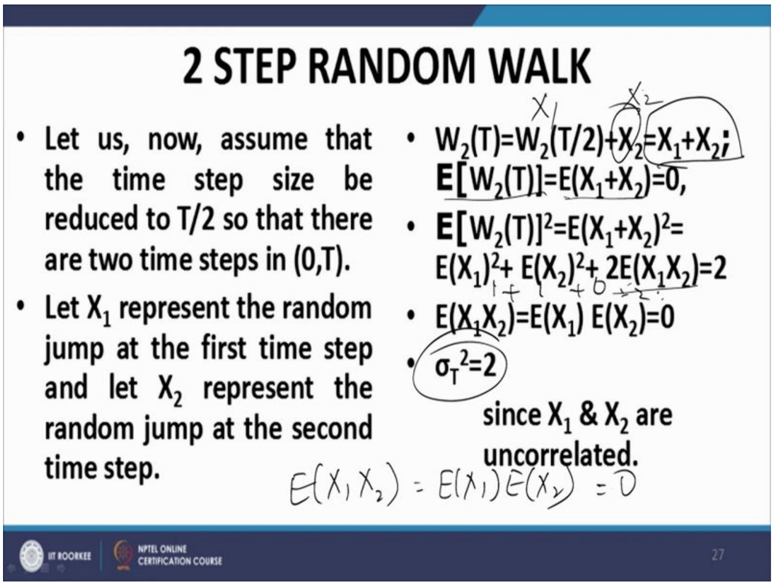

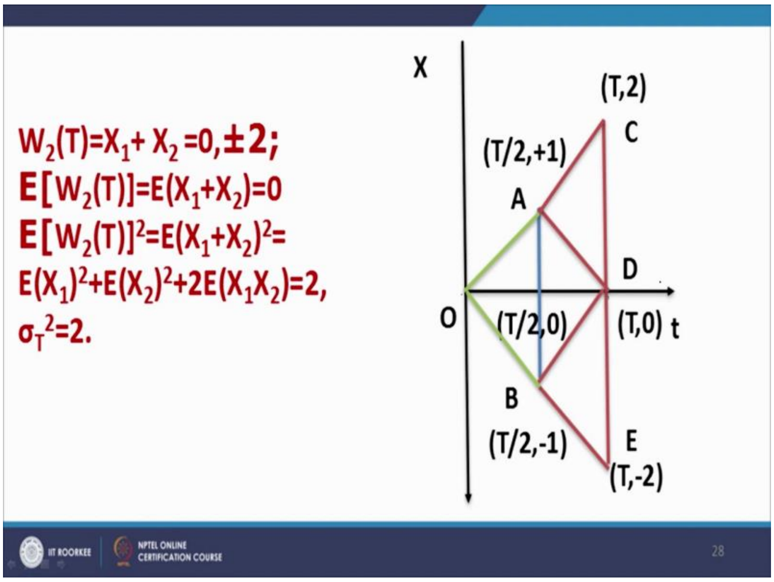

Let us know move forward to the 2 step random walk. Now what do we do? Let me go to the diagram straight away. What are we doing? We are splitting this interval 0 to capital T which was the single step that we had in the previous case into two steps that means that there be an intermediate transition and that transition will occur midway between 0 and capital T that is at t equal to capital T by 2.

So now the process will undergo two transitions, one at t equal to capital T by 2 and the second at t equal to capital T. The first transition will either be up by 1 unit and down by 1 unit. So the coordinates after the first transition will be plus 1 or minus 1.

And then the second transition will depend on where the system was at the end of the first transition. If the system was at the point A, then it could go either upwards by 1 further unit that is it could go to the point 2 and or it could go down 1 unit to the point 0 that is the point D here.

And if the system was at the point B at the end of the first step that it had a value of minus 1 at the first step, then again it could go up to the value 0 at t equal to capital T or it could go down to the value t equal to the value minus 2 that is the point E here. So we are having 2 step, this is a 2 step model. The first step is at t equal to T by 2. The second step is equal t equal to capital T.

Now, we can model the first step by a random variable X1 which can take the values plus 1 or minus 1 with equal probabilities and we can model the second step by another random variable X2 which can take values plus 1 or minus 1 with probabilities 1 by 2 again. And the value of the process at t equal to capital T will be the sum of X1 and X2.

First of all, it will jump at t equal to T by 2 which is represented by X1 to either X equal to plus 1 or X equal to minus 1 and from wherever, from wherever it is at t equal to T by 2, it will again jump either upwards or downwards which will be represented by X2.

The first jump is represented by X1. The second jump is represented by X2 and on that basis the value of the process at the end of the time capital T can either be 0 or it can be plus 2 or it can be minus 2 as you can see in the diagram in the right hand panel.

So let us now assume that the time step size is reduced to T by 2 so that there are two steps. Let X1 represent the random jump at the first step so that is at T by 2 and let X2 represent the random jump at capital T, then W2(T) is equal to W2(T by 2) plus where it is at t equal to capital T by 2.

From there it goes up or down which is represented by the random variable X2 here but where it is at T by 2 is represented by the random variable X1. So this is represented by X1. This is represented by X2. Therefore, W2(T) is represented by X1 plus X2 where both X1 and X2 are random variables which can take the values plus minus 1 with equal probabilities.

And so the expected value of W2(T) is equal to E of X1 plus X2 is 0 because E(X1) is 0, E(X2) is 0. W2 E of W2T whole squared is equal to E of X1 plus X2 whole square that is equal to E of X1 plus E of X2 square plus 2E(X1X2).

Now this E(X1X2) is equal to E(X1) E(X2). Why? Because X1 and X2 are independent of each other and this is equal to 0 because E(X1) is 0 and E(X2) is 0. Hence the variance of this process at t equal to capital T is equal to 2. Why? Because this is equal to 1. This is equal to 1. This is equal to 0. So this is 2.

So this is for the 2 step model. So what have we concluded? There is a very important conclusion that we have already arrived at using the one step model and a 2 step model, the variance of the one step model was 1, the variance of the two step model is 2.

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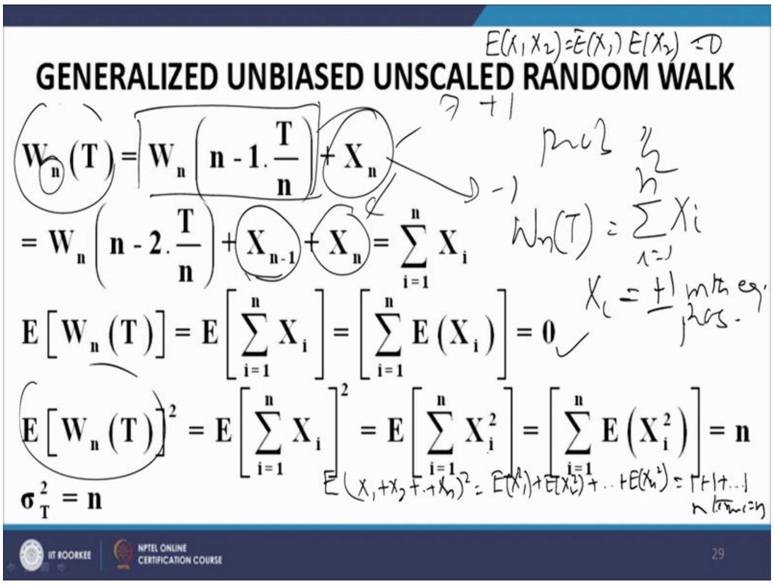

And now let us look at the generalized unbiased unscaled random walk, the n step random walk. The same represents the number of steps having a time length of capital T. The total time length of the entire process of n steps, one step, one step, two steps, three steps, four steps, up to n steps, the total time length is capital T.

So what will be the evolution of the process at time t equal to capital T? That will be equal to what? That will be the evolution of the process at t equal to the n minus 1 th step and then the nth step can be modelled as a 2 step, as a random variable with the values X1 with the value Xn which can take the values plus 1 and minus 1 with equal probabilities.

So we can write this as the position of the process at the n minus 1th step. This is the position of the process at n minus 1 step and then the nth step is represented by this random variable Xn which can take the values plus 1. This can take the value minus 1, probability 1 by 2. Now where the process is at n minus 1 step can be modelled as where the process is at the n minus e n minus 2th step and then with a random jump corresponding to the n minus 1 step.

So, that therefore we can write this Wn n minus 1 T upon N as Wn n minus 2 T upon n plus X n minus 1. This represents n minus 1 step. This represents the nth step. Same thing here and where both X n minus 1 and X minus 1 Xn are random variables which can take the values plus 1 and minus 1 with equal probabilities.

So continuing this iterative process, what we find is Wn(T) can be represented as a sum of n random variables where n is the number of steps. Please note n is the number of steps. Each of these random variables can take the values plus 1, minus 1 with equal probabilities.

So now let us work out the cardinals of the general process. E of Wn(T) is quite elementary and straightforward, so it is equal to 0. It is can be seen as prime of (())(13:13). What about E(Wn) squared? This is equal to E of summation Xi squared. This is equal to E of summation Xi squared.

Now please note here, what are we doing? We are doing this X1 plus X2 plus plus Xn whole square is and you take the E of this. Then this is equal to E of X1 squared plus E of X2 squared. Why are we able to do this? We are able to do this because the cross terms do not contribute to the summation.

Remember, what do we have? We have E(X1 X2) is equal to E(X1) E(X2) that is equal to 0. Why is this 0? Why we can write it like this? We can write it like this because X1 and X2 are independent and E(X1) and E(X2) are 0. That is obvious because they can take the values plus minus 1 with equal probabilities. So the expectation is 0.

And therefore, the cross terms do not contribute to this result and therefore we can write it like this. And each of this term has a value of 1. So this is equal to n. So that means what? That immediately gives us that the variance of the process t equal to capital T, at t equal to capital T, what is the process? Wn(T) equal to capital T that is the n step unbiased random walk as a variance equal to n. n step random walk has a variance equal to n. Unbiased random walk has a variance equal to n.

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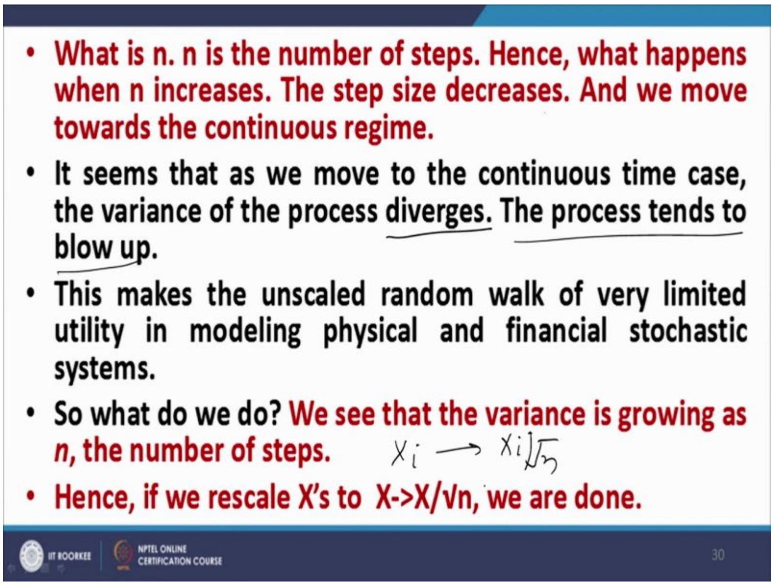

So what is in n? n is the number of steps. Larger the number of steps, larger the variance. Hence, what happens when n increases? The step size decreases and we find we move towards the continuous regime.

So, there are two things which happens as n increases. Number one, the variance is increasing and number two the number of steps is increasing. What does it mean? It means that the step length is decreasing and we are moving from the discrete regime to the continuous regime.

And in fact, we will arrive at the continuous regime in the limiting case when n tends to infinity. So what is n? n is the number of steps. Hence, what happens when n increases? When n increases, the step size decreases, the step length decreases and we move from the discrete regime towards the continuous regime.

We will model the continuous regime as the situation scenario in which n tends to infinity. In the limiting case when n tends to infinity, we move from the discrete regime to the continuous regime because the step size will then become infinitesimally small.

It seems that as we move from to the continuous regime, that is as n is increasing, the variance is also increasing. Please note this point. So there are two things,

However, the second important part is that the variance of an n step random walk is n. So as n increases, the number of, the variance of the process also increases. So that seems that as we move to the continuous time case, the variance of the process diverges.

And n tends to infinity means what? Means the various tends to infinity because variance is n. The process tends to blow up. This makes the unscaled random walk of very limited utility in modelling physical and financial stochastic systems. So, what do we do?

We see that the variance is growing as n in number of steps. Therefore, if we replace or if we rescale X's all the Xi's to Xi upon root n, we are done because then the variance will become 1. But we do not want that either.

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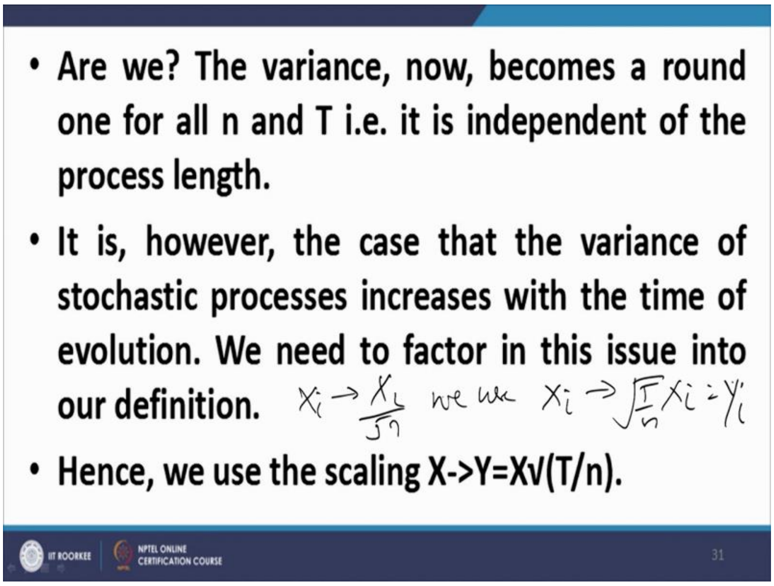

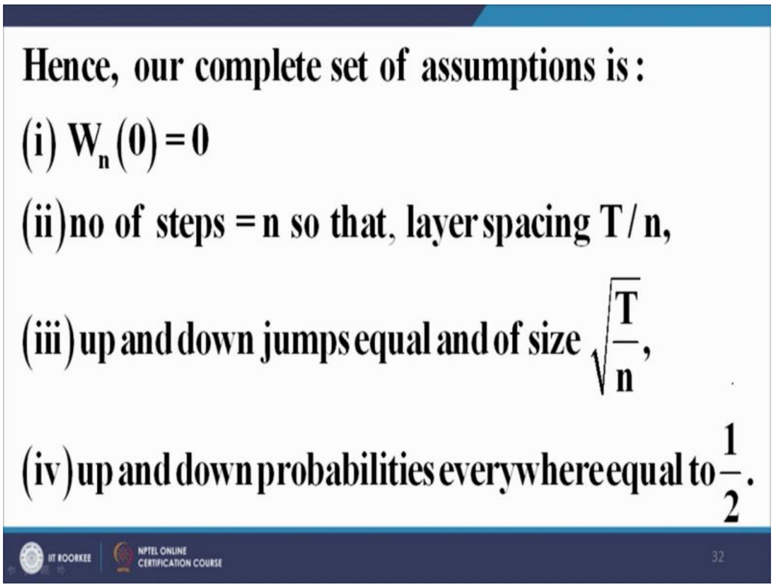

Why we do not want that? Because now the variance becomes a round figure of 1 for all n and therefore it is independent of the process length which is also not acceptable. It is empirically found that most stochastic systems show linear scaling of variance with respect to time. As time increases, variance increases. Therefore, in order to accommodate this empirical fact, in order to this nature of stochastic processes, we revise our scaling scheme.

What do we do? Instead of using this; what we do is we use Xi goes to under root T upon n into Xi. Let us call this Yi. So that is the revise scaling that we do. Hence, our complete set of assumptions is Wn(0). We start at the origin. So Wn(0) is equal to 0. Number of steps is equal to n. So that layer spacing is equal to capital T upon n. Up and down jumps are equal of size under root T upon n.

Please note this, this is no longer plus minus 1. It is now under T upon n because we have scaled the variable X which could take the values plus 1 and minus 1 to the variable Y which can take the values under root T upon n into X and where x can take the values plus minus 1. Therefore, Y can take the values under root T upon n with the plus sign or under root T upon n with the minus sign with equal probabilities. And up and down probabilities everywhere are equal to 1 by 2.

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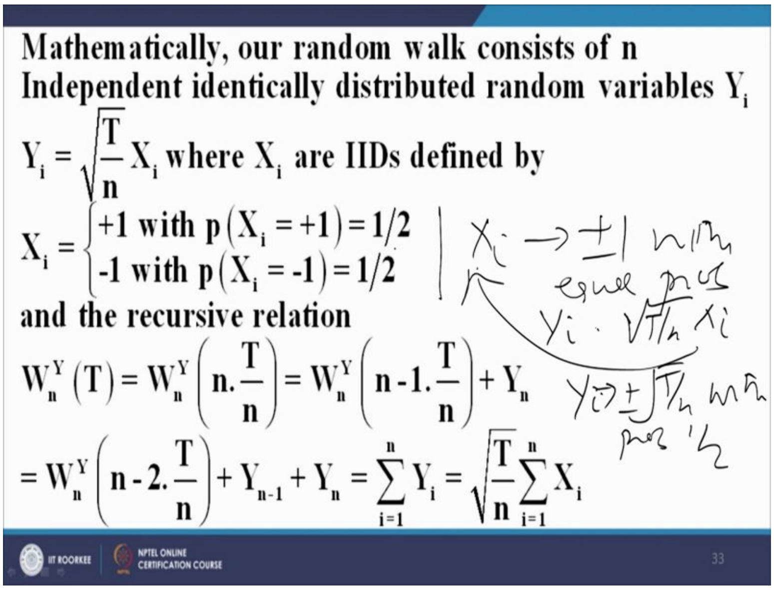

Mathematically, our random walk consists of n independent identically distributed random variables Yi. So now we are talking about the scaled random walk where we have revised the size of the step, we have revised the size of the jump from plus minus 1 to plus minus under root T upon n. This is the Y random walk. This is not the X random walk.

X random walk was a sequence of X variables each of which was a binomial in the sense that it could take the values plus minus 1 with equal probabilities. Now we are talking about the Y random walk which can take the values plus minus under root T upon n with equal probabilities.

So our Y random walk is a sequence of identically independent of a independent, please note, Y1, Y2, Y3 continue to be independent because of X1, X2, X3 being independent. So the future evolution of the stochastic process, the random walk process that we are talking about is independent of the previous values. So that is important.

The independent identically distributed, all of them can take two values plus under root T by n minus under root T by n with probabilities 1 by 2. So they are identically distributed and obviously, they are random variables.

So Yi which is equal to under root T upon n. Xi where Xi's are IIDs. IIDs is the short form of independent identically distributed random variables defined by. This is for Xi. Please note this. This is for Xi. So Xi's contains the values plus minus 1 with equal probabilities.

Correspondingly because Yi is defined by under root T upon n Xi and Xi is defined by this expression. We have Yi's can take the values plus minus so this is what is given here and this is the recursive relation which is absolutely the same as what we had for the X random walk.

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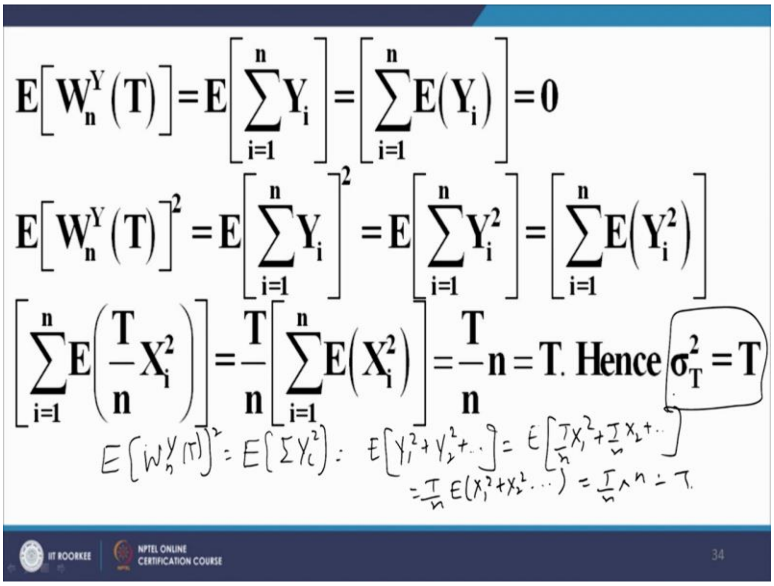

Now let us work out the cardinals of the Y random walk. As far as the mean value is concerned, there is no change. The mean value will remain 0 because the mean value of each of the Y's continues to be 0 under root T upon n is the size jump into 1 by 2 is the probability of that jump and the other value that the process can take, the random variable can take as minus under root T upon n with probability 1 by 2. So the expected value is 0 straightaway.

As far as the expected value of the square of the process, E of WnY(T) square, we again note that the random variables are independent. Therefore, what do we have? We have E of Wny(T) squared is equal to summation E of summation Yi square into E of Y1 square plus Y2 square.

So the value of the square of the process that is E[Wn(T)] square is equal to capital T and therefore the variance of the process; this is important, this is a fundamental important variance of the process is equal to the time length of evolution of the process.

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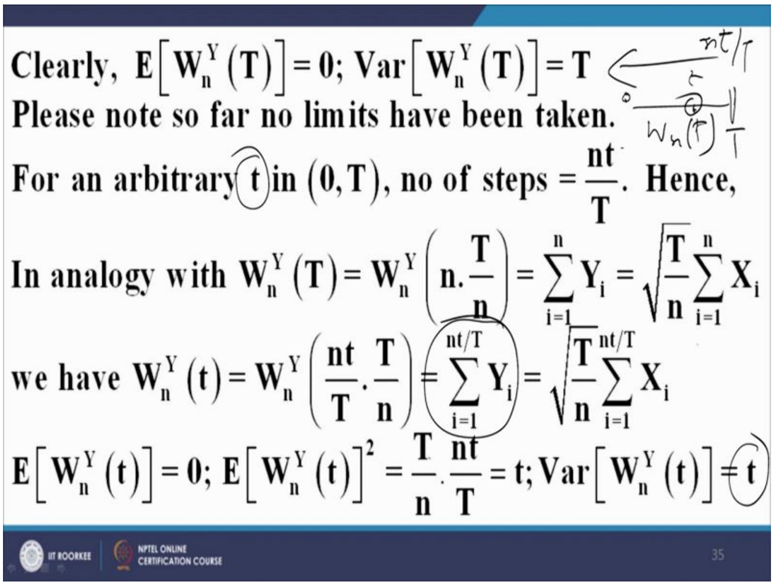

So this is what we have here. In the first line on the slide. Please note so far no limits have been taken; no limits have been taken so far. Please note this point. For an arbitrary time t between 0 and capital T. So please note, so far we have been working on the premise that we are making the observation at capital T.

We are talking about Wn capital T where n is the total time of every evolution of the process. This is capital T, okay? Now let us talk about any intermediate point. We have shown what? We have shown that the variance of the process is equal to the total time of evolution of the process.

What if I make an observation at any intermediate point, then what happens? That is our next question. Now, if t is at small t is any arbitrary point between 0 and capital T, the number of steps that we need to reach the point small t is given by n small t upon capital T because each step length is of T upon n.

So (())(25:24) a small t divided by capital T upon n that is equal to n small t upon T is the number of steps that we need to take to reach from this t equal to 0 to the small t. This 0 to small t. This is equal to nt n small t upon capital T number of steps.

Now, if that is the case, what happens? Then it would be the number of Y variables that we need for modelling, the number of steps that we have up to this point small t is equal to nt upon capital T.

In other words, the number of variables Yi Y1, Y2, Y3 up to Y what? Up to a certain Y that we need for modelling the number of jumps is equal to the number of jumps in fact. The number of Yi's we need is equal to the number of jumps. And how many are the number of jumps? nt upon capital T. So, the number of Y variables that we need is nt upon capital T. That is precisely what is given here.

The rest of it is as in the previous case. What we find is that the mean is 0 and the variance is equal to small t. So the formula that we arrived at in the context of the entire evolution of the process holds for any arbitrary point in time as well during the evolution of the process because here Yi is what?

It is the same Yi but the number of variables that we need, a number of random variables that we need for modelling of the process up to small t is equal to the number of steps and that is equal to nt, n small t upon capital T. By using this expression, using this part of the analysis, we can clearly arrive at the fact that the variance of the process up to small t is equal to small t.

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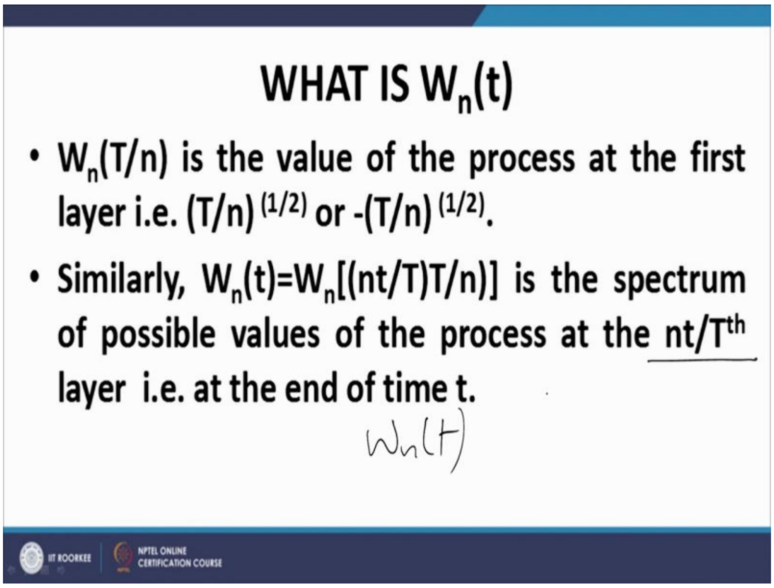

What is Wn(T)? Wn(T upon n) is the value of the process at the first step that is equal to T upon n plus under root T upon n or minus under root T upon n. Similarly, Wn(T) is equal to Wn nt upon T into T upon n that is the spectrum of possible values of the process at the nt upon capital Tth layer that is at the end of small t.

Therefore, Wn(T) is the spectrum of possible values that the process could take at t equal to small t.

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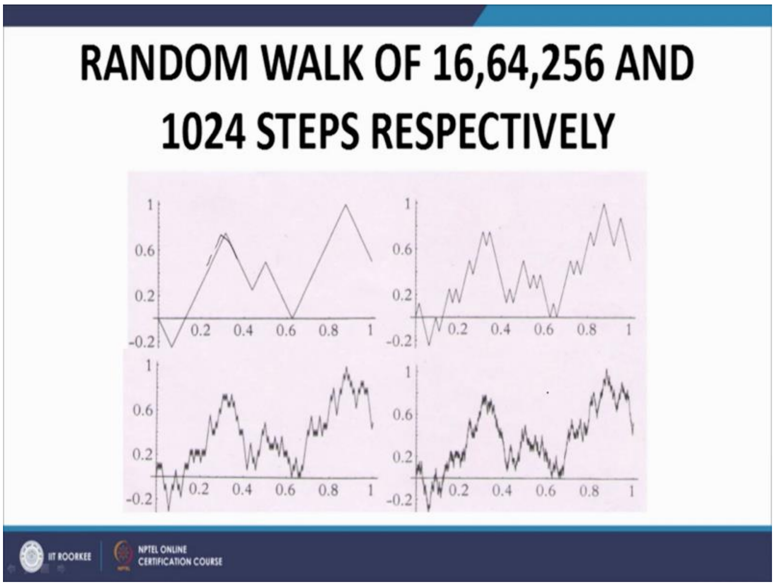

This is the illustration of the random walk. The first one is a 16 step random walk. Second is a 64 step random walk within the same time range then we have the 256 step random walk and then the 1024 steps random walk compressed in the same time range.

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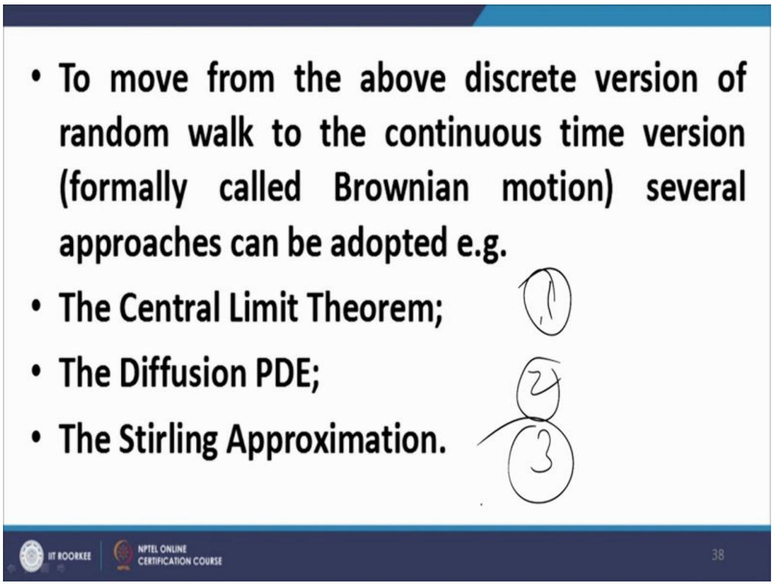

Now to move from this above discrete version of the random walk to the continuous version, we can adopt any of these three approaches: number one, we can use the Central Limit Theorem. Number two, we can use the diffusion partial differential equation and or we can use the Stirling approximation.

Number one, number two, number three. We can use either of them. I shall confine myself to using the central limit theorem which I will take up in the next class. So in the next class, my objective is to move from this discrete random walk to the continuous random walk which is given a special name the Brownian motion and which is the, which epitomizes the same importance as a straight line as in the context of deterministic trajectories. Stochastic trajectories or trajectories of stochastic processes can be modelled using the concept of Brownian motion which I will explain in the next lecture. Thank you.