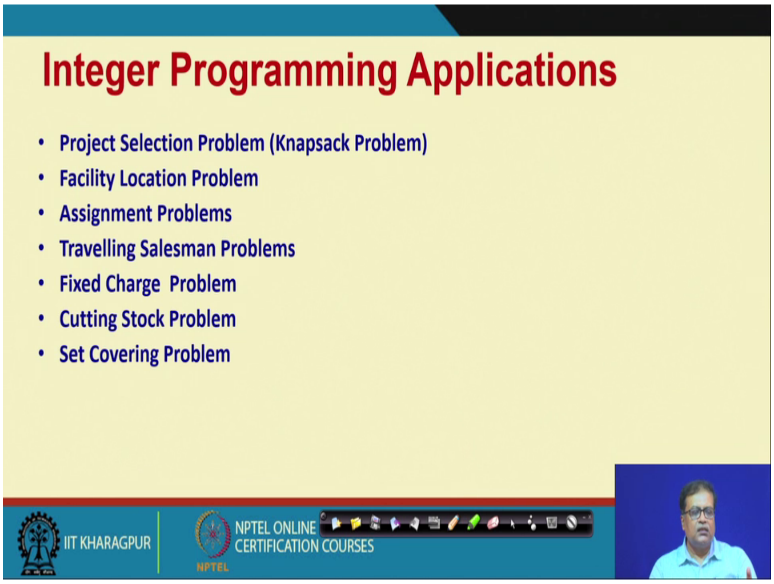
Selected Topics in Decision Modeling
Prof. Biswajit Mahanty
Department of Industrial and Systems Engineering
Indian Institute of Technology, Kharagpur

Lecture - 12
Integer Programming Formulation

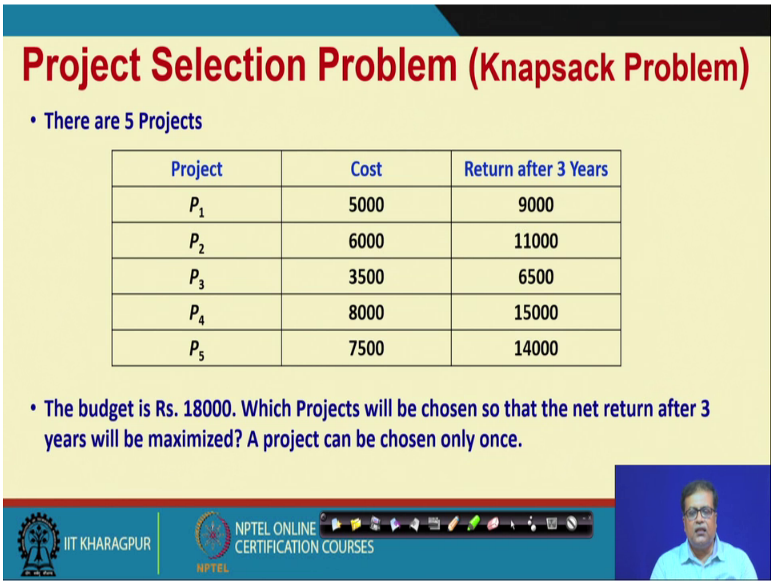
Welcome to you all. And in this particular lecture, we are going to discuss the formulation of Integer Programming Problem. In our previous lecture, we have discussed different types of integer programming problems and we have seen what are their pros and cons, what are certain difficulties, and how they actually put before us certain formulation challenges right. So, in this lecture, we take up some of them and you know try to see how do we formulate such problems.

(Refer Slide Time: 01:00)



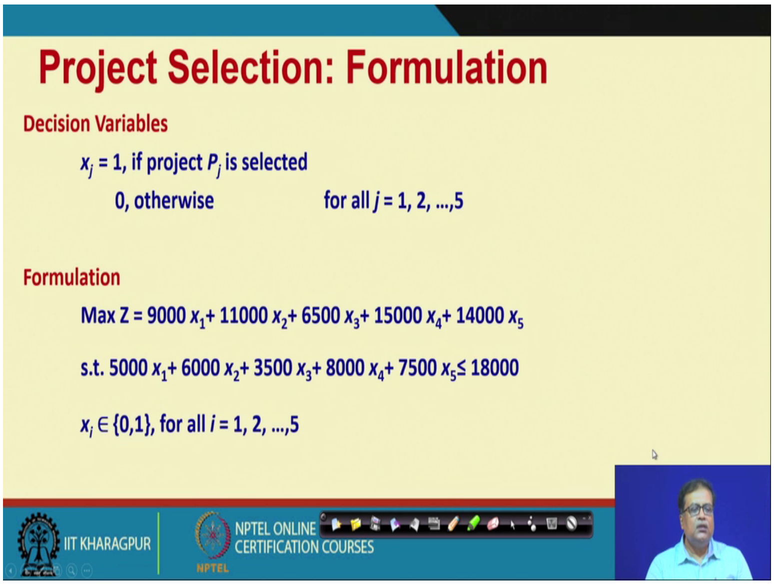
So, let us take to begin with the first you know we discussed these type of problems like project selection problems, facility location problems, assignment problems, travelling salesman problem, fixed charge problem, cutting stock problem, set covering problems and so on.

(Refer Slide Time: 01:17)



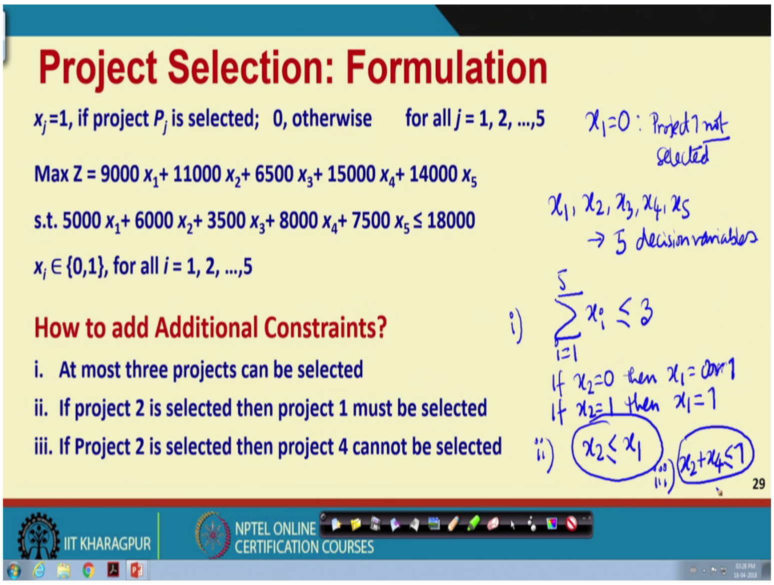
So, let us take the project selection problem which is like a knapsack problem. So, you know there are 5 projects as we discussed in the previous class that there are five projects, the cost are given and the return after 3 years are also given. And if our parts that is the budget is 18000 rupees, which projects will be chosen so that the net return after 3 years will be maximized. A project can be chosen only once right. So, what should be our decision variable for such a problem? We have discussed in our previous lecture, so you know by this time the decision variable has to be that whether a given project is chosen or not, is it ok, so that means, x i or x j equal to 0 or 1 x j is 0 means given project j is not chosen, x j equal to 1 given project j is chosen.

(Refer Slide Time: 02:40)



So, having said that you know these are our decision variables. And our formulation should therefore be the maximize the return, so you see what are our various values that is 9000, 11000, 6500, 15000 and 14000 those are our returns. So, maximize the returns from this project. Subject to the initial cost that is these values 5000 x 1, 6000 x 2, 3500 x 3 less than equal to 18000 where x j is between 0 and 1 for all i 1, 2 dot dot 5 right, so that is very simple we formulate this problem that is a project selection problem. And these formulation is very similar to the knapsack problem, is it not. So, if you recall what you used to happen in knapsack problem that maximize the value and subject to the individual items and you know x 1, x 2, x 3, x 4, x 5 are their numbers and within the total weight limit. So, this is our project selection formulation.

(Refer Slide Time: 04:00)

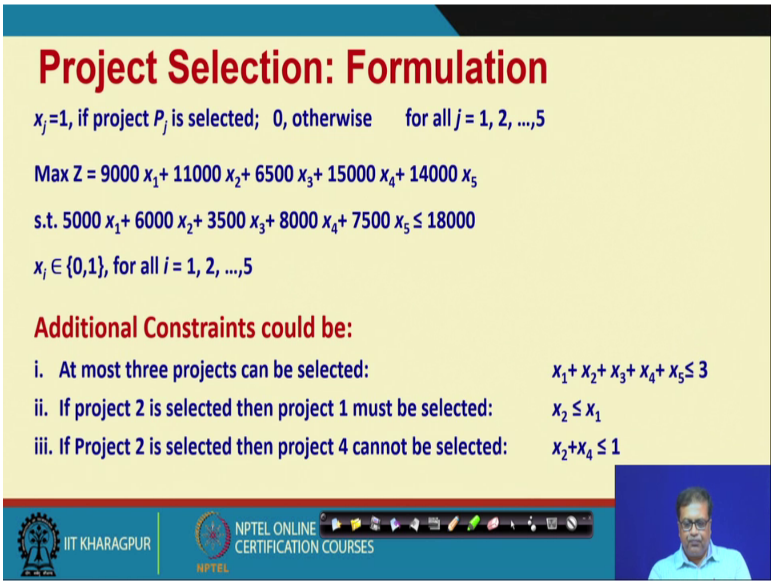


But the additional question that we also raised at that time that at most 3 projects can be selected. So, how do you formulate these? So, you see you know this x 1, x 2, x 3, x 4, x 5 these are our 5 decision variables. Now, if x 1 equal to 0, that means project 1 not selected. And obviously, x 1 equal to 1 means project 1 selected. So, if you say at most 3 projects can be selected, how do you put that, if I add all the 4, 5 you know x i s then what should I get you see not more than three of them should be one is it not that is the maximum. It cannot be 4 or 5, is it ok, that means, the sum over i equal to 1 to 5 x i should be less than equal to 3 right. So, at most 3 projects can be selected.

And a project 2 is selected then project 1 must be selected. How do you do this? What does it mean, it means that if x 2 equal to 0 then that means project 2 is not selected then x 1 could be 0 or 1 is it not. But if x 2 equal to 1, then x 1 should be what, it should be 1; that means you know you can easily understand that if x 2 equal to 0, then x 1 could be 0 or 1. But if x 1 equal to 1, then x 2 equal to 1, then x 1 should be equal to 1, that means, you can see this x 2 should be less then equal to 1; that is the solution to the second problem. This is the solution to the first problem.

And if project 2 is selected, then project 4 cannot be selected that means if x 2 equal to 0, then x 4 could be 0 or 1 that is fine; but if x 2 equal to 1, then x 4 should be 0 is it all right. So, what would be what can you comment about x 2 plus x 4, x 2 plus x 4, its maximum value should be 1. Why, because let us write it down number three x 2 plus x 4 would be less than equal to 1. You know what does it mean that if x 2 is selected let us say x 2 equal to 1, then x 4 should be 0 right, because if x 4 is also 1 then x 2 plus x 4 will be 2 right, in that case you know it will violate the constraint. But if x 2 equal to 0, x 4 could be 0 or 1 right, so there is no difficulty there. So, this is how we actually can take care of such constraints.

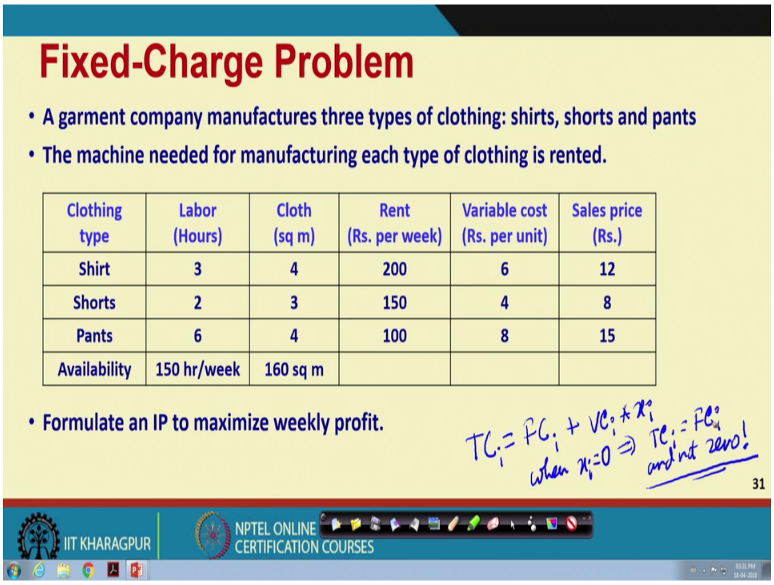
(Refer Slide Time: 08:36)



So, you know let us see how this you know here it is written that x 1 plus x 2 plus x 3 plus x 4 plus x 5 should be less than equal to 3. And x 2 is less than equal to x 1, and x 2 plus x 4 is less than equal to 1 right. So, this is how all these additional constraints could be really done. So, you know there could be many such questions you know, you can raise that if you select 3, then you cannot select 1 or 2; out of 1, 2 and 3 only 1 can be selected. So, like this you can think of various you know if then else conditions which can be actually modeled with you know the 0, 1 variables very nicely right.

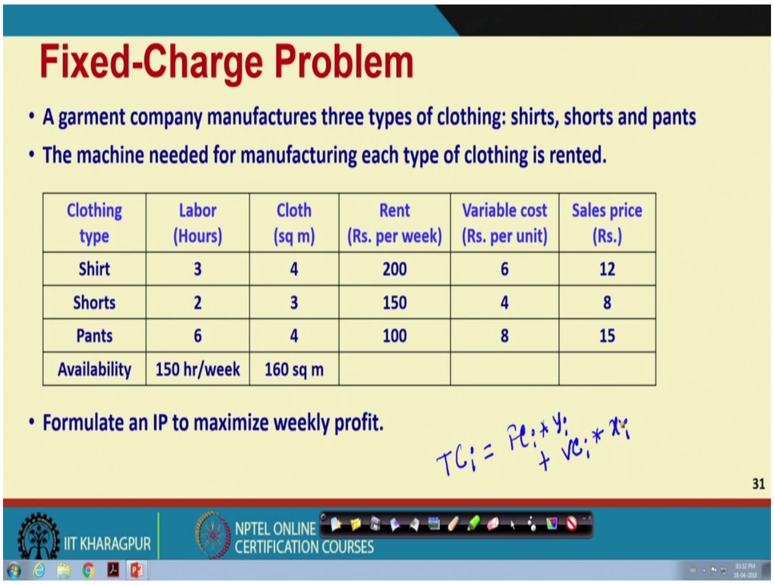
Assume that if you know in a different problem where I have these x i is not 0 or 1, x i's are actually integer numbers. So, if x i's are integer numbers then you know these additional constraints cannot be done in this manner. So, you know in that case you have to define another set of binary variables is it not. So, if you know those binary variables will be such that if a given project is selected, then the binary variable is 1, it does not matter whether 1 or 2. If it is 0, then those variables are 0; but if you know variables are more than 0, then this binary variable will be 1. And then these relationships are to be there put in terms of those binary numbers right.

(Refer Slide Time: 10:22)



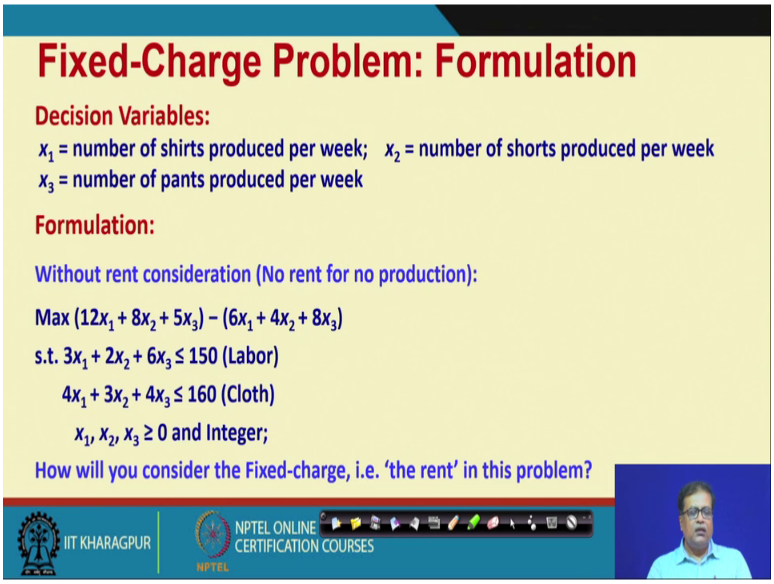
So, having said that lets look at the fixed charge problem. So, how do you go about this, how do you go about putting the fixed cost variable. So, look here I explained it in the last lecture that if you really take the total cost let say TC you know if you take total cost TC is equal to FC i plus VC i star x i right. So, x i is whether a given type of you know clothing type shirt or shorts or pants are taken or not. So, if you take this way, the problem is when x i equal to 0, then TC i equal to FC i and not 0. So, this is the problem, that when x i equal to 0 then TC i equal to F C i are not 0. But what we really required is that TC i should be equal to 0. So, what is done we take another variable y, y i right.

(Refer Slide Time: 11:53)



And we make the model in a different way that is we make TC i equal to FC i star y i plus VC i star x i. So, moment you take it you know you have to ensure that when x i equal to 0 then y i is also equal to 0 you have to ensure that, is it all right. Now, question is how do you do that? So precisely that is what is modeled and let us see how you know this is modeled.

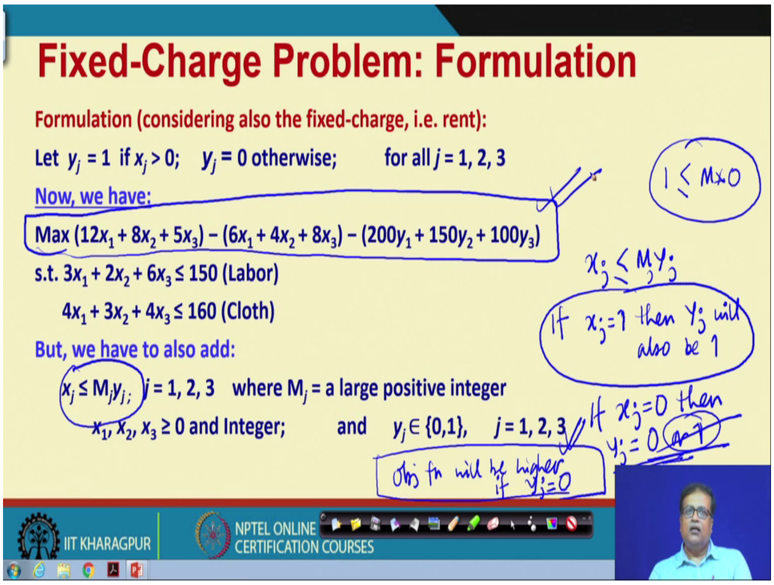
(Refer Slide Time: 12:40)



You see what you know you can do the rest of the things are easy to model that you know without rent consideration you can you can simply take the 12 x 1 plus 8 x 2 plus 5 x 3 which are like profit and 6 x 1 4 x 2 and 8 x 3 they are like cost. So, the profit minus cost and then sorry the 12 x 1 8 x 2 that is like the revenue, the revenue minus cost that is a profit and that is will be maximized and subject to the labor constant or the clothing constant that is the 3 x 1 plus 2, x 2 plus 6, x 3 less than equal to 150, and 4 x 1 plus 3 x 2 plus 4 x 3 less than equal to 160, they are like the usual linear programming formulation. And x 1, x 2, x 3 they should be greater than equal to 0, and they should be integers right.

The question is how do we consider the fixed charge. So, here we do what we really do is that we take large positive number M j and we put and constant that is x j should be less than equal to M j y j. What does it mean? Where the y j is a is a 0, 1 variable is a binary variable. So, why does it work? Please think over what really happens is that when you have added you know these additional variables 200 y 1, 150 y 2 and 100 y 3, now what are these 200, 150 and 100 this are the rents right. So, these 200, 150 and 100 these are rents.

(Refer Slide Time: 14:35)



So, what you do you are you know taking that rent also. So, profit will go down by that amount. And in other words, these two terms this is the fixed cost; this is the variable cost; this is the revenue. So, revenue minus variable cost minus fixed cost. The idea that is you know x j should be less than equal to M y j see the point is that we are maximizing a function.

So, when we area maximizing a function tell me you know what really happens is let us let us understand this carefully. Supposing x j is less than equal to M y j or M j y j or where M j is a large number. Let us say if x j equal to 1, let us say x j equal to 1 an integer number other than 0. So, if x j equal to 1 what will be the possible value of y j, think over. If y j equal to 0; see y j is a binary variable related to the rent, if y j equal to 0 then what will happen it will become 1 less than equal to M multiplied by 0. Is it possible? It is not possible, 0 cannot be greater than equal to 1.

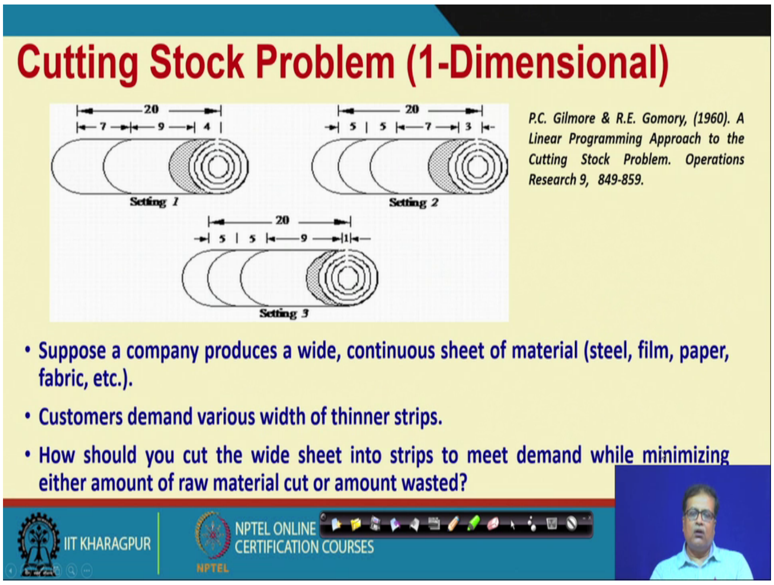
So, if x j is equal to 1 then y j will also be 1. So, you see this particular relationship that x j less than equal to M j y j will ensure that at any time if x j equal to 1 that means, that a particular activity is there you know it is produced that means, the clothing machine is used and we produce some clothing let say shirt or whatever then y j should be 1 that there will be a rent. The rent component will be definitely there.

But what happens if x j equal to 0 that means, the activity is not there, then you know from the type of equation, you can see y j will be 0 or 1. We do not know, because both will be valid you know, it will be y j can be 0 that is also going to satisfy y j equal to 1 that is also going to be satisfied. So, how do you take that particular thing out here? You see the interesting thing is that we have an objective function which is maximizing. So, this is a maximizing function.

Now, in this function tell me if y 1 or y y 1 is 0 what will be the functional value and if y 1 equal to 1, which will be the functional value. Can you see that objective function will be higher if y j equal to 0, you see if x j equal to 0 that this activity is not there then it is a maximization problem? Now, if y 1 equal to 0, then this term will be 0 that means, the maximization will be whatever quantity we get from other consideration, this term will not be there, but if y 1 equal to 1, then this 200 y 1 will be 200. So, the objective function value will be reduced by 200 which is not a very good idea, because the objective function goes down. So, objective function will be higher if y j equal to 0 and since it is a maximization problem that is what will be selected.

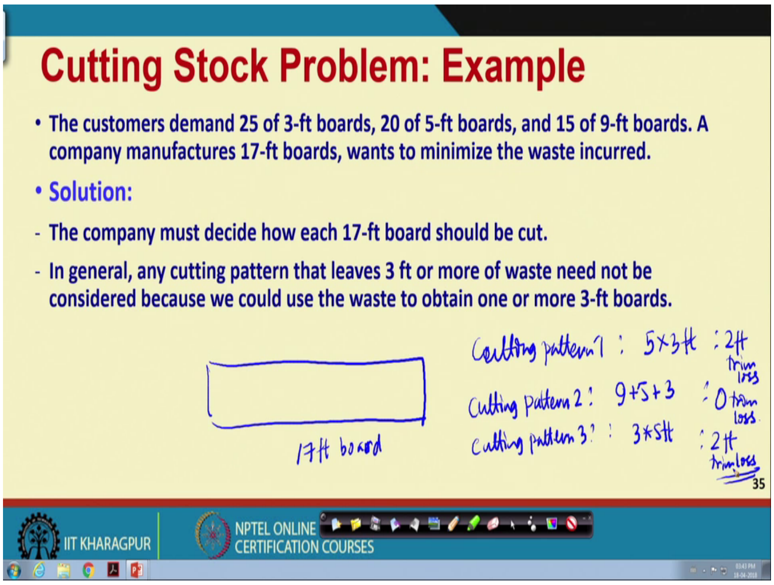
So, if x j equal to 0, then by the nature of the problem y j will be 0 also. So, we get what we desire that if x j equal to 0, then y j will be 0. And if x j equal to 1 then y j will be 1 and that is what we want right. We want the rent to vanish if there is no activity that is x j equal to 0, and we want rent to be there if the activity x j value is other than 0; that means, if activity is there rent will be there is it all right; and if activity is not there rent also will not be there. So, both are accomplished. One by the condition that we have taken and the other by the nature of the objective function is it all right. So, that is how we model what is known as the Fixed-Charge Problem.

(Refer Slide Time: 20:16)



So, let us see how do we model our next one; that is the cutting stock problem. So, you know the cutting stock problem which you know a picture has been depicted from Gilmore and Gomory's paper which is one of the very fundamental work on you know such cutting stock problem. But whatever you know we have understood that there is a continuous shift and customers demand various width. So, how should it be cut into strips to meet demand while minimizing either amount of raw material cut or amount wasted.

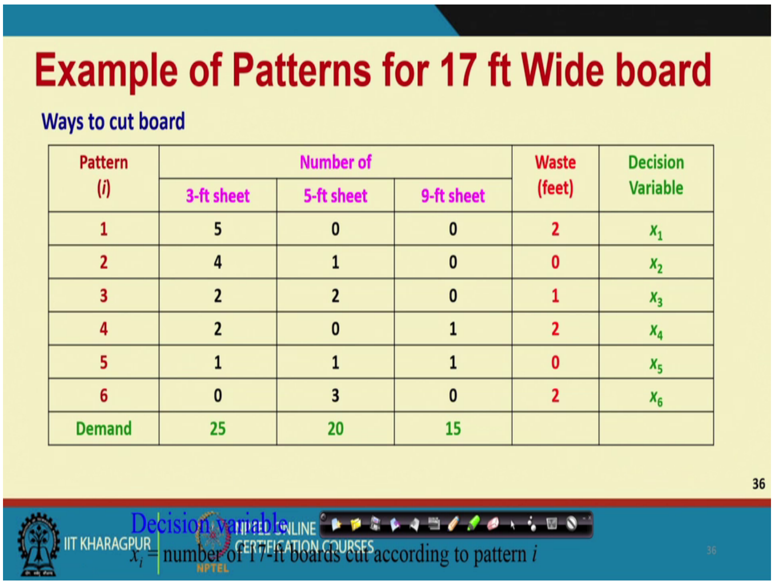
(Refer Slide Time: 21:00)



So, that is a example we had taken that customers demand 25 3-feet boards, 20 5-feet boards, 15 9-feet boards and company has got different 17-feet boards. So, idea is how do I minimize the waste. So, what I discussed in previous lecture that we can think of different cutting patterns right. Like what I said that from 17-feet board you know from a 17-feet board. So, if this is my 17-feet board a given cutting pattern. So, cutting pattern one is 5 into 3-feet right cut 5 3-feet, but then it leads to 2-feet loss. Then there was cutting pattern 2; that was you know what we said that 9 plus 5 plus 3. If you do that, then 0 trim loss, there was no trim loss.

Think of another cutting pattern cutting pattern 3. How about 3 5-feet right; again we have 2-feet trim loss. Like this you can think of different cutting patterns and when you cut in those different cutting patterns you know you can have different kind of patterns.

(Refer Slide Time: 23:00)

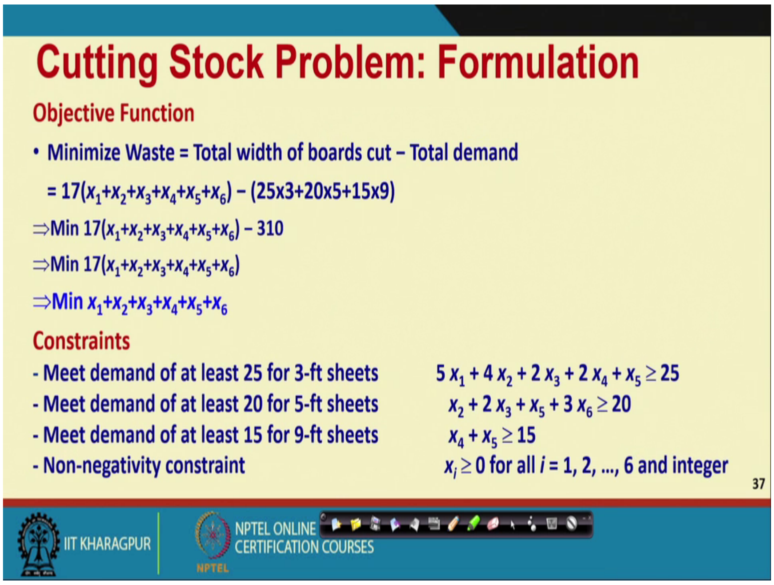


Pattern (i)	Number of 3-ft sheet	Number of 5-ft sheet	Number of 9-ft sheet	Waste (feet)	Decision Variable
1	5	0	0	2	x_1
2	4	1	0	0	x_2
3	2	2	0	1	x_3
4	2	0	1	2	x_4
5	1	1	1	0	x_5
6	0	3	0	2	x_6
Demand	25	20	15		

So, you know this is how they may look. So, here is some different you know at least 6 cutting patterns that has been thought about. So, supposing you can have 5 3-feet sheet card in that case the waste is 2, because 17-feet wide, 5 3-feet sheet. We will take 15-feet waste is 2. Let say how many of these cutting patterns we use, let say x 1. Another pattern could be 4 3-feet, 1 5-feet then there is no waste.

Then 2 3-feet, 2 5-feet that also is possible in that case waste is 1. Then 2 3-feet and 1 9-feet again waste is 2. 1 1 1 waste is 0; 0 3 0 waste is 2, is it alright. It is not given because it is very much loss making that is could be 1 9-feet, no, it is not possible then you do not waste 8-feet you know that is ok. So, these are the six patterns in which we have we can cut the 17-feet wide board into different you know cutting sheets as per customer demand.

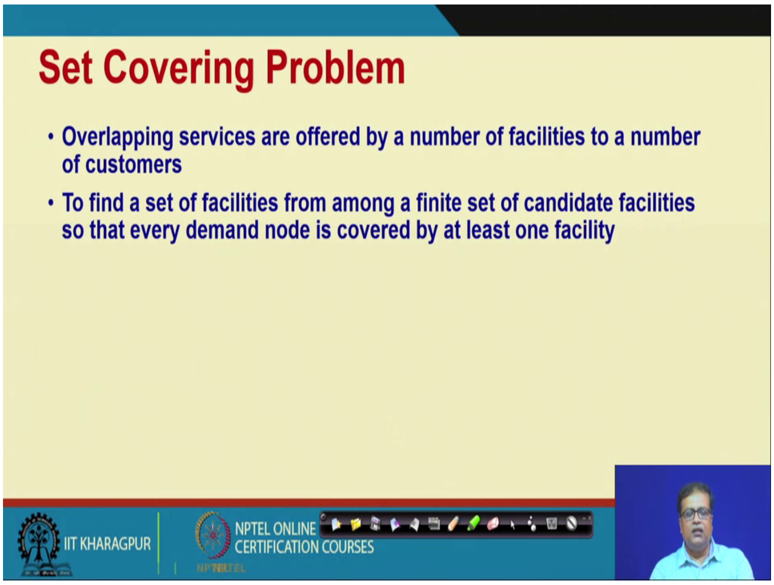
(Refer Slide Time: 24:25)



So, you can see what will be the objective function? The objective function will be minimized waste total width of boards cuts minus total demand, so 17 into x 1, x 2, x 3, x 4, x 5, x 6. So, you see total width of boards cut. So, x 1 number look here, x 1 number of pattern 1, x 2 number of pattern 2, x 3 number of pattern 3, so into each is 17-feet. And these many products are to be finally, obtained because we do not want any more customer demand only these much customer demand. So, total customer demand is 310, so that is the minimizing, but this is a constant amount. So, as an objective function, these much is sufficient; in other words this is sufficient right because that is only a multiplication.

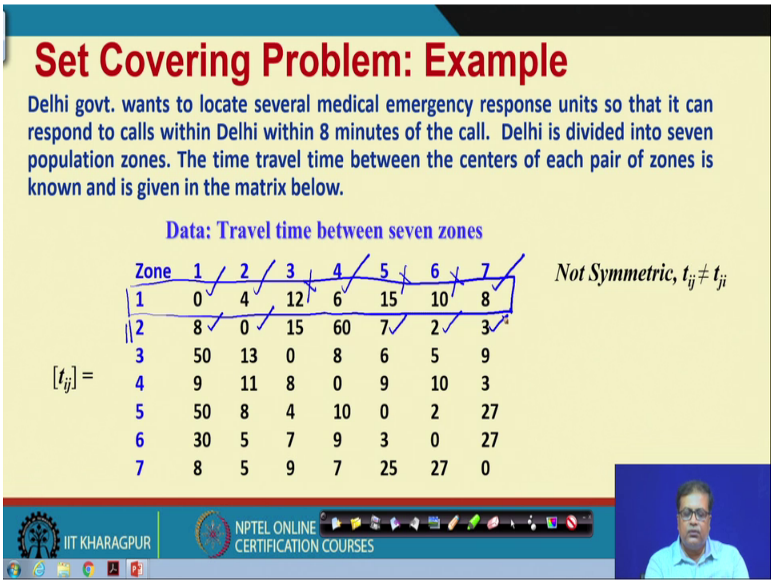
So, if I take these as my objective function that should be to make this formulation. Now, the idea is that look here in the first pattern, I get 5 3-feet, second 3, third 2, forth 2, fifth 1, sixth 0, so 5, 4, 2, 2, and 1 that should be 25 that much should be available, is it all right. The next x 2 plus 2 x 2 plus x 5 plus 3 x 6 that should be more than 20 that is for the 5-feet and x 4 and x 5 should be 15, because 9 is only on x 4 and x 5, so that should be there. And each x i should be greater than equal to 0 for all i as integer, so that is how you can actually formulate the cutting stock problem right. So, you understood how do I you know formulate these cutting stock problems. But the question is they are all integers.

(Refer Slide Time: 26:27)



Now, the set covering problem already I have discussed what is that set covering problem.

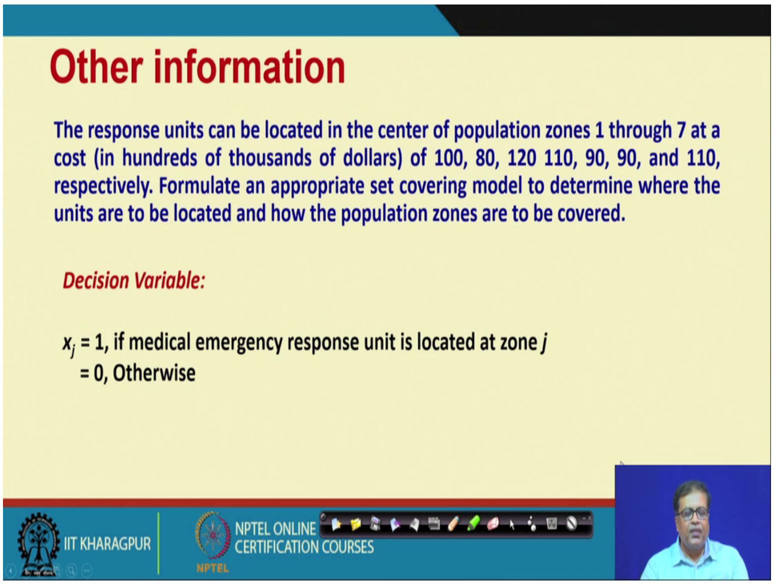
(Refer Slide Time: 26:29)



So, let us say government wish to locate several medical emergency response units so that it can respond to calls within the city with 8 minutes of the call. Now, the city is divided into 7 population zones. The time travel between the centre of each pair of zones is known and is given in the matrix below. Look here it is not symmetry. So, you see if I locate my team at zone 1, then what will happen. Look at zone 1, this is our zone 1. Can you tell from zone 1, if I look you know if I put my medical emergency team facility, then it can serve to which locations; it can serve to location 1, location 2, location 4 and location 7.

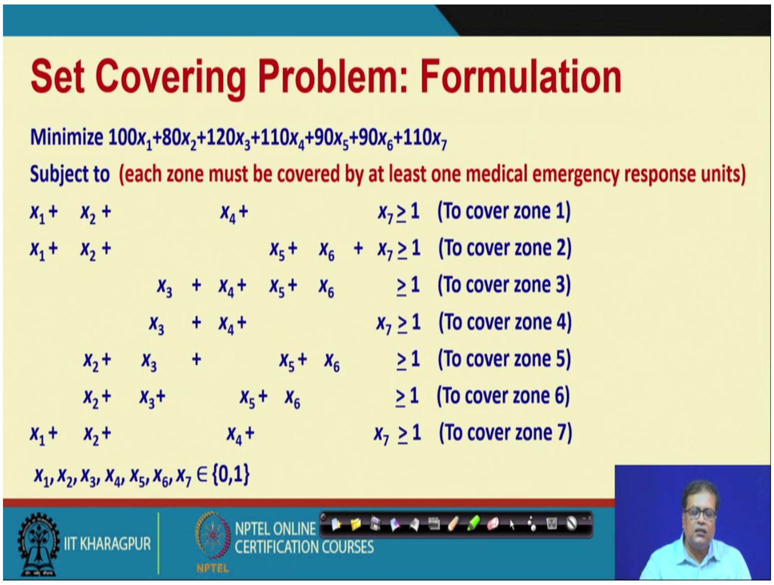
It cannot serve location 3, location 5, and location 6, why, because those distance are higher than 8 all right. Can you now tell if I locate in location 2, what are these you know locations or in zones where service could be given. It could be in 1, in 2, in 5, in 6 and 7 right, so that is how we can start looking into it. The essential idea is where to locate this emergency response units so that all the zones could be covered within 8 minutes that is the idea. So, it is a kind of facility location problem, but it should cover the entire set that is the essential idea right.

(Refer Slide Time: 28:35)



So, if that is so now let us look at what really happens further that x j equal to 1 in medical emergency response unit is located at zone j and 0 otherwise.

(Refer Slide Time: 00:48)



So, look here that is how the set covering problem can be formulated, the minimize 100 x 1 plus 80 x 2, now what are these 100, 80 and all that, you see the 100, 80 and all that is the response units can be located with a cost of 100, 80, 120 etcetera. So, these are the cost. So, you know we have to minimize the total cost. So, if I locate x 1 number you know in the first so that will be the total cost will be this that should be minimized. And our idea is x 1 plus x 2 plus you know x 4 plus x 7 should be greater than equal to 1, is it ok. And x 1 plus x 2 plus x 5 plus x 6 plus x 7 should be greater than equal to 1 to cover zone 2, is it all right.

Similarly, you know in all of these now how why it should be x 1, x 2, and x 4, and x 7 look here what we found in the first zone it can cover 1, 2, 4 and 7. So, you know that is why 1, 2, 4 and 7, is it all right, so that means, that if each zone must be covered at least by one medical emergency responsive unit. So, we have x 1, x 2, x 4 plus x 7 that should be with one to cover zone 1. And similarly for all of them when we put and you know we get the solution will give that what should be our overall coverage. And obviously, each one is 0 or 1 binary because that is how we go ahead with it.

So, you know with this what we do that you know we have seen how we model different kinds of at least four types of different problems, how do you formulate right. In our next lecture, we will continue this formulations and we shall see how do we formulate you know another kind of problems like either or constant or k out of p problems, the 0 1 problems the binary problems or mixed integer problems, and you know go ahead from there to how to solve integer programming problem right.

Thank you very much for this lecture.