Introduction to Operations Research
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Module - 02
Graphical and Algebraic Methods
Lecture - 01
Graphical Method (Maximization)

In the 2nd module we move on to study linear programming solutions, in the earlier module we looked at linear programming formulations. Now, we will study solution to linear programming problem, specifically the graphical and algebraic methods of solving linear programming problems.

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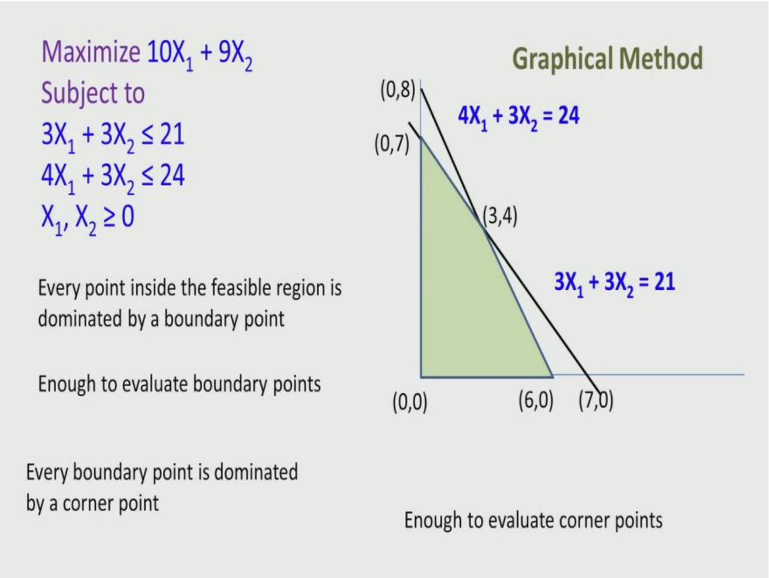

We will first look at the graphical method to solve a given linear programming problem and the problem that we will consider is a maximization problem, maximize 10 X 1 plus 9 X 2 subject to 3 X 1 plus 3 X 2 less than or equal to 21, 4 X 1 plus 3 X 2 less than or equal to 24, X 1, X 2 greater than or equal to 0. There are two variables X 1 and X 2 and since there are two variables, it is possible to represent these two variables in a graph.

So, we first try to represent the constraints and the objective function using a graph. So, we first start drawing the graph by defining the origin and by drawing the X and Y axis. Usually X and Y axis divides the graph into 4 quadrants, but when we solve linear programming problems, we look at only one out of the 4 quadrants. Because, X 1 and X

2 have to be greater than or equal to 0, then X 1 and X 2 are greater than or equal to 0 we look at the first quadrant and we do not look at the other three quadrants which are here, where at least one variable takes negative values.

So, in all linear programming problems when we use the graphical method, we will be restricting ourselves only to the first quadrant, where the two variables are greater than or equal to 0. Now, we try and draw, the X axis will now represent the first variable which is X 1 and the Y axis will represent the second variable which is X 2, we now try to draw the two constrains into the graph. So, we take the first constraint which is 3 X 1 plus 3 X 2 less than or equal to 21 which has an inequality.

So, what we do is, we draw first the corresponding equation and then we find out that area in the graph which maps to the inequality. So, the first constraint will be 3 X 1 plus 3 X 2 equal to 21 and that is drawn this way. So, the constraint or the line 3 X 1 plus 3 X 2 equal to 21 is drawn here, we require two points to draw this line and these two points are 7 comma 0 and 0 comma 7 and with these two points we can draw the line 3 X 1 plus 3 X 2 equal to 21 or X 1 plus X 2 equal to 7.

Now, this line divides the graph into two regions, one region which is below this line which I am showing, the other region is above this line which is this space. Now, one of these two is less than or equal to and the other will become greater than or equal to. Now, to find out which area is less than or equal to we take a convenient point which is the origin and then, we substituted into this equation and we realize that the left hand side is 0 and the right hand side is 21, left hand side is less than the right hand side therefore, the origin is in the region which is less than or equal to 21.

And therefore, this triangle which is made out of these three points is the region corresponding to 3 X 1 plus 3 X 2 less than or equal to 21. Having drawn the first constraint, we now draw the second constraint which is 4 X 1 plus 3 X 2 less than or equal to 24. And this is how the line will look like, first we will draw 4 X 1 plus 3 X 2 equal to 24 and to draw that we need to mark two points 6 comma 0 and 0 comma 8 which satisfy 4 X 1 plus 3 X 2 equal to 24.

So, if the line 4 X 1 plus 3 X 2 equal to 24 is drawn. Once again, this line will divide the region into two portions, one is this portion, the other is the portion on the other side. One of which will correspond to 4 X 1 plus 3 X 2 less than or equal to 24 and the other will correspond to the greater than or equal to, as we did before we take a convenient point which is the origin and evaluate it. So, when we evaluate at 0 comma 0 which is the origin 4 X 1 plus 3 X 2 becomes 0 and is less than the right inside value of 24.

Therefore, the region formed by these three points the triangle formed by these three points 0 8, 6 0 and 0 0 forms the region 4 X 1 plus 3 X 2 less than or equal to 24. So, now, what we do is, we now look at this region which is common to both the regions. We first saw that for the first constraint, this triangle was there it was the region between 0 7, 7 0 and 0 0 and for the second constraint it was the region between 0 8, 6 0 and 0 0 and the common region which is now shaded and shown is the region, where both these constraints are satisfied.

Since, we want to satisfy both the constraints we are now looking at that region, where both the constraints are satisfied and the shaded region is where both the contains are satisfied. And we also have this point of intersection which is 3 comma 4 and the shaded region is called feasible region, we use the word feasible to represent that the constraints are satisfied. For example, let us look at other points. If we look at the points 7 comma 0, the point 7 comma 0 satisfies the first constraint, but the point 7 comma 0 does not satisfy the second constraint and therefore, it is infeasible.

So, any point which violates even one constraint and does not satisfy even one constraint is called infeasible. So, 7 comma 0 is infeasible, now 0 comma 8 satisfies the second constraint, but violates the first constraint therefore, 0 comma 8 is infeasible. If we take a point which is somewhere here which could be 7 comma 5 or something like that which is somewhere here, then it violates both the constraints. So, any point other than the point which is in that feasible region, violates at least one constraint.

So, all the points in this space are not feasible are infeasible, because they have negative values for X 1. So, it is important to identify the feasible region and we have identified the feasible region for this linear programming problem. Now, this feasible region is the shaded region that we have shown and any point inside the feasible region including these corner points and the boundary points.

For example, a point here like 3 comma 0 is a boundary point for example, a point here which is 0 comma 3 is also a boundary point. So, all the points which are the in the corners, in the boundaries and inside these region are all feasible points they belong to the feasible region and they are called feasible solutions. When we use the word solution, we use the word solution to indicate values that the variables take. For example, 0 comma 0 is a solution for example, 1 comma 1 which would be somewhere here is a solution.

Now, 0 comma 0 and 1 comma 1 are feasible solutions, because they lie in the feasible region, 7 comma 0 is also a solution it represents X 1 equal to 7, X 2 equal to 0, but it is an infeasible solution. Since, we are interested in solving the linear programming problem using the graph, we now want to have X 1 and X 2 which satisfy the two given constraints and X 1 and X 2 to satisfy the non-negativity. Now, the two axis basically represent the region, where the non- negativity is satisfied the first quadrant.

And then, we have added the two constraints into that first quadrant to get a feasible region, where are all the points in the feasible region satisfy the two given constraints as well as satisfy the non-negative restriction. So, anyone or everyone of the points in this feasible region is a solution that satisfies all the constraints and we want to find out that solution or that value of X 1 and X 2 which lies inside the feasible region which maximizes 10 X 1 plus 9 X 2.

When we take a close look at the feasible region, we quickly realize that there are infinite points in the feasible region and we should now have to find the way by which we are able to get the best point out of infinite possible solutions that are feasible. Now, to get the best solution, the value of X 1 and X 2 that has the highest value of 10 X 1 plus 9 X 2, we now do a few more things.

We first start by saying this every point inside the feasible region is dominated by a boundary point. So, let me explain this, now let us take a point which is inside the feasible region and let this point be 1 comma 1. Now, the value of the objective function is 10 plus 9 17, when X 1 is equal to 1 and X 2 is equal to 1. So, if we assume that we are going to make 1 unit of X 1 and 1 unit of X 2 the revenue that we will get is 19.

Now, what I do is I can simply extend this point increase the X 1 value up to it reaches a boundary point. So, here what will happen is I will be getting something like 5 point 5 comma 1 or something like that this is the boundary point which will be here. So, this boundary point when I evaluate 10 X 1 plus 9 X 2 for this boundary point, the value of

10 X 1 plus 9 X 2 for this boundary point will be higher than the value of 10 X 1 plus 9 X 2 for 1 comma 1.

So, if I move this point to the right which means I keep increasing X 1 by keeping X 2 the same and I keep moving this point to the right till it touches a boundary point I realize that the value of the objective function or the revenue is increasing. In a similar manner if I keep this 1 comma 1 and I keep increasing the X 2 value and push it upwards it will reach 1 comma 6 and at that point it would still be feasible. Now, when I take 1 comma 6 the point that is here which is a boundary point, the value of the objective function will be more than 19 it would be 64 in fact.

Therefore, for every point inside this region, there will be a boundary point that will give a better value of the objective function. I use the term better, because if the objective function was were to be minimize, then when I move it leftwards I will get a better value, if it is to minimize I will move it down to get the better value. But, depending on whether it is maximize or minimize, depending on whether sum of these coefficients are positive or negative, it is always possible to move this point towards the boundary and we could do that in many ways.

But, clearly 4 ways by shifting it to the right by moving it to the left by moving it to the top and moving it to the bottom and we will realize that out of these 4 one or more will give better values of the objective function. Therefore we summarize that I do not have to evaluate points strictly inside the feasible region, it is enough to evaluate points which are on the boundaries. So, it is enough to evaluate points which are on the boundaries, but then we realize that this is the boundary of this region starting from 0 0 to 6 0 to 3 4 and 0 7.

But, then we also realize that there are infinite boundary points. Now, we make another observation that every boundary point is dominated by a corner point. For example, if I take the point 1 comma 6, the point 1 comma 6 which is a boundary point has a value 10 plus 54 equal to 64. Now, I can try and move this boundary point along the boundary region and try to come to either 0 comma 7 or 3 comma 4. If I do that I am still feasible I am satisfying all the constraints.

So, if I do that 0 comma 7 will give me 63 and 3 comma 4 will give me 66. So, by moving downwards towards 3 comma 4, I will be able to get higher value of the objective function. So, every boundary point is dominated by a corner point, because when we have two variables and two lines, we can always move the boundary point in along the boundary till it touches a corner point and one of the corner points will give a better value, if it is maximization a higher value, if it is minimization a lower value it will give a better value of the objective function. So, every boundary point is now dominated by a corner point and therefore, it is enough to evaluate only the corner points.

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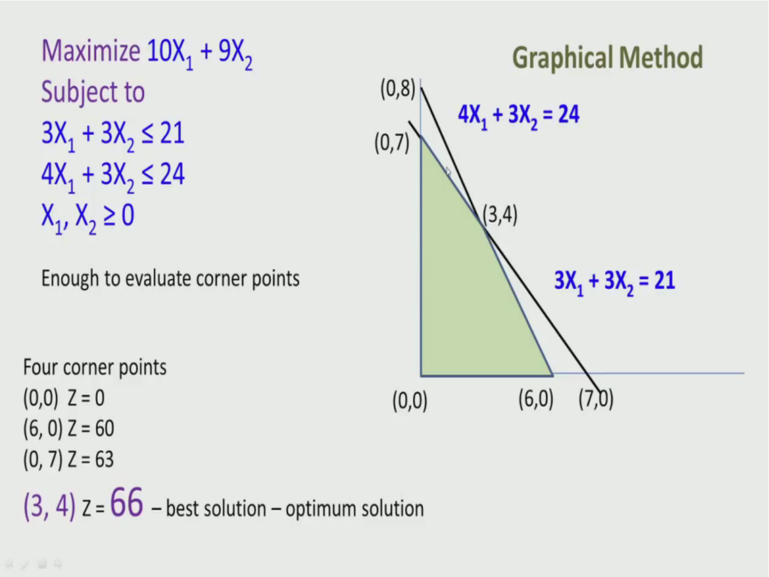

Now, we realize that there are 4 corner points 0 comma 0, 6 comma 0, 3 comma 4 and 0 comma 7. So, there are 4 corner points in this and now we evaluate the objective function for the 4 corner points. The corner point 0 comma 0 z is the value of the objective function. So, at X 1 equal to 0, X 2 equal to 0, 10 X 1 plus 9 X 2 will be 0 and therefore, z is equal to 0 for the corner point 6 comma 0, 10 X 1 plus 9 X 2 will be 10 into 6 plus 9 into 0 which is 60.

For the corner point 0 comma 7, 10 into 0 plus 9 into 7 will gives us 63 and for 3 comma 4 we will get 10 into 3 30 plus 9 into 4 36 which is 66 which is the best solution, which is the one that gives us the highest value of the objective function or the revenue. Now, the best solution is called the optimum solution, so in summary for the graphical method we first plot the constraints, find the feasible region which is the common region, find the feasible region and evaluate find the corner points enough to evaluate the corner points and that corner point which has the best value of the objective function is the best solution or the optimum solution. We will see more aspects of the graphical method and some more aspect in the subsequent classes.