Industrial Safety Engineering
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Lecture – 30
Quantification of Basic Events – Weibull Distribution

Hello. Today we will discuss Weibull distribution.

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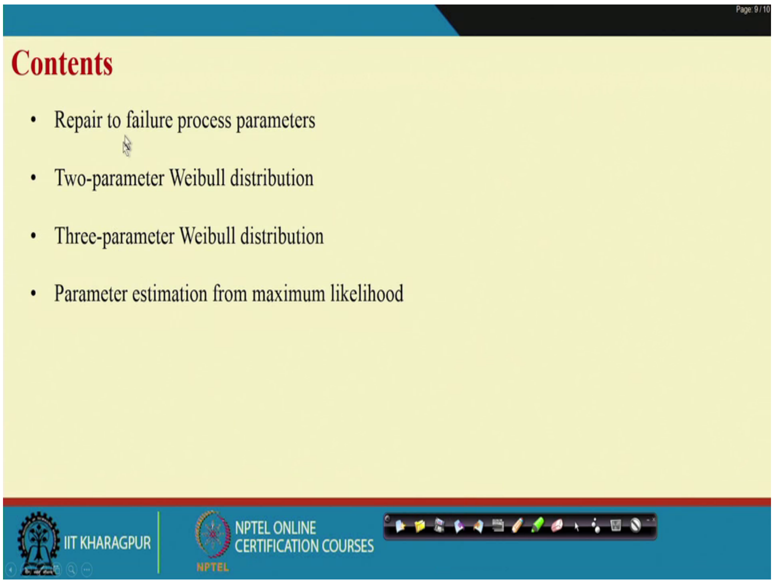

And the contents of today's presentation is, we start with the repair to first failure process parameters, then we discuss two-parameters Weibull distribution, followed by three-parameter Weibull distribution and then parameter estimation by maximum likelihood method, as well as graphical method.

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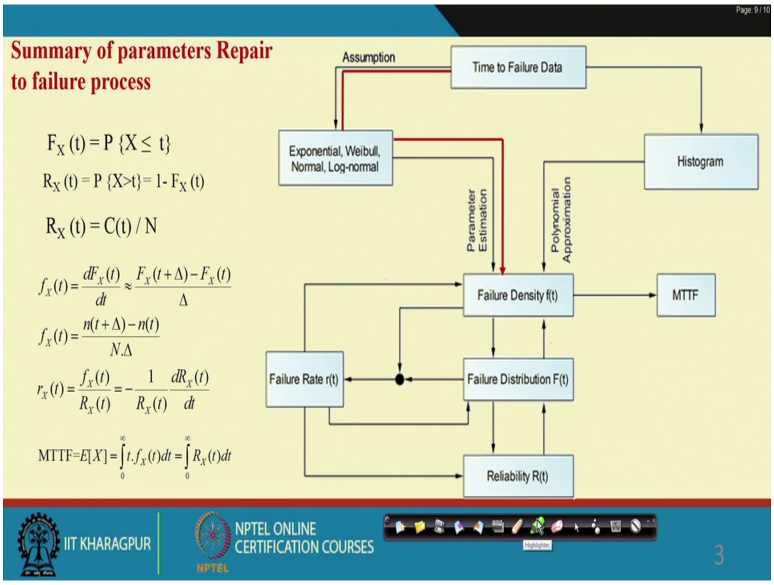

You all know this. You have seen this one several times, that I do not want to discuss it further, it is known to you, only I want to tell you that if you have time to failure data, then if you have sufficient information and it can be assumed that Weibull distribution may be a fit to the data and that is what is today's presentation.

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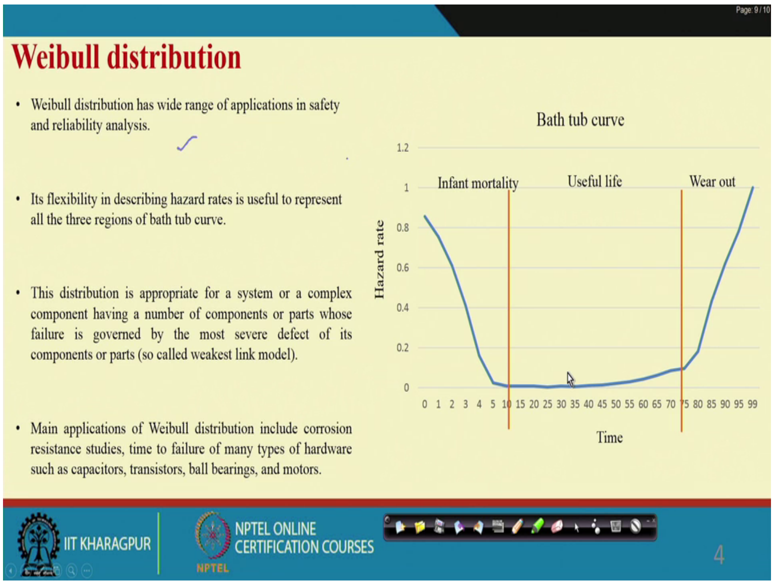

So, let us see the some of the basics of Weibull distribution, it is a interesting distribution and it is a continuous distribution. And it has good relation with hazard rate. Because it can describe the hazard rate with it is beta parameter. So, you see that we discussed earlier what is hazard rate. And then we have plotted the hazard rate in terms of bathtub curve. And in the bathtub curve there are 3 regions, infant mortality useful life and wear out phase, or burn in useful life and burnout phases.

Now, Weibull distribution can feed this infant mortality or burn in phase it can also model the wear out or burn out phase, it can also model the useful life or constant rate hazard rate phase. So, when it becomes constant, it explained the constant is nothing but it is basically a exponential distribution that mean exponential is a special case of Weibull distribution. Now, that is means it is a very, very much a flexible distribution. This distribution is appropriate for a system of complex component having number of components or parts, whose failure is governed by most severe defect of it is components or parts.

So, the if you see the application of Weibull distribution, then you will find out that it is extensively used in reliability and safety studies. And many failure analysis like corrosion resistance studies, and a or in many of the electronics equipment like hardware transistors and also mechanical equipment like ball bearing motors everywhere this is used and the Weibull of interest can be time to failure.

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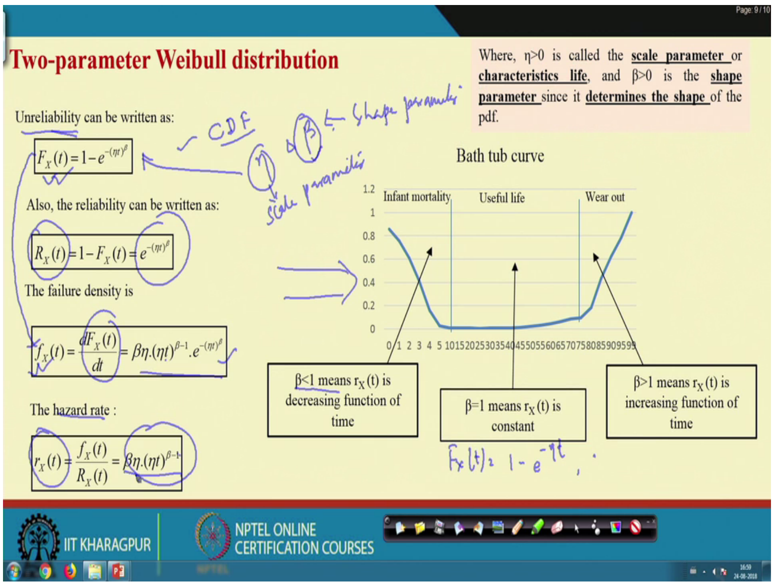

Now, let us see the different parameters of Weibull distribution. The first one is unreliability, and you all know: what is unreliability that is basically the cumulative distribution function CDF. So, CDF is 1 minus e to the power minus eta t to the power beta. So, if I look into the CDF so, you are getting 2 parameters one is eta another one is beta. Eta is known for known as scale parameter, and beta is known as shape parameter. Eta a is also known as characteristic life. And beta is a parameter it basically determined the shape of the Weibull distribution.

And if you your component or a system failure follows Weibull distribution, and then for the exam for the time being if you if you consider that we have n identical component well which follows the same Weibull distribution means the eta and beta parameter are same for the all the component failure distribution. So, then the fundamental approach what we have adopted at the beginning; that means, n identical components under the test and if you find out the find out the number of item survives after certain time intervals. So, ultimately from that data, from that experimental data you can very easily develop this distribution. Later we will see that how eta and beta can be estimated using graphical method.

Now, if F x t is this obviously, R x t will be 1 minus F x t which is nothing but e to the power minus eta t to the power beta. So, then if you take the derivative of this the del F x t by del t this you will be getting this. You will be getting that for one it is 0 now for this one it will be beta eta eta t to the power beta minus 1 and e to the power minus eta t into t to the power beta. So, if you take derivatives of this here you are seeing you will get this. So, these are standard calculus so, I am not going into the how after derivation F x t is like this, it is easy and you can you will be able to do it.

So, similarly we have seen earlier that the hazard rate or instantaneous failure rate is that R x t which is nothing but the density function divided by it is survival function. So, F x t by R x t, then this f x t by R x t this will give you this equation. So, it is obvious, because here it is e to the power minus eta t to the power beta, that will be cancelled out, then beta eta eta t to the power beta minus 1 this will be the hazard rate function or hazard rate or hazard function. And as I told you, that it has very good relation with your bathtub curve. So now, see suppose you have data and you have computed eta and beta, and after that you found that your beta value is less than 1. It means this region, infant mortality region; that means, that hazard rate is decreasing function of time.

Now, if it is constant, beta is constant, then this is basically talks about that the useful life beta is constant. And when beta greater than 1, it is basically wear not phase or burnout phase. So, that mean if you know the beta value, you know, the component is operating under which phase whether it is in the infant mortality phase or in the useful life or the worn out phase. So, if you put your beta equal to 1, then what will happen to F x t? F x t will be 1 minus e to the power minus eta t, because beta equal to 1. So, what will happen to then you really this one reliability? E to the power minus eta at lambda t. Sorry, eta t and similarly F x t beta equal to 1. So, it will be also similar and this equal to even this equal to 0; so, then eta e to the eta e to the power minus eta t; which is nothing but the expression for exponential distribution.

So, that mean when beta equal to 1, all those that distributions starting from unreliability to hazard rate distribution, all will be all will be related will be related to exponential distribution. What will happen here if I put beta equal to 1, then eta t to the power beta minus 1, this will beta equal to 1 this will be 0. So, it is a beta and eta which are the 2 constant of that Weibull distribution through hazard rate is constant. So, it is it gives good insight about the health of the component which follows; however, Weibull distribution.

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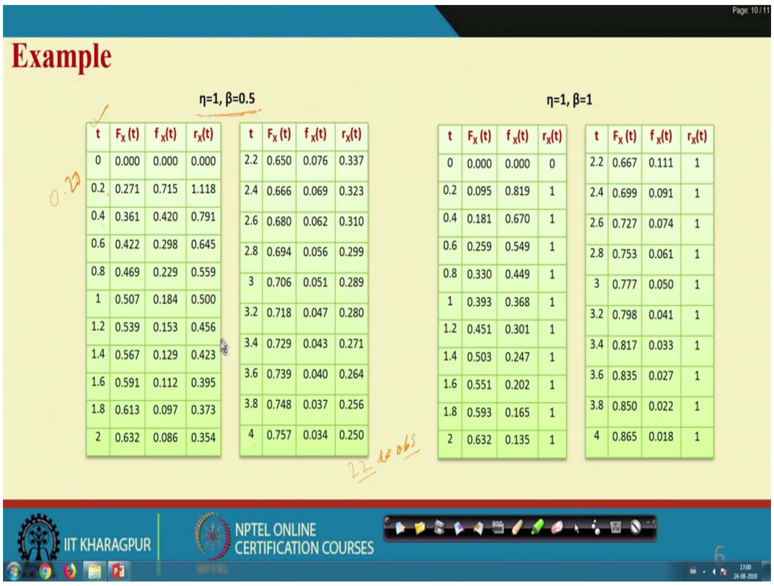

Now, we have create it is hypothetical examples what we have done here, we have first taken t values; t 0 to we increased it by 0.2 only, point to 0; so, 0 0.2, 0.4, 0.6 like this up to 4. So, as a result we have created 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, plus 11, 22, 22 data points, 22 observations or other way I can say that failures. So, then we assume that this data follows Weibull distribution. And also we assume that this coming from the Weibull distribution with eta equal to 1 and beta equal to 0.50. Please remember that, this data may not be coming from Weibull distribution. At the and even if it is coming from Weibull distribution may not be the parameters to parameters eta equal to 1 or beta equal to 0.5.

Let us assume that we have Weibull distribution, and then which basically fall which who whose parameters are like this. Then we have created some t values, and we have fitted to that distribution. For example, here like this is this is your Weibull distribution, 1 minus e to the power eta to the power beta. So, we assume that eta equal to one and beta equal to 0.5, then we created the distribution. And then we assume t values from 0.2 like this, some increment values we have taken this. So, let us consider in this way; that means, we have assumed a Weibull distribution with eta equal to 1 and beta equal to 0.5.

And in that case if we change the t values with increment point with increment sorry, yes with increment 0.2, 0.2 what will happen to F x t which is cumulative distribution function, density function and hazard rate function, that is what is plotted here. Then what we have done again? We have basically that we have consider we have not changed eta rather we have changed beta here. That beta is point beta is one instead of 0.5, and that in that case rest of the things remain same, and how the cumulative distribution function on reliability function density function and hazard rate changing that also we have computed.

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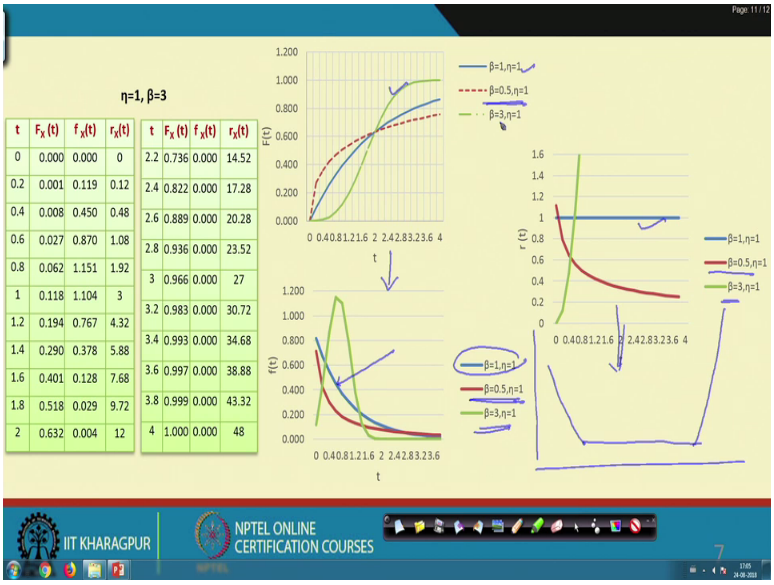

And in the same manner, we have computed it, we have again changed the beta value to 3, eta value we it kept it 1, and again for the same t values that we found out the 3 functions. And then we have plotted all those things here. So, you just see: what is the plot here, the middle one which is basically the blue one is the first one; where beta equal to 1 and eta equal to 1. When beta equal to 1, it is basically exponential distribution. And it is if you see the f t is look like exponential distribution. And then the top one is beta 3 eta 1, you see the see that basically the cumulative probability distribution or unreliability, how quickly it is increasing here, change is increasing. And in the other 1.5 and eta equal to 1 this 1.

When we see the PDF, that mean density function for all those 3 cases, you see that density function, this is the first density function for this, beta eta 1, and the second density function when beta equal to 0.5 like this, and third one is when beta equal to 3, like this. And if you see the hazard rate function for this all the 3 cases. So, beta one eta one it is constant, the reason is, it is basically exponential distribution. And this one, you see it is the decreasing failure rate; which is the infant mortality part of the bathtub curve. And here beta equal to 3 which is increasing failure rate.

So now this one if I manipulate, if suppose I write this portion fast, then this fast, then if I write the second one, then second one, and then third one, third one. So, it is basically bath tub curve line. So, that means, depending on the beta values, the different phases of component failures can be modeled. So, in order to show this, we have done this hypothetical case study, hypothetical case. Ok, I hope you understand this, now what will happen if we change the eta values. So, eta values if you change you, you will find out the change in the characteristics life. What is characteristic life? Basically two third of the observations or the failure within that particular unit of time: eta unit of time.

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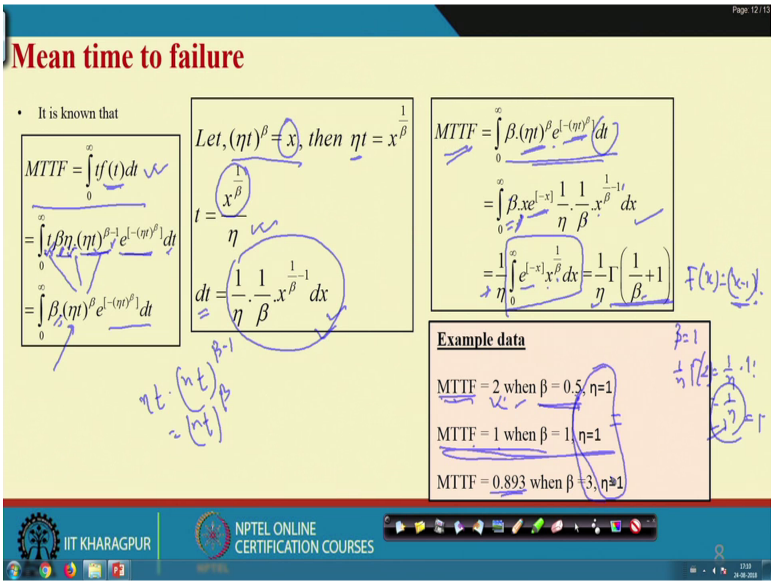

Now, we will calculate the mean time to failure. So, in order to compute the mean time to failure, you know what is that what is the equation? Equation is TFT dt integration 0 to infinite, that is mean tend to failure. Straight curve you put t as it is, f t value you know, beta eta eta t to the power beta minus 1 and e to the power minus eta t to the power beta and dt.

So, then this one, if I write the t and eta, eta t here then this into eta t into e to the power beta minus 1, then this is nothing but eta t to the power beta, what I mean to say this t this eta and eta t be minus 1; that mean, eta t into eta t to the power beta minus 1 which is eta t to the power beta. That is what we have written.

So, they mean this one this one and these 3 combining getting this one. Remaining one is beta beta is here. Now this portion remain as it is. So, in order to integrate, you just put eta t to the power beta equal to x. Then eta t will be x to the power 1 by beta and t will be x to the power 1 by beta by eta; if you take the derivative both derivative with reference to x, then dt will be 1 by eta 1 by beta for these into x to the power 1 by beta minus 1 into dx. So, if t stand if t is 0 x will be 0, if t is infinite x will also be infinite. So now, we are putting this conversion in this equation. So, beta as it is eta t to the power beta is nothing but x. So, you put here x, e to the power minus eta t to the power beta is nothing but e to the power minus x put here.

Now, dt you found out the dt is this value. So, put dt 1 by eta 1 by beta x to the power 1 by beta minus 1 into d x; so, then 1 by eta 0 to infinite, now what happened e to the power minus x. Here 1 x is there x to the power 1 by beta minus 1 is there. So, it is x to the power 1 by beta. And this beta this beta cancelled out, 1 by eta is out of the integral. And then this into dx this particular quantity 0 to infinity to the power minus x, x to the power one by beta; that dx this will basically is is the gamma function.

So, gamma 1 by beta plus 1; so, there in the meantime to failure in a in case of Weibull distribution with parameter eta and beta. So, it is basically 1 by eta in gamma 1 by beta plus 1. So, gamma function, were suppose gamma a gamma x equal to factorial x minus 1. So if you know beta and eta, then you know what will be the value of mean time to failure. For example, you have seen that beta beta equal to 1. So, then what will be the value here; 1 by eta gamma 2 which is 1 by eta into x minus 1 factorial 1 so, 1 by eta.

Now, in that case if we find a if which we choose value of eta equal to 1, then mean time to failure will be one which is given here. So, that mean depending on the value of eta and beta the mean time mean time to failure will change. Mean time to failure is 2 when it is decreasing failure rates like beta equal to 0.5. When it is increasing failure rate it is basically with; obviously, it is eta is the eta is important parameter, but if you consider eta equal to 1 what is the change in behavior in meantime you are finding out?

So, eta is 1 then MTTF will be eta which is 1 when beta is constant. Beta equal to 1 which is exponential and it will be when decreasing failure rate beta equal to 2 MTTF equal to 2 when it is increasing failure rate MTTF is less than equal to 1, keeping that eta equal to 1 characteristics life is 1.

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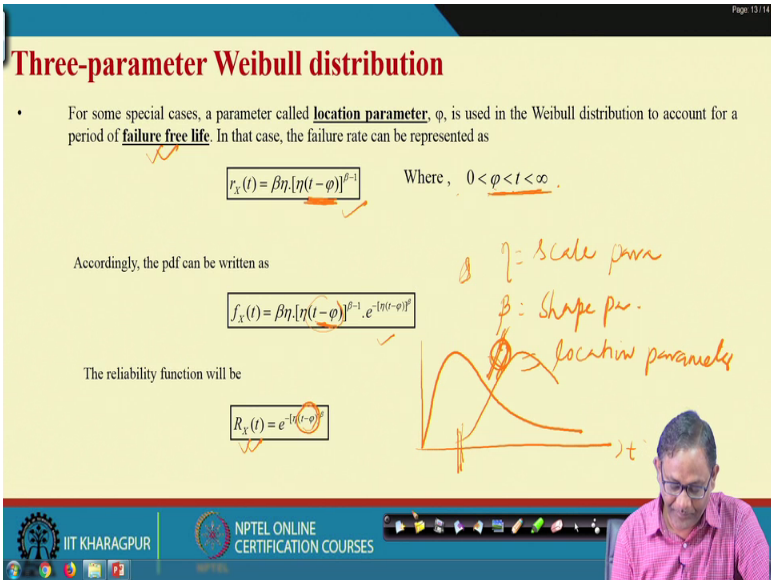

Ok, now we will see the 3 parameter distribution in 3 parameter distribution. Apart from the that is scale parameter. So, what we have seen? Eta equal to scale parameter, characteristics life, then your beta equal to shape parameter, shape parameter. And then we will use another parameter called gamma; which is location parameter. This what is the location parameter? You have seen the shape and scale parameter. Location parameter it basically talks about failure free life.

That mean it will not the t will not start from 0, here you just see the t is that phi is the location parameter, t value that phi to infinite not from 0. It is something like this, when your phi equal to 0 you may think that it is Weibull distribution, but if phi equal to phi it may start with from this. So, this is what is your what is the location parameter. If this location parameter is used: what actually happening the t minus phi that location parameter. Here, here I have written gamma, later you write phi here. So, then R x t is this, and PDF will be this, you just see that with reference to the 2 parameter whatever you got. The changes is only instead of t you are writing t minus phi.

So, in 2 parameter model what we have written about this reliability? E to the power minus eta t to the power beta; where in these 3 parameters location parameter is coming, we are writing eta t minus phi. So, t is replaced by t minus phi; so, similarly density case also replaced by this, replaced by this. So, most of the time we see that this phi phi we consider as 0, and we model it, but there are situations where you may find that failure field life will be consider, and then the phi phi value appropriate phi value will be chosen. Otherwise, once data is given maybe you can use the appropriate parameter estimation methodology, and find out whether it is a 3 parameter model or a 2 parameter model.

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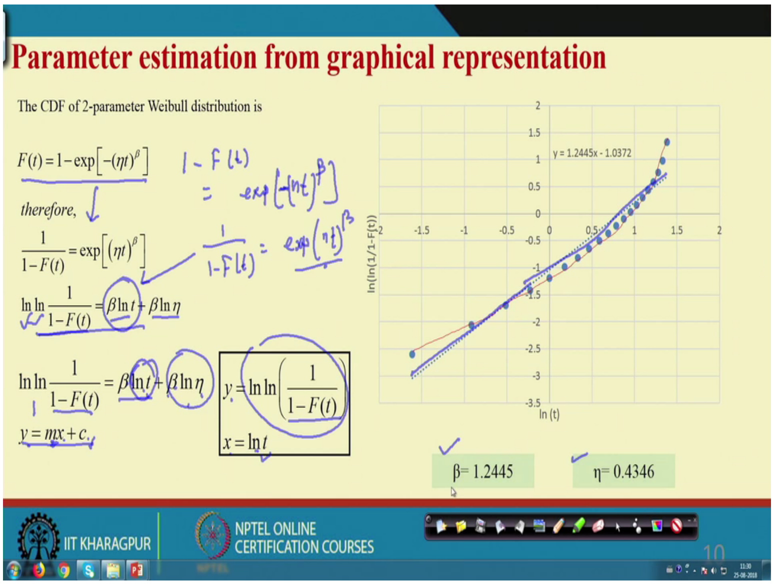

Now, let us see that how parameter can be estimated using graphical methods. So, we will consider the 2 parameter Weibull distribution. And you know the distribution function for 2 parameter Weibull distribution we have already seen this one. Now let us see that how it can be represented in linear form. So, what you do? You first take manipulate this equation in this form.

So, f t equal to 1 minus e to the power minus eta t to the power beta if I do something like this, I can write 1 minus f t equal to e to the power or exp to the power minus eta to the power beta, that can be written. So, this minus is there I want to make it plus. So, I can write 1 by 1 minus f t equal to this is e to the power eta t to the power beta. So, if you now take log, if you take first log of this, then you will get eta t to the power beta. If you take another log, then you will get beta eta t l log t and beta log eta.

So, now then this equation that log log 1 by 1 minus f t equal to beta log t and beta log eta. So, t is the variable here, but beta and eta are the parameters. So, it looks like y equal to m x plus c; where y is the left hand side log log 1 by 1 minus capital f t, and m is basically here your m x, x will be your log t m x. So, x will be log t m is equivalent m equal to beta, c equal to beta log beta.

So, essentially you got a linear equation; which is a straight line y equal to m x plus c. And now if you if when you have the data. So, when if you plot that time to failure data, then you and plot between this y and x; where y is log log 1 by 1 minus f t and x is log t, then if the data come from the 2 parameter Weibull distribution, then you will get this kind of straight line. So, the data we have given that 0.024 that the increment 0.2 we have generated earlier. So, the same data we have used, and we found out that the that the log log of this curve these points are like this. So, 2 sense if I join all the points it is something like this, something like this.

So, it is a little deviated from the linear or the straight line, but a most approximate straight line is this one. That the best straight line is this one; which is the dotted line here so, from this straight line, you are able to get the slope of the straight line which is our aim. And also intercept of the straight line which is our c, now if we see the equation m is equivalent to beta, and c is beta log eta once you know the beta value and c value, then you will be able to find out the eta value by algebraic manipulation.

So, in this manner we have developed, and then we found out that the beta value is 1.2445 and eta value is 0.4346. So, it is it is basically beta is more than 1, and eta is less than 1, whatever may be, but this is the this is the way of graphically a estimating the parameters, for 2 parameter Weibull distribution. So, when it will be 3 parameter distribution, then the things will become complicated. And at the same time, the graphical method you see that the approximation is there. So, people will be interested to know what is the other method. So, that like the maximum likelihood method. We will give you the glimpse of maximum likelihood method how it will be applied in this case, and then we will finish to today's lecture.

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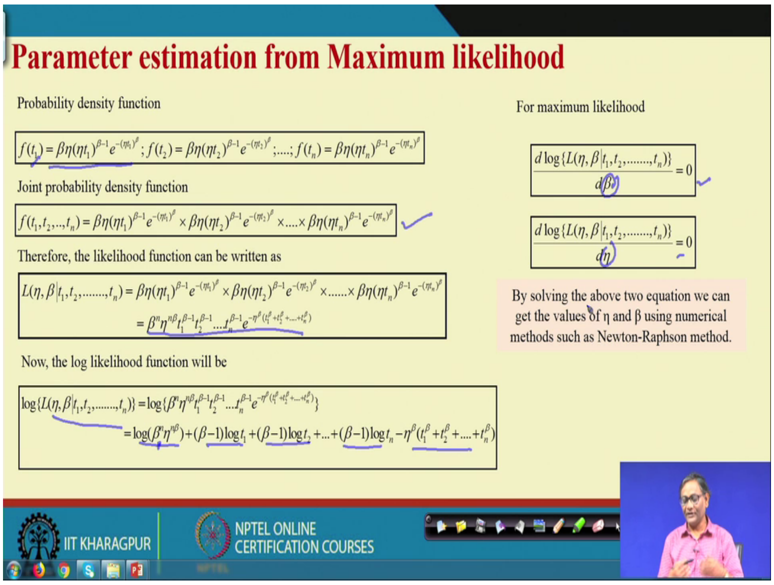

All of you know that whatever observation you collect; they are independent. And as the observations are coming from Weibull distribution so, each of the observation will follow this Weibull distribution with the parameters of the distribution. So, that means, every time values time to failure values, this values can be thought of coming from the Weibull distribution. And as a result the failure the density function will be, will be same with the means the parameter will be same, but only t will be replaced by t 1, t will be replaced by t 2 like t n when you are collecting in or you are planning to collect in data points entity a values.

So, then their joint probability distribution will be will be the multiplication of the individual probability distribution, because every variables every time bit to failure values are independent to other. And as a result their joint probability distribution will be the multiplication of their individual distribution; so, this is the second one. So, that n number of pdf port density function is multiplied here. Then you take the likelihood that therefore, the likelihood will be this what is this all those multiplication, if you if you just do algebraic manipulation you will be getting this.

Then you take the log of these, because it is a multiplicative 1 if you take log then you are getting some kind of additive equation. And then now you have log n log likelihood, the log of likelihood, then you take the derivative of that log likelihood, put it to 0 with reference to beta first parameter the same parameter.

Similarly, with reference to eta the scale parameter. But what happened? You will find out that it is not a completely additive model here times are additive and multiple some additive terms are terms are additive, and within one term there are multiplication.

So, what is happening here? That is why even if you take the derivative of these. So, you will not get a closed form solution, that means, some kind of few linear equations which will be simultaneously solved, but it will it will be difficult to solve in this manner. You will get some equations, but it will not this thick card (Refer Time: 32:21). So, you will get 2 equations from here and here, 2 equations you will get, and those equations cannot be directly solved. So, you require numerical methods to solve it. For example, Newton Raphson method will be used to find out the parameters eta and beta.

So, if you consider 3 parameter model; similarly, you have to find out the likelihood and then take the log likelihood. And take the derivative there will be 3 equations, one with respect to shape parameter, another with respect to scale parameter another with respect to location parameter. And again you use numerical methods, like Newton Raphson method and then finally, get the get the most likelihood values for the parameters. And I hope that you have understood the concept here.

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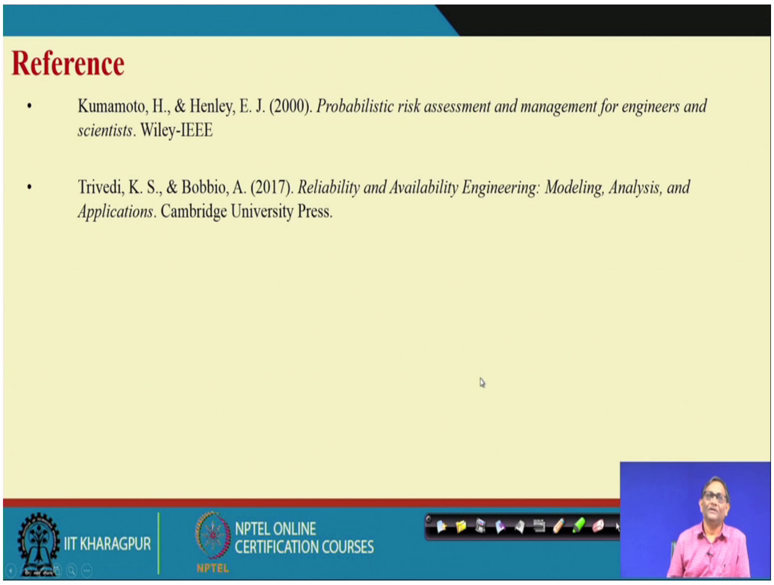

I already told you that we have we are basically following primarily the first book followed by the second book, when in developing the lecture material.

Thank you very much.