

Multivariate Procedures with R

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Lecture – 08

Matrix Operations

Hello friends, welcome to the course Multivariate Procedures with R. So, you can recall that in the last lecture we were talking about the matrix operations. We introduced the concept that how are you going to write the matrix elements in different ways in the R software. And we had discussed some basic elementary operations. Now, in this lecture we are going to continue further and we will try to understand that how to do the basic operations like addition, subtraction, multiplication etcetera in matrices inside the R software. So, once again I would say that we are assuming that we all know the matrix theory or at least the basic fundamentals of matrix theory and the rules of the mathematical operations in matrix theory.

What we need to say is that suppose if you want to add two matrices, then we know from the matrices theory that the rule is that both the matrices should have the same order. Similarly, if you want to subtract the two matrices, then both the matrices should have the same order. And on the other hand, when we try to do the matrix multiplication, then the number of columns in the first matrix and the number of rows in the second matrix, they have to be the same and so on. And similarly, when we try to find out the inverse of the matrices, there are certain rules, certain conditions, they have to be satisfied when we try to obtain the matrix operation.

For example, the first condition to find out the unique inverse of matrix is that the matrix has to be square matrix, this has to be a positive definite matrix. Well, I am not saying that if these conditions are not going to be true, then we cannot find out inverse of the matrix. We can still find out the inverse of the matrix. For example, if you have a rectangular matrix, then definitely in this case we cannot find out the unique inverse, but we can find out the generalized inverse. Those properties are beyond the objective of the course.

So, I will not consider them, but I will try to use the standard matrices theory or standard procedures, standard rules which are implemented in the matrix theory. So, let us begin our lecture and try to understand these basic operations. Okay. So, now first I try to take a very simple operation that if you want to multiply a matrix with a constant. So, you will see that I will be using the same matrices which I used in the last lecture.

So I try to create here a matrix which is of order 4 by 2, 4 rows, 2 columns and the data here is 1 to 8 and the elements are arranged by rows. So, this is your here this matrix by rows means 1 to 2 and then 3 to 4 and then 5 to 6 and so on.

So now suppose you want to multiply this matrix by a scalar, by a constant. So, in order to do it the operator here is simple star. This is the mathematical sign. You have to notice I am talking of the multiplication. So, there are two types of multiplications. If I have a matrix that has to be multiplied with a constant and the second option is a matrix is multiplied by a matrix. So, at this moment I am talking about the matrix multiplication where one element is a matrix and another element is a scalar or a constant. So, if I try to multiply this x matrix by 4 which is a scalar then it is all the elements in the matrices and in the matrix, they are going to be multiplied by 4.

So, you can see here this 1 into 4 this is here 4, 2 into 4 this is here 8 and so on. So, each of the element in the matrix you can see here this matrix this will become here 4, this will become here 8, this will become here 12, this will become 16 and so on. So, this is how we try to obtain results for multiplying a matrix by a constant. So, that is straightforward right, but you have to observe that what it is trying to do. And if you want to multiply a matrix by another matrix then the operator here is like this `%*%`.

And you know that if there are two matrices A and B which have to be multiplied then the condition here is that if A has got the order $m \times n$ that is m rows and n columns then B should have n number of rows and any value of the column A and B. So, these two numbers they have to match and the order of the output will be $m \times p$, right. So, this is what you have to keep in mind right. So, suppose if I try to take here the same matrix if I have taken here which is like this which is a 4 by 2 matrix x here is 4 by 2 and it is another matrix which is of order 2 by 4 right. So, one simple option just for the sake of illustration is that I can take its transpose.

So, transpose in the transpose the number of rows and number of columns are interchanged and in case if I try to obtain here $t(x)$ into x here, that is `%*%`, then $t(x)$ is going to be here if the order of this matrix and the order of this matrix will satisfy the matrix multiplication condition, right. So, now if you try to see here that I have taken here a matrix of order 4 by 2 and now when I am trying to take its transpose then the transpose will become of order 2 by 4 and x will become here 4 by 2. So, the output will become here 2 by 2. So, I have taken this example just to make sure that the number of

rows and number columns they are matching otherwise you can take any two matrices and can multiply them without any problem. My basic objective is to introduce here the matrix multiplication operator.

So, if you try to see here $t(x) \%*\% x$, this will give you here this matrix, right, and you know that how these matrices are multiplied, right. Suppose if I see here that I have matrix here 1, 2, 3, 4 and another matrix here there is 5, 6, 7, 8 then the multiplication is like this 1 into 5 plus 2 into 7 will be the first element this means this into this and then 1 into 6 plus 2 into 8 will be the second element 3 into 5 plus 4 into 7 will be the first element in the second row and 3 into 6 plus 4 into 8 this will be the element on the second row and second column. So, that is the matrix multiplication this is how it is done. It is different than the scalar multiplication, but anyway I assume that you are aware of such rules. Now, okay I would like to introduce here one more concept and that is why I have taken this particular example where I am trying to multiply here transpose of x , i.e., $t(x)$ and x , right.

In R there are certain built-in function and similarly if you want to find out such a product like x transpose x if x is a matrix that means you are trying to find out its transpose and then the matrix. Then there is a built-in function here `crossprod()` and inside the parentheses you write here x . So, this will give you here this outcome. So, this is actually same that you try to see here this number 84, 100, 100, 120 this is the same here 84, 100, 100, 120 and this is the same operation. So, the same operation which you have done here $t(x) \%*\% x$ this is the obtained directly by this built in function `crossprod()`.

Well, my objective if you inform you is that when these built in functions are written then they are written by those people who are good in programming and because in programming there is a concept that the same job can be done in different ways, but the procedure which can done it fast in a that is always preferable. So, to this function `crossprod()`, although it is trying to multiply the transpose of x and x together, but it execute the multiplication faster than the conventional method like this one. So, that you have to see depending on the case that whether such operations are possible or not. I am not saying that all the mathematical operations can be done through such built-in function, but anyway that you have to see. So, before we go to the topic let me try to show you these things on the R console.

So, that you get confident and we and then we move further. So, if you try to see here, I try to create here some matrix here x and now if I try to find out here if I multiply x with a scalar say 5, say $5 * x$ you can see here this is here coming out to be like this. And similarly, as we did in the y contour 4 into x this is coming out to be here like this. And similarly, if I try to define here transpose of x . So, you can see here this is x and this is

here transpose of x and if you try to multiply here transpose of x %*% x you will get here this matrix.

Whereas if you try to multiply here x with a transpose of x that means x transpose then what do you expect? What should be the order? Don't you think that the order should be 4 by 4? So, this will come here. So, similarly you can see here the same operation. Suppose if I take the cross broad say x. So, you can see here cross broad x is giving you the same outcome which was given here. So, this will make you more confident that whatever I had given in the slides this is also working in the R software.

So, now let us move forward and try to understand some more basic operations. So, now let me try to take here addition and subtraction. So, you know that if you are trying to add or subtract any two matrices then the first condition is that both the matrices A and B should be of the same order. So, if A has a order m by n then B should also be have the same order m by n right. So, let me try to give you this illustration that I try to consider here the same matrix here x which is here like this.

Now I need another matrix here which is also of the same order. So, one simple option is that I can multiply this matrix by 4. So, this will again remain as the same order 4 by 2 and then I try to add and subtract them. So, if you try to see here this if I try to use here addition and subtraction then the usual plus and minus signs, they are used in the matrix addition and matrix subtraction also. So, if I say here x plus 4 into x this will give us here like this and if I try to subtract $4x - x$ that is 4 into $x - x$ then this outcome will be here like this and how this operation is going to be.

So, if you try to see here your 4 into x in here like this and x here is say here 1, 2, 3, 4, 5, 6, 7, 8. So, when we are trying to add or subtract then it is becoming element wise. So, when I try to say here $4 * x + x$. So, this will be you are here the elements at the same addresses in the two matrices will be added like this 4 will be added with this 1. So, 1 plus 4 then the 8 and 2 will be added 8 plus 2 then simply here 12 plus 3 then 16 plus 4, 20 plus 5, 24 plus 6, 28 plus 7 and 32 plus 8.

And similarly if I try to do the subtraction. So, if I try to make it here like this then all this plus sign they will be replaced by the minus sign, right. So, this is how the element wise operation for addition and subtraction works here and you can verify this in this way also. You can see here it is not a very difficult thing actually, right. And now before I go forward let me try to show you these operations on the R software also.

So, if I see here x we already had defined. So, this is here x now I try to define here say 4 into x plus here x, we continue this is here becoming like this and you can see here $4x - x$ is here like this and x was here like this. So, you can now verify from this screen that whether this is working or not, right. So, if you see here 4 into 1 plus 1 is 5 and so on, right.

So, these things are working. Now, on the other hand means if you want to means do something else for example, if you will hear x let me try to clear the screen this is your here x and suppose if you want to define here another matrix here say y and if you want to see whether these things are working or not. So, right. So, there is a number.

So, let me try to make it 11. Yeah. So, you can see here x is your here like this and y here is like this, right. So, let me try to clear the screen so that you can see it clearly. So, now if you try to see here x plus y, now this element y addition will happen and x minus y will happen and if you try to see here y minus x then it will also happen, right. So, now you can be confident that this addition and subtraction in the R software they are not difficult and they can be done exactly in the same way, right. So, okay after this I try to take up another issue that whenever you are trying to have a matrix then many times we need to extract a particular row, a particular column or a sub matrix from the same matrix then how to get it done.

Actually, these operations are very important when we are trying to do the multivariate procedures because many times you want to extract a particular data set or you want to extract a particular data set to a particular row or particular column or that can be a sub matrix. So, it is important to understand this thing although I am not giving you here the basic rule but I am trying to take here various types of examples and yeah due to the limitation in the time I am taking here limited number of examples but these examples will surely give you a way out that to understand that how you can do it yourself and in different situation how these concepts can be implemented. So, now let me try to take another matrix now. Suppose I try to create here a matrix of order 5 by 3 and the data here is 1, 2, 3, 4 up to 15 and they are arranged by rows, right.

So, this is here my matrix X. Now I would like to do different types of operations to extract a particular row, particular column or a sub matrix. Now as I explained you earlier that what you have to keep in mind you have to observe that how the rows and columns are being indicated here. Suppose if I try to see this is your here first row. The first row is indicated by writing the square brackets and then this is the first row. Our rule is that we try to write down the row first and then the column.

So, the first row that means here it is 1, then here comma and then this is here blank. Similarly, if you try to go for the third row this will be here square brackets 3 that means third row and then here blank. Similarly, if you try to look for the columns also this is your here first column. So, you can see here first two try to write down the square brackets.

Now this is the first column. So, row will remain as blank or you do not write anything then comma and then here write 1. And similarly, if you try to go for the third column this will be your here the square brackets blank sign here comma and then here 3. And if

you on and on the other hand as I told you earlier if you want to access any particular means element suppose I want to access this 11. So, 11 is lying on the intersection of fourth row and second column like this.

So, this is how this is going to be addressed. So, these are the addresses. Now if you want to access any particular row or particular column then you simply have to give the right address. Suppose I want to access the first row. So, you have seen here now I am writing in this color red that you simply have to write down here the name of the matrices. If you want to access the third row then you simply have to write here the address of the matrix X and say 3 comma.

Similarly, if you want to find out here if you want to recall the first column then you simply have to write down here X and the address of the first column. And similarly, if you want to write the address of the third column you want to recall then the complete address is X inside the bracket square brackets comma 3. And similarly, if you want to address a particular element 11 whose address is at the fourth row and second column then you have to simply write down here X 4 comma 2. So, this is the one of the very simple rule by which you can write down these addresses and you can recall them inside any program. So, let me try to give you here some examples so that you can understand them clearly.

So, suppose I want to recall that further. So, I simply have to write down here X inside the square bracket 3 comma and it will give you here 7 8 9 and you can see here this is the matrix 7 8 9. And similarly, if you want to have the second column let me try to change my color. This is your here second column which is your 2 5 8 11 14. We can see here this is your here second column 2 5 8 11 14. And now if you want to access any sub matrices what do you mean by sub matrices? Sub matrices is a part is a part of the matrices that can be any row and a combination of any rows or any particular columns.

Suppose I want to access the matrix and I want to consider the fourth and fifth rows and second and third column. If you try to see here fourth and fifth rows are this and second and third column here is like this. So, what are the numbers on the intersection? They are here 11 12 14 and 15. So, I want to access actually here this sub matrices. So, other way around you can say that if I want to access this sub matrices then I have to write down the appropriate and correct addresses for the rows and columns.

So, if you try to do it here then you get here the same matrices. So, let me try to show you these examples on the R console first and then I will move forward. So, let me try to create this matrix. So, suppose if I want to access here the second row this is here like this and suppose if I want to access here the second column this is here like this. And suppose I want to create here a sub matrix where I want to take the second and third row and second and third column.

So, this will appear here like this 5 6 8 and here 9. Well, it is not necessary that you need to have only the continuous rows and columns suppose I want to choose here say first and fourth row. So, I can see here $c(1, 4)$ and I want to choose here the first and third row columns. So, I can see here $c(1, 3)$. So, you can see here this comes out to be here 1 and 4 means this one first row fourth row and then here the first column and the third column. So, this is how you can create different types of matrices.

Now, I would like to have your attention here to observe the following. I am trying to observe where I am trying to highlight. If you try to see here this is the second row and this is the second column and their outcomes are indicated like this 4 5 6 and this here is indicating the row number means row number on the R-GUI window not of the matrix or columns row number. So, this is also here 1 and then here 2 5 8 11 14. So, my question to you here is that if I hide this and this and if I give you here only this outcome 4 5 6 and 2 5 8 11 14 can you really identify whether this outcome is for user column.

If you try to see here, I can re-translate the same question here. This is here my third row and this is my here second column. And suppose if I hide this information and if I ask you that this is given as 7 8 9 can you identify whether this is row or column no. In some software it is possible, but in our software, you cannot identify the rows or column by looking at the outcome. If you want to know you simply have to use the correct command like as from the `as.matrix` and only then you will get it.

So, this is what you have to keep in mind. Now, I come to another the inverse of a matrix. So, in order to find out the inverse of a matrix in the R software the command here is `solve`. Solve some of you might be confused that why it is solved in many software it is like I and V etc. etc. So, those who are familiar with the matrix theory they also know that many times we try to find out the inverse of a matrix by solving a couple of equations.

So, that is why this name has been given, but anyway this is not our choice and if you want to find out the unique inverse of a matrix then you have to use the command `solve`. And all the basic rules for obtaining the inverse of a matrix they have to be satisfied here like as the matrix have to be square matrix or the positive definite matrix. Just for your information if these conditions are not as satisfied and if a matrix is a rectangular matrix, then also it is possible to find out the inverse of a matrix which is called as generalized inverse right. So, but the problem is this generalized inverse can be found in different ways. So, it is not unique there is only one way out which is called as Moore Penrose inverse which is which gives you a unique generalized inverse.

But anyway, we are not going into this detail, but I just wanted to inform you that when you are trying to do this multi-unit analysis then many times you may need it right. And in our software, we have built in packages available which can find out such generalized

inverse right. So, let me come back to our job. So, now we try to illustrate how you can use the solve function. So, suppose I try to create here a matrix of order 2 by 2 and where the data is here like 84, 100, 100, 120 right.

Well, you can ask me why I am always taking on the matrix of order 2 by 2 or 3 by 3. But so surely you can see that I have to prepare the slides to teach you and these slides limited space and I want to make the font size bigger so that you can see it easily. And when I am trying to do with the smaller number of elements it is easier for me to communicate rather than doing large matrices. So, now if you see with this command, I can create here the matrix Y here like this. Now if I want to find out the inverse I simply have to write down here solve(), S O L V E all in lowercase alphabets and inside the parenthesis, y.

And you can see here this is going to give you this outcome. So, let me try to show you these things in the R software also so that you can be more confident. So, if I try to create here this matrix here y you can see that this is your here y and if I say here solve here y. So, you can see here this is here like this. And if you say that if I try to multiply the solve y with y that is. So, you know that $A^{-1}A$ if A is a matrix is an identity matrix, but definitely it is not coming out to be exactly 1 or something and it should exactly be 0 because there is a reason that this R software uses different types of algorithms to find out the inverse of a matrices because usually I am giving you here the example of 2 by 2, but sometimes the matrix order may be very high.

So, R uses the algorithm and if you try to see this is exactly actually 1 this is exactly 1, but the off-diagonal elements they are something like 2.84×10^{-14} that is very close to 0. So, this is what you have to keep in mind when you are trying to do such things. And if you want to find out here like if you try to say here Y percentage star this will come out to be exactly 1.

So, this is what you have to keep in mind. Anyway, let us come back to our job and now let me try to give you here one more concept that how to do it in the R software. And so that you must be aware of the concept of eigenvalues and eigenvectors which is also called as characteristic roots and characteristic vectors. So, now these are going to be useful when we are trying to do this multivariate analysis particularly when we are trying to do this principle component analysis later on. So, well I am not going to give you here the details that how do you want to how do you find the characteristic roots and characteristic vector actually it is something like if you have a matrix $XA - \lambda I = 0$ you try to solve it and then you will get here different values of lambda and for each value of lambda you try to find out the characteristic vector. But anyway, I would request you to look at this concept in the book and but here I would like to show you illustrate that how you can find it in the this R software.

So, in order to find out the eigenvalues and eigenvectors of a positive definite matrix the command here is EIGEN eigen and inside the parentheses you have to give the matrix. So, if I try to take here the same example my Y matrix is 2 by 2 matrix like this which is a positive definite matrix. Now, if I say here eigen of Y eigenvalues of Y. So, this is of order 2 by 2. So, there are going to be 2 eigenvalues and corresponding to them there will be 2 eigenvectors.

So, as soon as you see here it is here values these are the 2 eigenvalues and corresponding to these 2 eigenvalues you have here eigenvector this is corresponding to the first eigenvalues and the second this row is corresponding to the eigenvector corresponding to the second values. So, these are eigenvalues and this is our these are here eigenvectors right. They have a role in doing the multivariate analysis. So, I wanted to introduce it, but there are many more things which are going to be useful, but that we will see from time to time.

Come to end to this lecture. So, now, you see that in this lecture I have taken couple of topics which are related to the basic operation, but I would like to tell you there are many more operations and R is capable of doing all of them. And these multivariate operations which we are going to learn in the future lecture forthcoming lectures, they would require this type of knowledge. Finding out the addition, subtraction, inverse, eigenvalues etcetera they will be required when we are trying to do the structural multivariate analysis. So, that was the reason that I tried to cover, but surely, I have not covered all the topic otherwise the whole course will be based only on the matrix theory. So, now, I have given you some basic idea that what is the level of matrix theory which I require and now on a case-to-case basis from time to time wherever we will need a particular type of operation.

I will explain you and you try to understand or I will request you to look into the books and try to understand that matrix process. So, I would request you that you try to brush up your knowledge and information about the matrix theory because matrix theory is going to be the foundation for the multivariate analysis and multivariate procedures and surely this will help you. So, now in this lecture I will stop with the topic of this different types of calculations either scalar, data vectors or matrices. I hope that this limited information will help you in following the forthcoming lectures. So, you try to have a look on these concepts particularly matrix theory and try to revise your concept, try to see those topics which I have not covered here how they can be done in the R software also that will make you more confident.

So, you try to practice it and I will see you in the next lecture. Till then goodbye. Thank you!