Path Integral Methods in Physics & Finance
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Lecture - 04
Generating Functions, Gaussian Distribution

Before the break, I discuss the characteristic function and the properties that need to be satisfied by the characteristic function, in order that it represents a represent the characteristic function of a probability distribution.

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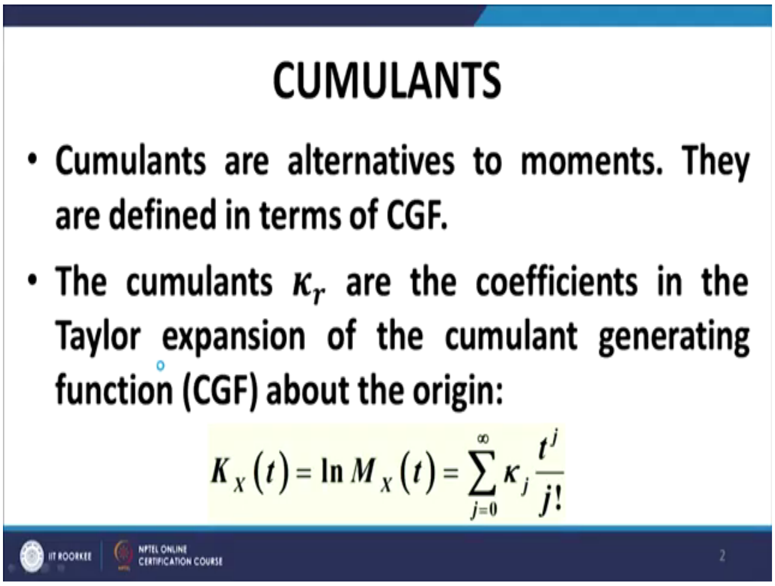

Now, we come to a new concept which is occasionally used in substitution of the concept of moment generating function which is called cumulants. Cumulants are very similar to moments, in fact, they are alternatives to moments and there exists direct correspondence between cumulants and moments.

So, let us see how we define the cumulants. The cumulants are defined by this particular equation; the cumulants K the cumulant generating function K X of t is given by the natural logarithm of the moment generating function. The cumulant generating function is the natural logarithm of the moment generating function.

So, and if we expand that as a power series by introducing a parameter t, we can write it as summation of K j into t K upon j factorial, we write it in this form. Then this coefficients K j kappa j rather the coefficients kappa j represents the cumulants of the distribution.

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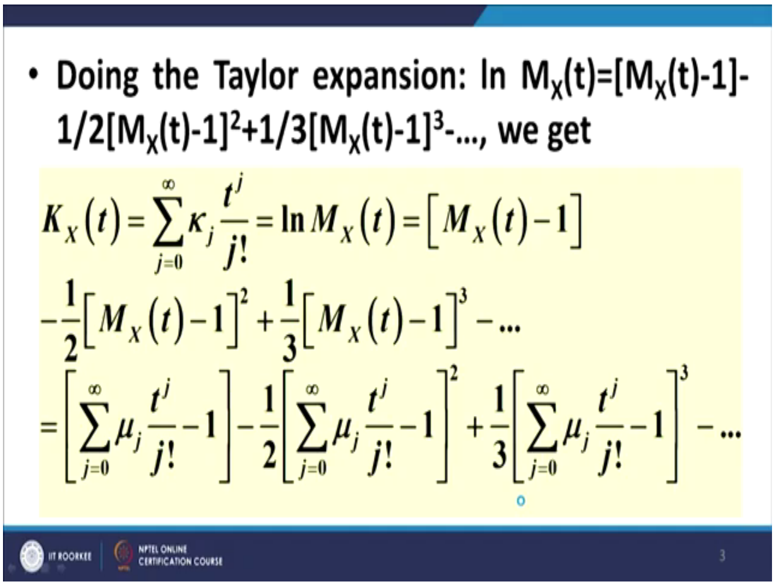

Now, if I expand the moment generating function you see the cumulant generating function is given by the log of the moment generating function. So, we know that log of 1 plus X the natural log of 1 plus X can be expanded as a Taylor series X minus x square upon 2 and so on.

Therefore, log of X can be expanded as M X minus 1, because on the left hand side we have got log of X. So, I have to subtract 1 on the right hand side from every term and I have to I expand it as M X t minus 1, minus 1 by 2 M X t minus 1 square plus 1 by 3 M X t minus 1 whole cube and so on.

Now, this is equal to the cumulative generating function which I have expanded as a power series in t in this form and the kappa j s represent the cumulant, so by comparing the coefficients. Now by comparing the coefficients of the t s, because M X t is nothing, but summation of the moment about the origin into t j upon j factorial summation of i cube. It can be written in this form, because it is the moment generating function.

So, now we have what do we have? On both sides this expression is a function of power series of t and this expression is also a power series in t. So, we can equate the power series the coefficients of each power of t and arrive at an equivalence, between the cumulant coefficients and the corresponding mus the corresponding moments about the origin.

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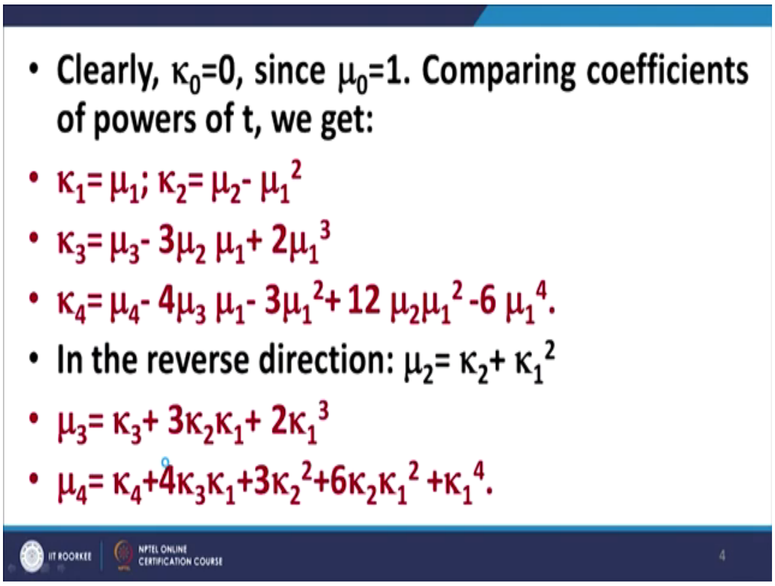

And this is what we get kappa 1 is equal to mu 1, kappa 2 is equal to mu 2 minus mu 3 square and minus mu 1 square and so on. These are the various relations between the kappa's which are the cumulants and the mus which are the moments about the origin. These moments have been obtained through from the moment generating function and the kappa's are been obtained from the cumulant generating function. The converse can also been worked out.

So, just to recap we define the cumulants as log of the moment generating function and then we expand the moment generating log of moment generating function as a Taylor series. And then substitute the values of M X t as a power series of the in t with the coefficients representing the moments about the origin.

Similarly, I can write the cumulant generating function as a power series in t with the coefficient representing the cumulants. Now, we got on two slides we got power series in t. We equate the coefficients of the respective powers of t and arrive at these relationships.

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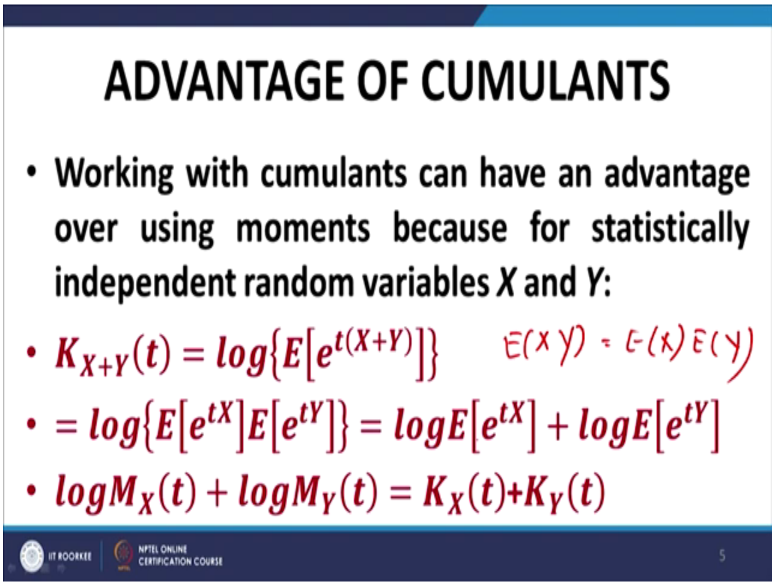

Now, the advantage of cumulant. Well if we have two random variables X and Y which are independent. Now, let us see how the cumulants we have the cumulating cumulant generating function of the sum of the random variable X plus Y as a log of E of this thing, because this expression is nothing, but the moment generating function.

Now, because X and Y are independent. Therefore, I can write log E of this expression the expectation of this e T X. This is a product of two random variables and I can write it as for two independent variables. We know that E of X Y is equal to E of X E of Y.

So, using that expression. I have written e exponential t X plus Y as e exponential t X exponential t Y which is equal to e exponential t X exponential t Y, all of them within the logarithm. Now, when I take the logarithm this product translates to a plus sign and each of this is a moment generating function.

The left hand side this expression log E of e to the power t X exponential t X is nothing, but the moment generating function of X. And the log E of exponential t Y is the moment generating function of log of moment generating function of Y. So and these are nothing, but the cumulant generating functions.

So, the net result is that the cumulating cumulant generating function of X plus Y has been expressed as the sum of the cumulant generating functions of X and the cumulating cumulant generating functions of Y. I repeat the cumulant generating function of X plus Y, where X and Y are independent random variables is equal to the sum of the cumulant generating function of X and the cumulant generating function of Y.

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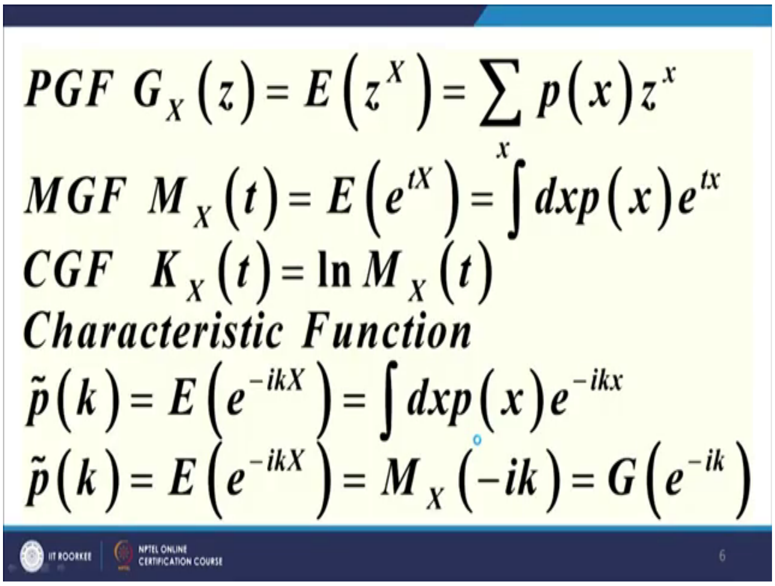

So, this is the summary of what we have discussed so far. The probability generating function, the moment generating function, the cumulant generating function, the characteristic function. All these are there on the slide for the purpose of reference.

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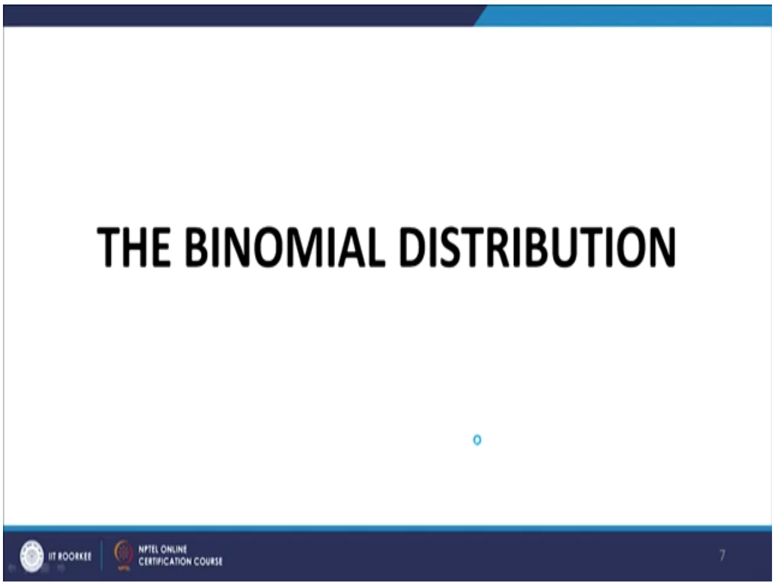

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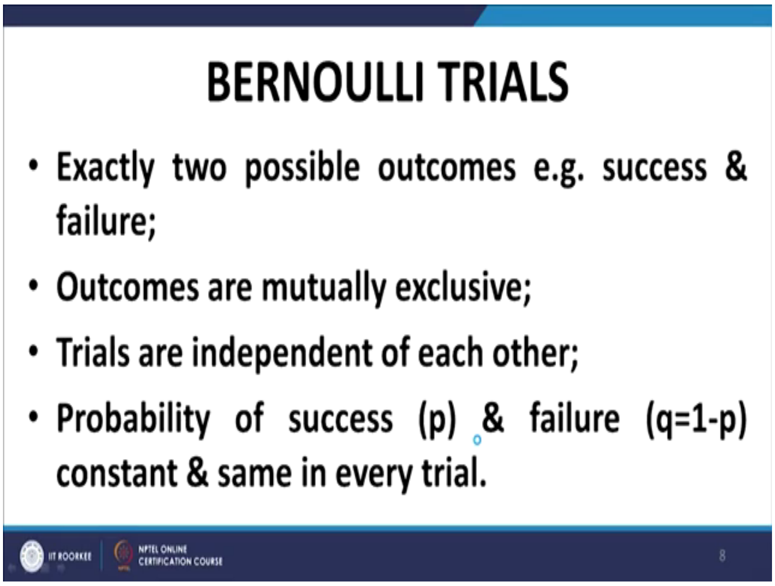

Now, we come to the binomial distribution. Before we talk about binomial distribution, because binomial distribution is directly related to an experiment that involves Bernoulli trials, let us understand what are Bernoulli trials? In a Bernoulli trial Bernoulli trial is a random experiment which has precisely two outcomes exactly two outcomes, let us call one of the outcome a success and the other outcome a failure.

The outcomes are mutually exclusive. In other words, there is no situation where both the outcomes can occur simultaneously. Either the success will occur or the failure will occur. Then the trials are mutually independent.

In other words, the one experiment or we will not influence any other experiment carried out in the series. And finally, the probability of a success p or the probability of a failure which is 1 minus p, that is q usually represented by q remain constant over all the experiments in a particular set of experiment particular set of trials. Now, this is the background in which the binomial distribution comes into play.

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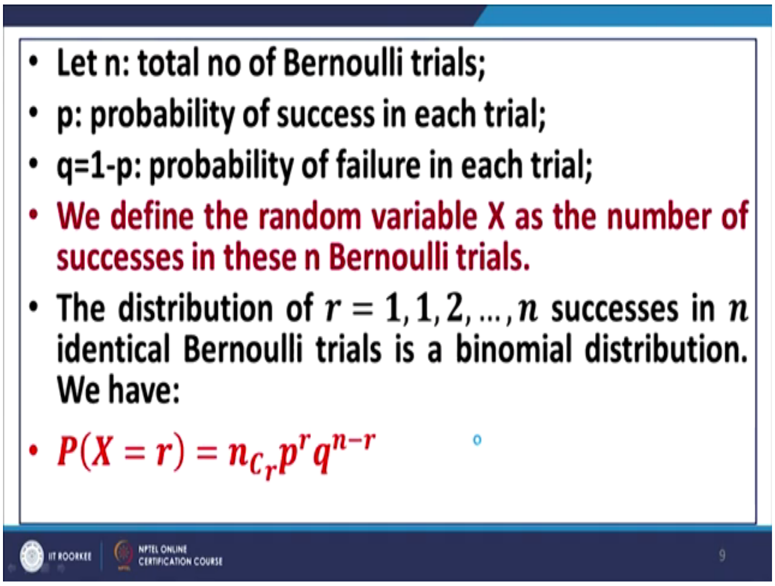

If you have set of n Bernoulli trials if you have a set of n Bernoulli trials let us say, you have coin tosses biased coin tosses. Coins are biased there is a. The probability of a head is p, the probability of a head of q of tail is q and they are not the same. They may be same they may not be same, but let us talk about the general case where they are not the same. Probability of head which is termed as a success is p, probability of a tail which is termed as a failure is q. And there are n coin tosses.

Then the probability of getting exactly r heads r successes is given by this term: p X equal to r. We have defined capital X as the number of heads. So, the probability of getting X equal to r means probability of getting r heads. So, probability of getting r heads is equal to n C r p to the power r q to the power n minus r.

This is the binomial distribution. If you conduct n Bernoulli trials with probability of success p, probability of failure q then the probability of getting precisely r heads is equal to n C r p to the power r q to the power n minus r.

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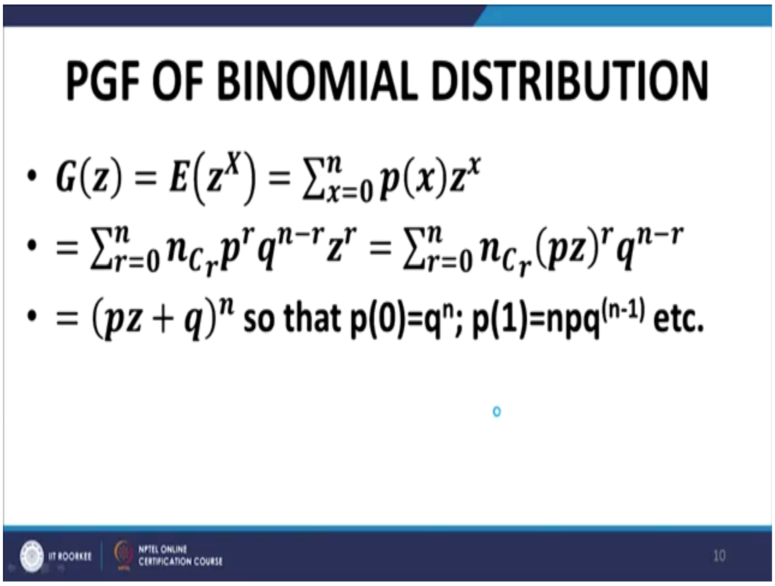

The probability generating function of this particular distribution is quite simple. It is given by p x z to the power x, p x in our case as just now we have seen is equal to n C r p to the power r q to the power n minus r. We take this p to the power r z to the power r in one term and this becomes a binomial distribution. And when you take the summation. So, that is equal to p z plus q to the power n.

Clearly, if you take. From this you can identify that if you put z equal to 0, you get q to the power n which is the probability of getting 0 heads probability of getting 1 heads. If you use 1 head and if you want to work out 1 head you differentiate it once and then put Z equal to 0. And you get the probability of getting 1 head as n p q to the power n minus 1.

Because, and you can check that you can check that, because n C 1 is equal to n p to the power r in this case is p q to the power n minus r is n minus 1. And on this side, you differentiate it with respect to z and then put z equal to 0. You get precisely this expression.

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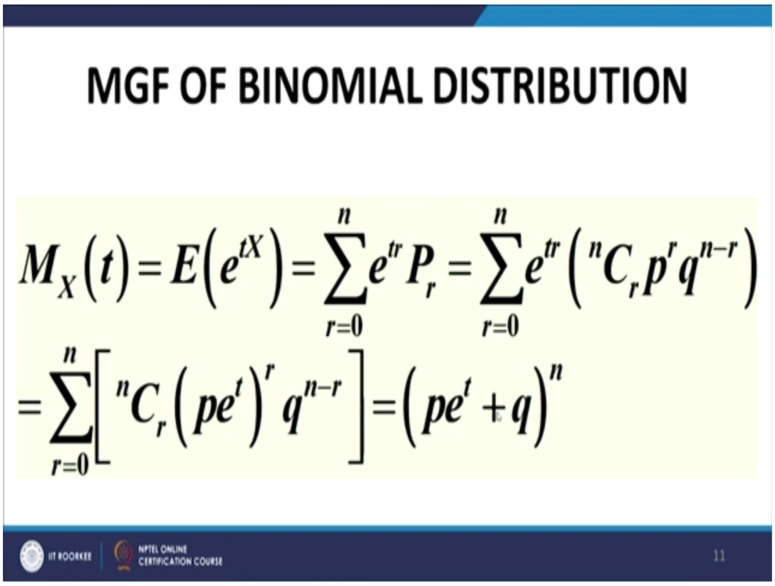

The moment generating function of the binomial distribution is X, it is the expected value of px. So, let us put t X. Let us sum over r all the possible values of X. So, we have because we are taking expectation. So, we have to sum over all possible values of r which is 0 to n, because there are total of n trials and the number of heads can vary from 0 to n.

So, e to the power t r p to the power r substitute the value of p to the power r as n C r p to the power r q to the power n minus r. And when you simplify this you get pe to the power t plus q to the power n. And you can verify that mu 0 is obtained by putting t equal to 0 in this and you get p plus q to the power n that is equal to 1.

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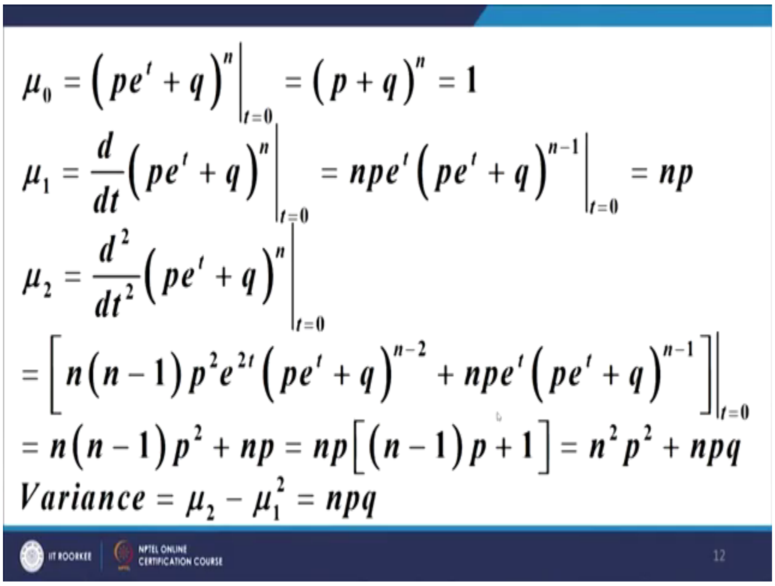

The mean of this particular expression mu 0 is equal to 1; sorry not the mean mu 0 is equal to 1 mu 1; mu 1 is equal to. If you differentiate it with respect to t you get this expression right. This expression when you simplify when you put t equal to 0 you get p plus q to the power n minus 1, which becomes equal to which becomes equal to 1.

So, this expression is 1, e to the power t becomes 1 and the mean becomes n p. So, mu 0 is 1 mu, 1 is equal to np and similarly you can work out mu 2, mu 2 you differentiate this again. Do a second differentiation put t equal to 0 and you end up with mu 2 is equal to n square p square plus n p q and that gives you a variance of n p q. So, the binomial with the cardinal parameters of the binomial distribution which you need again is n p for the mean and n p q for the variance.

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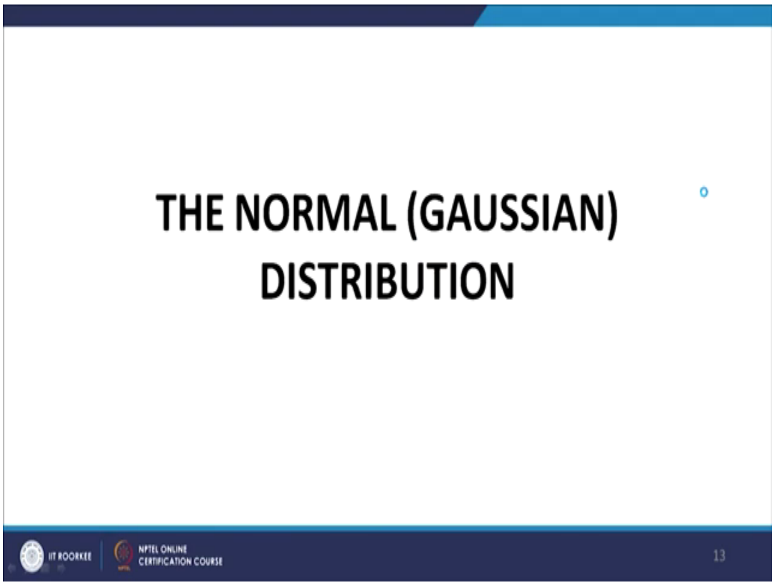

Now, we come to the normal distribution. The normal distribution is also known as the Gaussian distribution. In physics literature it is more commonly called the Gaussian distribution and statistical literature on statistics, we normally call it a normal distribution.

Here, we start with the binomial distribution. We know for the binomial distribution p x equal to r is equal to n C r p to the power r q to the power n minus r. Now, we look at a special case of the binomial distribution in which n tends to infinity; that means, the number of trials tends to infinity, but n p mean is finite. We introduce a new variable Z is equal to X; X was the binomial variable capital X was the binomial variable. We introduce a new variable Z defined by Z is equal to x minus np upon under root n p q.

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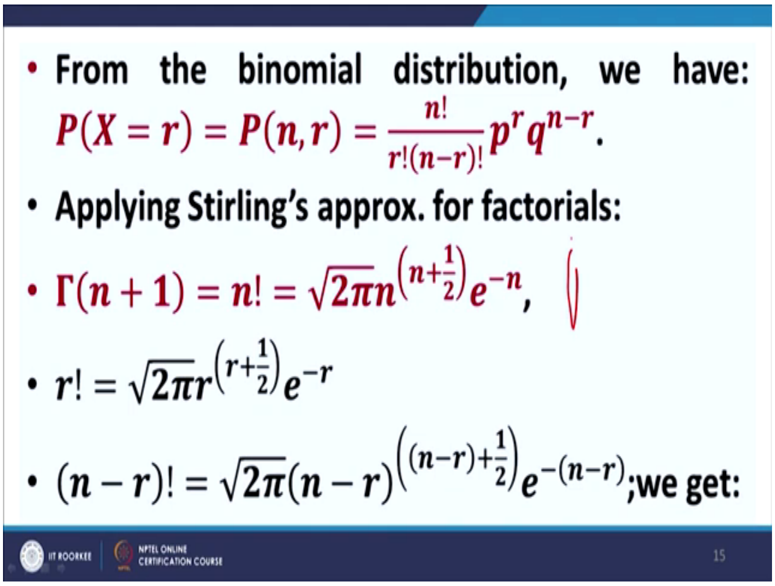

Now, this is the first expression is quite straightforward. This is nothing, but the binomial distribution n C r p to the power r, q to the power n minus r this is nothing, but P X equal to r. Now, we apply this Stirling approximation. The Stirling approximation is based on the concept of steep descent and it provides you an approximate expression for large factorials, the expression.

The formula is given by gamma n plus 1 is equal to n factorial gamma function gamma n plus 1 is equal to n factorial is equal to under root 2 pi n to the power n plus 1 by 2 e to the power minus n. This is the Stirling approximation for n factorial. This is what we have from the Stirling approximation.

We substitute these values here. For n factorial we use this expression. For r factorial we use the equivalence Stirling expression for the r factorial. And for n minus r factorial we use the expression with instead of n we have r here instead of r we have n minus r. Here, nothing else it is simply implementing the Stirling approximation.

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$$\lim_{n\to\infty} P(n,r) = \frac{e^{-n} n^{n+\frac{1}{2}} p^r q^{n-r}}{\sqrt{2\pi} e^{-r} r^{r+\frac{1}{2}} e^{-(n-r)} (n-r)^{n-r+\frac{1}{2}}} = \frac{A}{\sqrt{2\pi npq}} (say)$$

$$A = \left(\frac{np}{r}\right)^{r+\frac{1}{2}} \left(\frac{nq}{n-r}\right)^{n-r+\frac{1}{2}}$$

$$z_r = \frac{(r-np)}{\sqrt{npq}}$$

$$= \left(1+z_r\sqrt{\frac{q}{np}}\right)^{-\left(np+z_r\sqrt{npq}+\frac{1}{2}\right)} \left(1-z_r\sqrt{\frac{p}{nq}}\right)^{-\left(nq-z_r\sqrt{npq}+\frac{1}{2}\right)}$$

We substitute all the respective values. And we write it in the form which you will come to know why it is written this way, but it is written in the form A; where A is some expression upon under root 2 pi n p q. And what is A? A is this expression this is my A. And now, what I do is? I make a substitution z to the power z r z suffix r is equal to r minus np upon under root n p q. Remember we had made this substitution.

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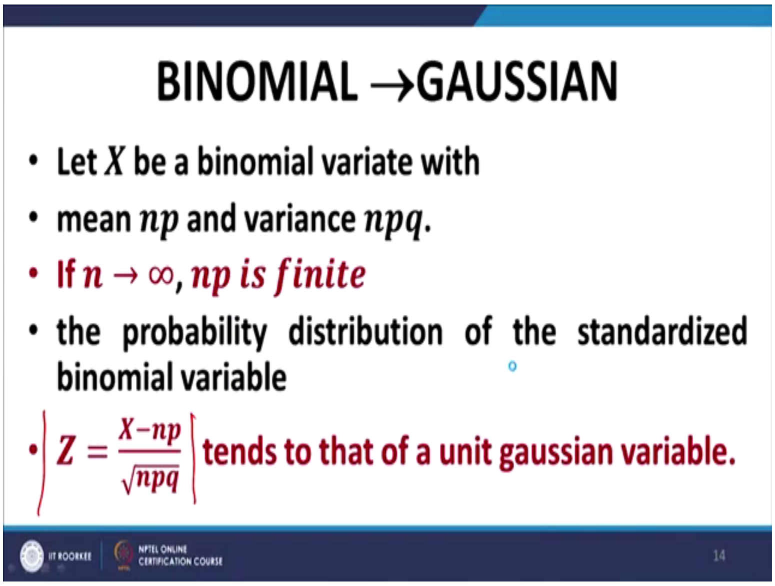

The random variable Z represented the random variable X which was a binomial variable minus n p upon under root n p q. Now, we are talking about a particular realization of Z corresponding to a realization of X given by X equal to r. If X is equal to r then Z is equal to Z r, which is given by r minus n p upon under root n p q.

We make this substitution in this equation; in this equation right. On making this substitution in this equation we get this expression this whole expression.

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where $z_r = \dfrac{(r-np)}{\sqrt{npq}}$ since

$$\left(1+z_r\sqrt{\frac{q}{np}}\right)=\left[1+\frac{(r-np)}{\sqrt{npq}}.\sqrt{\frac{q}{np}}\right]=\left(\frac{r}{np}\right)$$

The simplification that we have used is given in the slide as, I will simply a little bit of algebra. We have substituted r minus n p upon under root n p q equal to z r and thereby we have arrived at this expression. And this expression is equal to r upon n p. So, we started with this, we made the substitution and we arrived at this.

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$$\left(1-z_r\sqrt{\frac{p}{nq}}\right)=\left[1-\frac{(r-np)}{\sqrt{npq}}\cdot\sqrt{\frac{p}{nq}}\right]$$

$$=\left(1+\frac{np-r}{nq}\right)=\left(\frac{np+nq-r}{nq}\right)=\left(\frac{n-r}{nq}\right)$$

Similarly, for the other side we started with this, working backwards we arrived at this. So, using these two expressions we wrote the value of A in this form by making use of in this input.

Now, we work out log of A log of A the exponentials get multiplied by the log and we get this expression simply taking the log of A, nothing more. If you can recollect this, this is A this is the expression for A with taking the log of A. So, this comes this expression comes here and this will be log. Similarly, this expression comes here and this will be log. So, that is what we have done.

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$$\text{Assuming } |z_r| < \min\left(\sqrt{\frac{np}{q}}, \sqrt{\frac{nq}{p}}\right);$$

$$\log_e A = -\left(np + z_r\sqrt{npq} + \frac{1}{2}\right)\left[z_r\sqrt{\frac{q}{np}} - \frac{z_r^2 q}{2np} + \frac{z_r^2 q^{\frac{3}{2}}}{3(np)^{\frac{3}{2}}} + 0\left(n^{-2}\right)\right]$$

$$+\left(nq - z_r\sqrt{npq} + \frac{1}{2}\right)\left[z_r\sqrt{\frac{p}{nq}} + \frac{z_r^2 p}{2nq} + \frac{z_r^3 p^{\frac{3}{2}}}{3(nq)^{\frac{3}{2}}} + 0\left(n^{-2}\right)\right]$$

There is a lot of mathematics in this course, so we got to be careful. Now, we are expanding the log as well log A and so on in terms of the Taylor series for log.

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$$= -\frac{z_r^2}{2} + \frac{z_r(q-p)}{2\sqrt{npq}} - \frac{z_r^2}{4npq}\left(p^2+q^2\right) - \frac{z_r^3}{6\sqrt{npq}}(q-p)$$

$$+\frac{z_r^3}{6(npq)^{\frac{3}{2}}}\left(q^3-p^3\right).$$

As $n \to \infty$, the above expression $\to -\frac{z_r^2}{2}$.

Collecting terms we arrive at this expression collecting all the terms we arrive at this expression. Now, as you can clearly see from here, this first term is independent of n this term is independent of n. But if you look at all the other terms they have a factor of n in the denominator. So, when n tends to infinity we get the log of A is equal to this expression.

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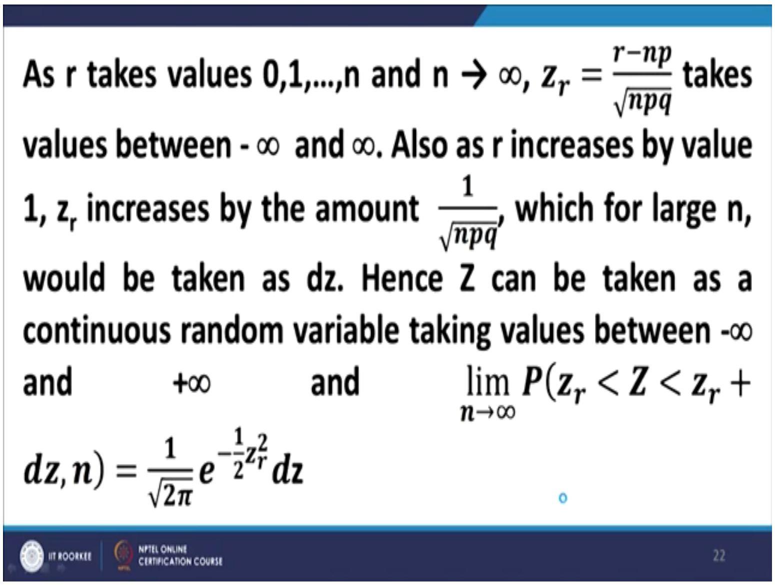

Therefore, in the limit that n tends to infinity. We get A is equal to exponential of minus z r square upon 2. Now, as r takes the value 0 1 2 3 4 Z r. you see r was the original variable r was the realization of the original variable binomial variable x and it can take integral values it can take integral values from 0 to n.

Now, as r takes values 0 to n what happens to z r? If you look at this if r is 0, it becomes minus n p upon root n and as n tends to infinity this approach is minus infinity, because as scales as minus root n and as n tends to infinity this approaches infinity.

Similarly, as r becomes very large then this expression tends to plus infinity. So, r varies from minus infinity to plus infinity. That is the range where that is the ranges where z r varies from I am sorry z r varies from minus infinity to plus infinity r varies from 0 to n; 0 to n integral values 0 to n. And when r is 0, the value of z r approaches minus infinity. So, the range of z r is from minus infinity to infinity.

And secondly, one more thing as z r as r changes by 1 value 1 unit , the z r changes by 1 upon under root n p q as. I repeat as r changes by 1 unit for example, from 1 to 2 or 0 to 1, z r changes by 1 upon under root n p q. Now, as n tends to infinity this expression becomes smaller and smaller.

And we can approximate this by the infinitesimal d z. In other words putting all these things together we can write that, in the limit that n tends to infinity probability that z lies between z r and z r plus d z, z lies between z r and z r plus d z is given by under root 2 pi e to the power minus 1 by 2 z r square dz.

Where recall I have used this expression, we have to go back a long way. This expression: under root 2 pi has gone there, A has become equal to e to the power A has become equal to e to the power minus 1 by 2 z r square, under root 2 pi is there and 1 upon under root n p q is our dz.

So, that is how we get that expression 1 upon under root 2 pi e to the power minus 1 by 2 z r square d z. And this is the probability that the random variable z lies between z r and z r plus d z correct.

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$$M_X(t) = E\left(e^{tX}\right) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx - \frac{1}{2\sigma^2}(x-\mu)^2} dx$$

$$Now,\ tx - \frac{1}{2\sigma^2}(x-\mu)^2 = \frac{2\sigma^2 tx - (x-\mu)^2}{2\sigma^2}$$

$$= \frac{2\sigma^2 tx - x^2 - \mu^2 + 2\mu x}{2\sigma^2} = \frac{-x^2 - \mu^2 + 2x\left(\mu + \sigma^2 t\right)}{2\sigma^2}$$

$$= -\frac{\left[x - \left(\mu + \sigma^2 t\right)\right]^2 + 2\mu\sigma^2 t + \sigma^4 t^2}{2\sigma^2}$$

$$= \mu t + \frac{1}{2}\sigma^2 t^2 - \frac{\left[x - \left(\mu + \sigma^2 t\right)\right]^2}{2\sigma^2}$$

So, that is this is the; this is the definition or this is the probability density function of the normal distribution which is at the point worked out at the point z r. This is the pdf of the normal distribution function worked out at the point z r. Now, if we want to work out the normal distribution function for a general variable, this was the standard normal variable. The standard normal variable which has a mean of 0 and a variance of 1.

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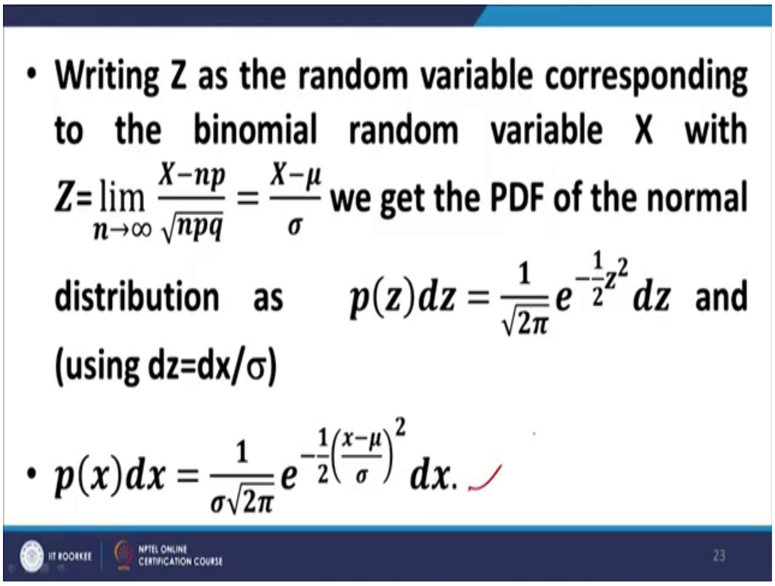

Now, suppose I write this as z as x minus mu upon sigma, then I can get the pdf of the general normal variable and that turns out to be this, because d x is equal to z sigma d z and on putting this value we get this expression. So, this is my this is my general normal distribution and this is my standard normal distribution.

Now, we come to the moment generating function of the normal distribution that is given by the mean value of t to the expected value of t e to the power t X. And when I substitute the pdf of the normal distribution it becomes 1 upon sigma root 2 pi minus infinity to infinity e to the power tx this tx this is my X into whatever is the minus 1 upon 2 sigma square into x minus mu whole square.

This is the this portion with the E exponential of this portion comes with the pdf of the normal distribution and e to the power t X is this e to the power t x. Now, I have to integrate this.

Now, in order to integrate this, what I do is I convert the exponential of the exponent of the exponential I am sorry exponent of the exponential into a perfect square. The process is simple. We simply convert it to a perfect square and with outcasts or take out a term which is independent of the integration variable x.

So, we break it up into two parts: one part which is a function of t, sorry which is a function of x and the other part which is independent of x. The part that is independent of x can be taken outside the integral and the part that is within or the part that contains x has to be integrated over within the limits minus infinity to infinity.

The exponent can be written as this in this expression. As you can see by making a few manipulations it can be written in this form. Now, you look at it carefully. This part is independent of x and this part depends on x. So, we can split this whole exponent into two parts: This part I will take out outside the integration I have taken it outside the integration and I have left the other part inside the integration.

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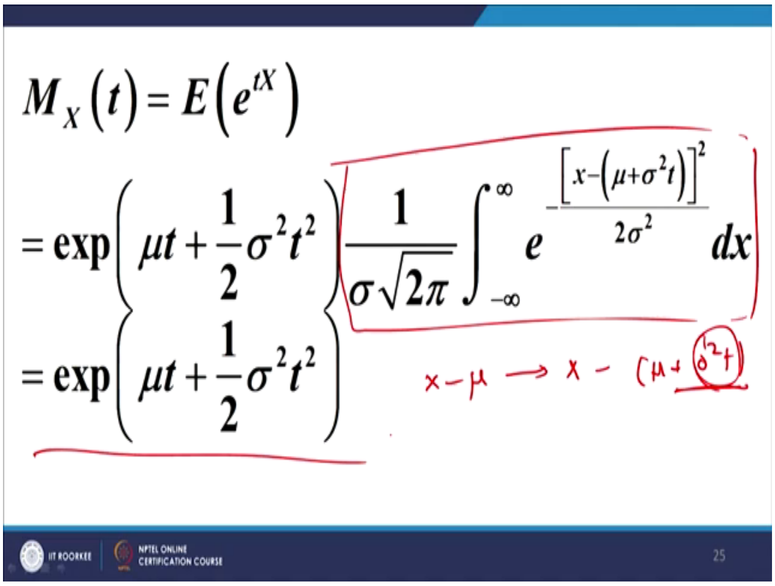

Now, look at this integration. This integration is simply a normal distribution. It is a normal distribution node and it has a variance of sigma square that is also clear. And it has a mean of mu plus sigma square t.

In other words, it is a shifted normal distribution. You started with a normal distribution with a mean of mu. Now, we have a normal distribution with a mean of mu plus sigma square mu minus mu plus sigma square t, that is the mean of the shifted distribution.

But the important thing here is important thing here is the important thing here is if I put x equal to minus infinity or plus infinity the. Whatever happened earlier to the earlier distribution continues to happen in this new distribution it is only a shifted distribution thus shape of the distribution the limits of the distribution do not change, because it is simply translation nothing else only translation. From X minus mu I have moved to X minus mu plus sigma square t.

In other words, this part I have shifted. I have shifted this part. The origin has shifted this part this much otherwise there is no change. But due to the shift of this origin there is no change in the integration limits, because when extended to infinity here also it the function value tends to infinity. And here minus infinity also the thing remains the same right. If X tends to minus infinity. You see this addition is a positive number it is only a number sigma square t.

So, it is simply shifts the origin that is it. And minus infinity and infinity do remain intact the limits of integration remain intact. And therefore, this part will give you 1, because it is an integration of a normal density function over its entire spectrum of values minus infinity to infinity right.

So, what do we end up with? We end up with a moment generating function of exponential mu t plus 1 by 2 sigma square t square. This is what we get as the outcome of this whole analysis. This is the moment generating function of the normal distribution.

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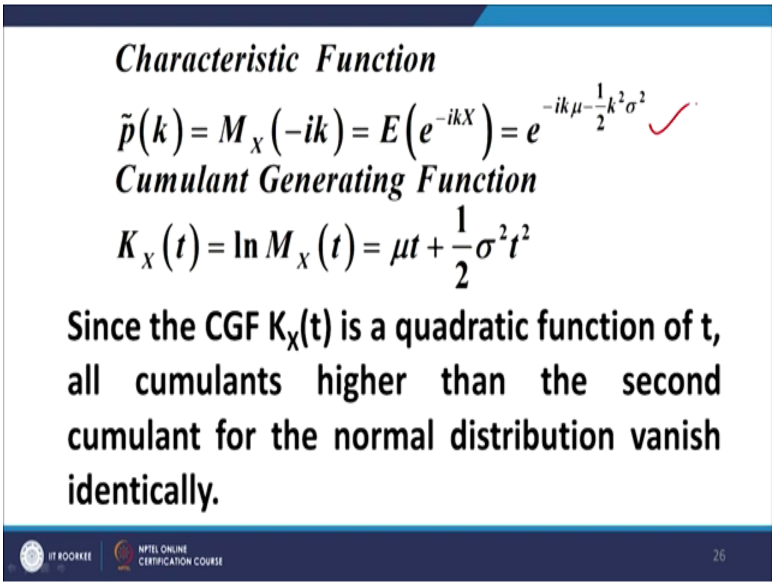

And the characteristic function is given by M of M X of minus i k and that is given on this expression quite straightforward. And as far as the cumulant generating function is concerned, it is the natural log of the moment generating function. So, it is given by mu t plus 1 by 2 sigma square t square.

Now, another interesting feature here, because the cumulant generating function is quadratic in t. So, what does it imply? It implies that all cumulants higher than the second cumulant will naturally vanish for the normal distribution.

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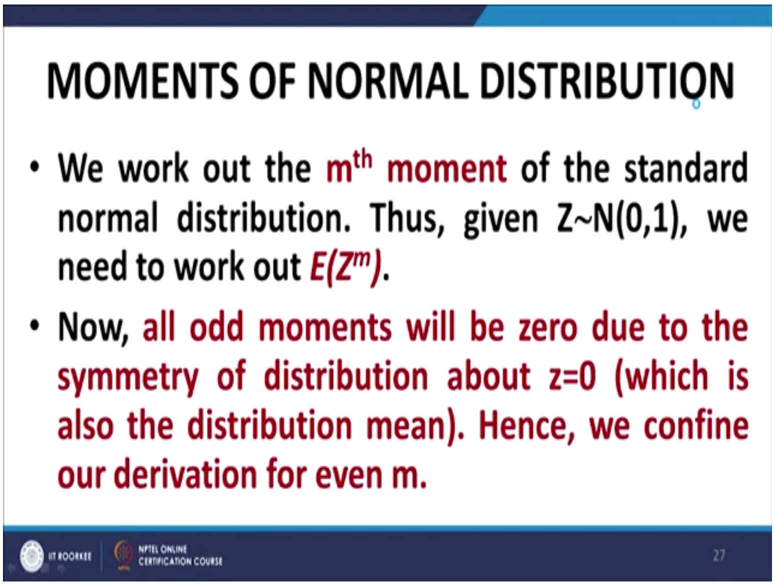

Now, we work out the general moments of the normal distribution. All odd moments should be 0 which can be worked out numerically, but from symmetry considerations also we can say that all odd moments should be 0, because we were talking about the standard normal distribution because the curve is symmetrical about the origin and as a result of which, when you integrate between minus infinity and infinity any odd function that results in a 0 value. So, all odd moments will give you 0, but let us look at the even moment.

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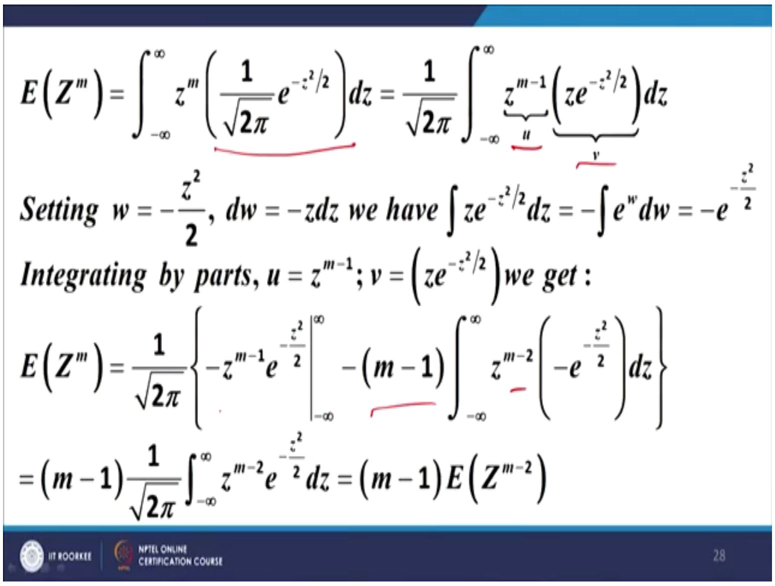

In fact, we can also numerically work out the odd moments and show that they are 0, but we will focus on the even moments. Now, let us say we want to work out exponential expected value of Z to the power m. Putting the standard normal distribution here, this is this pdf of the standard normal distribution. Integrate between minus infinity and infinity and what do we get? We get this expression. And I make this we need to do this integration. It is as simple as this we need to do this integration.

Now, 1 upon 2 under root 2 pi is a constant, you take it outside the integration. I split up the integrand z to the power m e to the power minus z square upon 2 into two parts. I write it as z to the power m minus 1 and then z e to the power minus z square upon 2. I take this as u I take this as v and then I integrate by parts.

The integral of this expression v the integration integral of v is quite simple. If you put minus z square upon 2 s w you end up with the integral you end up with is minus e to the power minus z square upon 2 right.

So, now we do the integration by parts. What does integration by parts give us? First function into integral of second function. First function u as it is integral of second function minus integral of derivative of first, which is this expression into integral of second. Now, if you look at this carefully. When I put the limits minus infinity and infinity this expression vanishes at both the both the extremes this expression vanishes.

So, what I am left with is minus minus plus. So, I am left with this expected value of Z to the power m is equal to m minus 1 1 upon root 2 pi exponential integral z to the power m minus 2 e e to the power minus z square n by 2.

If you look at this very carefully this part if you look at this carefully, this is nothing but the normal distribution probability function e to the 1 upon root 2 pi e to the power minus z square by 2 multiplied by z to the power n minus 2. And that is nothing, but the m minus two-th moment. So, what do we have? m minus 1 this m minus 1 comes here E to the power or sorry expectation of Z to the power m minus 2. So, what is our final conclusion?

Our final conclusion is expectation of Z to the power m is equal to m minus 1 exponential of Z to the power m minus 2, but exponential the expectation of Z to the power 0 is 1.

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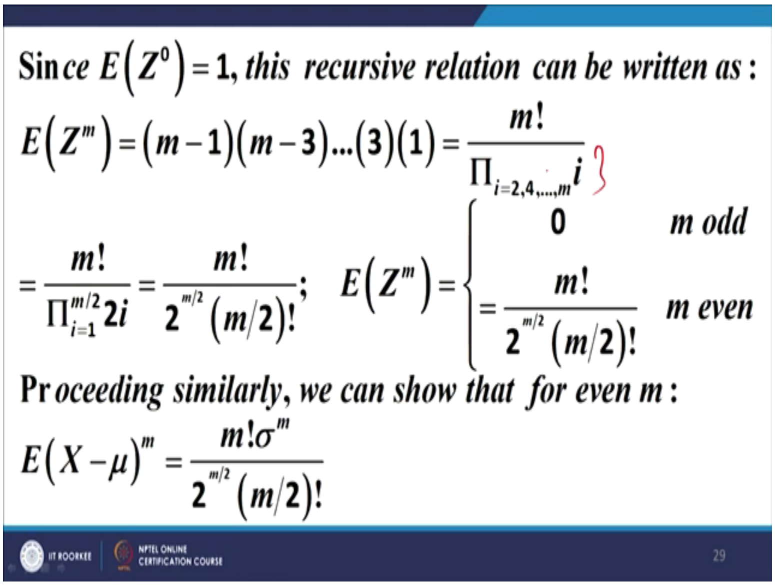

Therefore, this recursive relation can be written as exponential of Z to the power m is equal to m minus 1 m minus 3 and so on. 3 1 that is nothing but m factorial divided by the even numbers and product of the even numbers 2 4 6 and so on up to.

So, when this expression is simplified this expression is simplified up to 2 4 6 up to m. This expression is simplified the denominator is simplified we get m factorial. And we can write it as the product of 2 into 1 2 3 4 up to m by 2 and that becomes 2 to the power m by 2 into m by 2 factorial.

So, we can write this a summary of this expression as 0 if this is odd the E expectation of Z to the power m is 0 if m is odd and is given by this expression if m is even.

Thank you.