

Data Analysis and Decision Making - III
Professor Raghu Nandan Sengupta
Department of Industrial & Management Engineering
Indian Institute of Technology, Kanpur
Lecture No. 49

Welcome back, my dear friends a very good morning, good afternoon, good evening to all of you wherever you are in this part of the globe. And this is the DADM 3 course which is data analysis and decision making 3 course under the NPTEL MOOC series, the total duration spread is basically 12 weeks and which basically contact hours is 30, which means each lecture which is 60 in total number is half an hour. And as you can see from the slide this is the 49th lecture which is in the 10th week and with the 50th lecture we will wrap up the 10th week again another 2 weeks will be left. So, after each week you have basically 5 lecture of half an hour each, you take the assignment you have already completed and taken 9 assignments. After the end of the 10th week we will take the 10 assignment and after the completion of the totality of the course which is 12 weeks you will basically appear for the final examination.

And my good name is Raghu Nandan Sengupta from the IME department at IIT Kanpur. So, if you remember in the last 2 lectures which was in the 47th and the 48th went into detail explaining the branch and bound method for the integer programming problem. Now it will be a little bit different type of problem which is binary that is 0 1 problem, point 1. Point number 2 is that in the first problem which we discussed in the 47th and the 48th lecture it was purposefully taken which I am again repeating it was purposefully taken to be of 2 dimension in order to make our life simple and make them understand. While in this problem we will purposefully take it as a higher dimension in order to make things much better and clear in the higher dimension case, I will solve the branch and bound.

Also now an additional part would be there that at each and every stage I will be discussing how the branches happened and highlighting, I will basically draw the whole diagram for the branch and bound at 1 go but keep highlighting where I am at which stage and which direction I am going so it will be easy for you to basically take 2 things, the macro view about the total branch and bound in the totality and what are the branches and how they are going and also in the micro view where you are at that stage considering what is the integer programming or the binary programming in this case, how the linear relaxation has happened and how you going to proceed

to solve the problem. Now the procedure of solving the problem remain exactly as it is in the linear programming case and we can basically consider that the solution techniques we have already considered in the linear programming, in the simplex method will be utilized time and again, concept of dual prices, slag, surplus, artificial variable, everything will be considered as it is as we have discussed.

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Branch & Bound (B&B)
Example # 02

▶ $Max Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$
s.t.:

$$6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$$
$$x_3 + x_4 \leq 1$$
$$-x_1 + x_3 \leq 0$$
$$-x_2 + x_4 \leq 0$$

x_1, x_2, x_3, x_4 are binary

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DADM 02 42

Now this is the branch and bound example number 2, but now the difference is 2, they are binary variables and we will draw the branch and bound at each stage and it is a higher dimension. So, we want to basically solve a problem maximization of $9x_1$ plus $5x_2$ plus $6x_3$ plus $4x_4$ all are either 0 and 1. And the constraints are like this $6x_1$ plus $3x_2$ plus $5x_3$ plus $2x_4$ is less than 10 this is the first constraint, the second constraint is x_3 plus x_4 is less than equal to 1, so there is no x_1 and x_2 in the second constraint. The 3rd constraint which does not have x_2 and x_4 is as follows, minus x_1 plus x_3 is less than equal to 0. And the 4th constraint which does not have x_1 and x_3 is as follows, minus x_2 plus x_4 is less than equal to 0. And x_1, x_2, x_3, x_4 are all binary.

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Branch & Bound (B&B)
Example # 02

LP relaxed problem

$$\text{Max } Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

s.t.:

$$6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$$
$$x_3 + x_4 \leq 1$$
$$-x_1 + x_3 \leq 0$$
$$-x_2 + x_4 \leq 0$$
$$0 \leq x_1, x_2, x_3, x_4 \leq 1$$

Now here the problem starts, the problem means the solution starts how we proceed. The problem is there we will solve it. First as usual we will basically relax the problem and make it into a linear programming problem, so what happens, in the LP relaxed formulation now here it is important to note that at each and every stage they would come out to be different formulations but the general concept remains the same, only as the 0 and 1 integers some are included some are excluded depending on which direction we are going in the branch and bound, the whole problem objective function, constraints they will keep changing but do not get confused.

If I proceed, if I think I do not want to make it more explicit I will basically write down in details in each and every slide which I will keep adding as we proceed. So, the LP relaxed problem formulation is the same maximization $9x_1$ plus $5x_2$ plus $6x_3$ plus $4x_4$ is the objective function, the constraints remain the same, which is $6x_1$ plus $3x_2$ plus $5x_3$ plus $2x_4$ is less than or equal to 10, which is the 1st one. In the 2nd constraint you do not have x_1 and x_2 so it is x_3 plus x_4 is less than or equal to 1, in this 3rd constraint you do not have x_2 and x_4 it is minus x_1 plus x_3 is less than or equal to 0 and in the final constraint which is the 4th one you do not have x_1 and x_3 , it is minus x_2 plus x_4 is less than or equal to 0.

And the relaxed concept is this, now x_1, x_2, x_3, x_4 are no more binary, they are continuous between 0 and 1 they can take any values. So, technically it was a 2 dimension it would have been a square where the search space would have been 3 dimension, it would have been a cube

and in the 4th dimension is basically hypercube. There are basically 4 dimensions x_1 x_2 x_3 x_4 and the cube I have the dimensions that is the values are between 0 and 1.

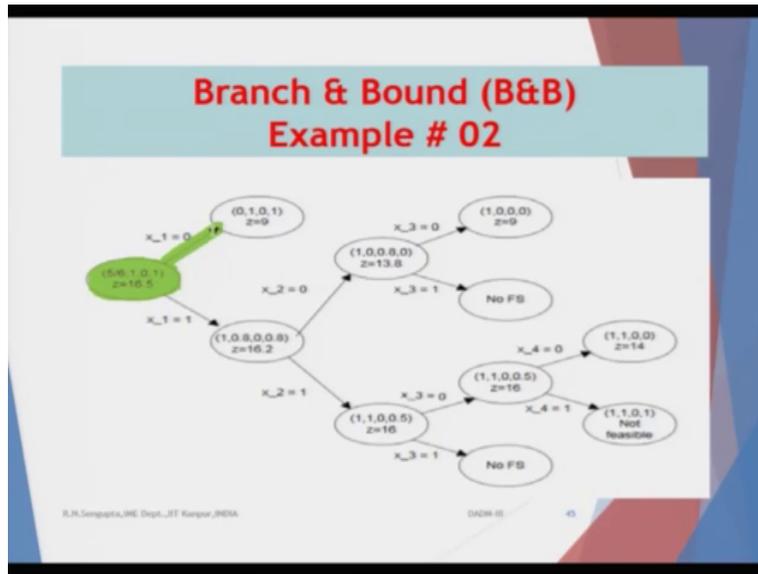
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The slide features a light blue header with the text "Branch & Bound (B&B) Example # 02" in red. Below the header, a blue arrow points to the text "The solution is Z=16.5 and". Underneath, the variables are listed: $x_1 = \frac{5}{6}$, $x_2 = 1$, $x_3 = 0$, and $x_4 = 1$. The fraction $\frac{5}{6}$ is circled in red, while the integers 1, 0, and 1 are each circled in green. At the bottom left, there is small text: "R.N. Sengupta, IIT Kharagpur, IIT Kharagpur, INDIA". At the bottom right, there is small text: "DADM 05 44".

So, you solve it, solve it using any problem formulation. And the problem formulation it is like this, so you have let me check weather if I am able to do that, I will continue doing it as. So, when you solve the solution is as this, it is 16.5 for Z and the corresponding values are x_1 is 5 by 6, technically it is linear programming problems so 5 by 6 is applicable there, not in the binary case so this is not done, not allowed. x_2 is 2 which is allowed, x_3 is 0 which is allowed, x_4 is 1 which is allowed. Now technically what you would have done is that you would have basically taken, if you have follow the concept of the branch and bound, x_1 is 5 by 6 so you would have taken 0 and 1 and proceeded accordingly, so this is the general concept.

Now comes the branch process, first I will tell you I am drawing the whole branch process but here I am going to highlight where I am, I again saying the macro picture will be given, the micro points will be highlighted where I am, in this like whole the map of say for example New Delhi is given and where I am, I may be at cannought place may be at Dwarka, that will be highlighted and then from where, form cannought place how I am moving that would basically be given by the branches.

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So this is whole map, look at it very carefully, I am starting from the left going to the right for the branches, I could have basically started considering the analytical high key process I could have started from the top and then gone down, so whichever it is fine. So, I will keep highlighting where I am by the green color and from where I have come I will basically be highlighting (with the red color so red colors) I will highlight this in blue colors, let me use I will use this concept, blue colors from where I have come, where I am is green. So, I am going to start at the solution which is as it is, so when I solve the problem linear programming, remember my actual solution is this, I have solved it I have the values of x_1 as 5 by 6, x_2 as 1, x_3 as 0, x_4 as 1 the Z value 16.5, so now I want to branch.

Where do I branch, remember I told that few minutes back I will branch for the values of x_1 where x_1 is 0 or x_1 is 1, so these are the branches, so there is the blue one where I am starting. So now I will change the color to green, so I have already solved the linear programming problem, relaxed version. Now I can take 2 direction, I can take the first path the first branch, the second path the second branch and highlighting the points. The x_1 variable which was 5 by 6 is 0, (x_2) x_1 variable which was 5 by 6 now it is 1. Now you may be asking that what if x_1 was 5 by 6 and x_3 was also a non integer then you will basically go x_1 and then take the 2 branches of x_3 x_1 and again take the two branches of x_3 so you could have done it. So, it will be a little bit more intense in the branching but you will basically follow the same procedure. So, I am here, I have solved the problem and then let us proceed.

(Refer Slide Time: 10:34)

Branch & Bound (B&B)
Example # 02

▶ **Branch # 01 ($x_1 = 0$ or $x_1 = 1$) and we consider $x_1 = 0$**

$Max Z = 5x_2 + 4x_4$
 ~~$9x_1 + 5x_2 + 6x_3 + 4x_4$~~

s.t.:

$$3x_2 + 2x_4 \leq 10$$
$$-x_2 + x_4 \leq 0$$

x_2, x_4 are binary

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Now we consider branch number 1, we can go the branches as I had mentioned if you remember x_1 as 0 or x_1 as 1. And now we are considering x_1 is 0 that means we are going in this direction. So what is the problem, interestingly check here one thing the actual problem formulation was $9x_1$ plus $5x_2$ plus $6x_3$ plus $4x_4$ so if you check this I will write it down. $9x_1$ plus $5x_2$, so it is $9x_1$ plus $5x_2$ plus $6x_3$. So if you see the objective function it seems to be changed, it is not changed because x_1 is 0, so obviously this will vanish, x_2 is there because yx_2 is here, x_2 value is 1 it can not vanish so he has to be there, so x_2 is there which is right, x_3 is not there, go back check x_3 value was 0 so we have just eliminated it, we could have written it but we are eliminating it. x_4 was what, x_4 was 1 so it should be there and rightly it is there. So, this equation is exactly equal to the actual objective function which I have written. So I told you there would be changes, and how I write the objectives functions but I will highlight and each every stage. So I will erase this. So, the objective function part is clear.

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**Branch & Bound (B&B)
Example # 02**

► **Branch # 01 ($x_1 = 0$ or $x_1 = 1$) and we consider $x_1 = 0$**

$\text{Max } Z = 5x_2 + 4x_4$

s.t.: $6x_1 + 3x_2 + 5x_3 + 2x_4$

$3x_2 + 2x_4 \leq 10$

$-x_2 + x_4 \leq 0$

x_2, x_4 are binary

Now let us come to the constraint, so I will try to do that each and every stage, so if it is very simple I will not highlight it you will understand, but I will try to go a little bit slow as I did for the first problem in the branch and bound in the 47th and the 48th lecture. So, what are the constraints, $6x_1 + 3x_2 + 5x_3 + 2x_4$, 6 3 5 2 so it is like this. Let us see, x_1 was 0 so it vanishes, fine, x_2 is 1 which remains in the constant which is fine, x_3 is 0 it vanishes so its fine, x_4 is 1 it remains the constraints which is fine. So this constraint and this constraint are also same. Now I come to the other 3 or 4 constraints, so I will go one by one, other constraints are $x_3 + x_4$ is less than 1, so x_3 is 0 x_4 is 1 so 1 is less than equal to 1 is does not make sense to write so we omit it. So 1 is equal to 1 is very simple to understand, what is the 2nd constraint, 2nd constraint is $x_1 - x_1 + x_3$ is less than equal to 1, so x_1 is 0, x_3 is 0 so 0 is less than equal to 0 which is also fine.

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Branch & Bound (B&B)
Example # 02

► **Branch # 01 ($x_1 = 0$ or $x_1 = 1$) and we consider $x_1 = 0$**

$Max Z = 5x_2 + 4x_4$

s.t.:

$3x_2 + 2x_4 \leq 10$ ✓
 $-x_2 + x_4 \leq 0$ ✓
 x_2, x_4 are binary

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So what is the 4th constraint, 4th constraint minus x_2 plus x_4 . so x_2 was basically considered as 1 and x_4 , it could have been 0 and 1. But here it is 1 so x_2 and x_4 is possible so that will come into one of the constraint. So, minus x_2 plus x_4 is 0 is the 4th one which is true, 2nd and 3rd are gone, 1st one has been reduced in this form. Now here you see only 2 variables because x_1 is 0 as per the concept here we are in that branch, what is, which is read by, I will be switching from slide to slide so please bear with me so this branch we are the green highlighted arm. And x_3 is 0, x_2 and x_1 are binary, so technically they are binary but we will solve them using the relaxed linear programming that is what I keep highlighting at each and every stage.

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**Branch & Bound (B&B)
Example # 02**

▶ Branch # 01 ($x_1 = 0$ or $x_1 = 1$) and we consider $x_1 = 0$

LP relaxed problem

$$\text{Max } Z = 5x_2 + 4x_4$$

s.t.:

$$\begin{aligned} 3x_2 + 2x_4 &\leq 10 \\ -x_2 + x_4 &\leq 0 \\ 0 \leq x_2, x_4 &\leq 1 \end{aligned}$$

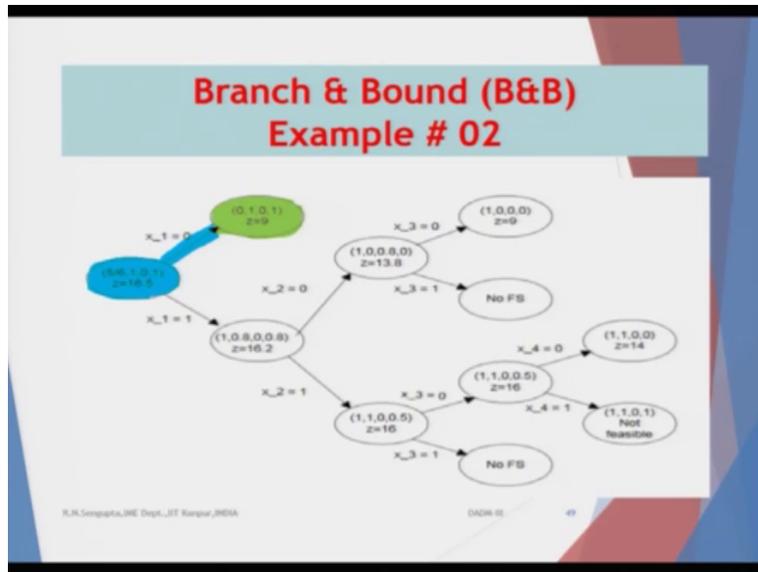
R.A.Sempurna, ME Dept., IT Kampus Jember DAAM 02 47

**Branch & Bound (B&B)
Example # 02**

▶ Branch # 01 ($x_1 = 0$ or $x_1 = 1$) and we consider $x_1 = 0$

The optimal solution is $Z=9$
and $x_1 = 0, x_2 = 1, x_3 = 0, x_4 = 1$

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So here it is, we are in this branch where x_1 is 0, problem formulation is maximization of $5x_2$ plus $4x_4$ you want to maximize. The first constraint remains which is $3x_2$ plus $2x_4$ is less than 10, the 2nd and 3rd are not to be considered very simplistically, I have mentioned that the 4th constraint remains which is minus x_2 plus x_4 is less than equal to 0 and now as we bring the relax formulation it becomes x_2 is between 0 and 1 both inclusive continuous. x_4 is between 0 and 1 both values inclusive continuous this has been highlighted here. So now we have the LP relax formulation of the branch where we gone the first step we solve it and now the optimum solution comes out to be like this. Z is 9, x_1 is 0, x_2 is 1, x_3 is 0, x_4 is 1.

If you remember x_3 was already 0 so it continuous to be 0, x_4 was 1 so it continuous to be 1, x_2 was basically 1 already and x_1 we have taken as one of the branches so this basically meets our criteria, there is a, binary variables are there, decision variables and the answer in 9. So where are we, let us see in the graph, graph means the branch. So, now you have remembered I mentioned from where we come, will be highlighted by blue, where we are we will be highlighted by green. So we have already come from here, we have taken this route and we are here, so blues would be we have already visited (consid) let us us be consider that.

So, now green one there is no branches here which is very simple because all of the variables are binary which is as per the norm of the problem and value is 9 which is fine, so we have reached one of the bounds and on the branches which is giving as a result. I will nor use the word optimum feasible it is obviously it is feasible whether optimum we will check that later on. So,

now you want to go in the other branches, so other branches are how did you proceed because we have taken x_1 as 0 so obviously the other branch would be x_1 is equal to 1 because x_1 was coming out to a value of 5 by 6 that is why we differentiate there into 1 and 0.

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Branch & Bound (B&B) Example # 02

► Branch # 02 ($x_1 = 0$ or $x_1 = 1$) and we consider $x_1 = 1$

$$\text{Max } Z = 9 + 5x_2 + 6x_3 + 4x_4$$

s.t.:

$$3x_2 + 5x_3 + 2x_4 \leq 4$$

$$x_3 + x_4 \leq 1$$

$$x_3 \leq 1$$

$$-x_2 + x_4 \leq 0$$

x_2, x_3, x_4 are binary

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Branch & Bound (B&B) Example # 02

► Branch # 02 ($x_1 = 0$ or $x_1 = 1$) and we consider $x_1 = 1$

LP relaxed problem

$$\text{Max } Z = 9 + 5x_2 + 6x_3 + 4x_4$$

s.t.:

$$3x_2 + 5x_3 + 2x_4 \leq 4$$

$$x_3 + x_4 \leq 1$$

$$x_3 \leq 1$$

$$-x_2 + x_4 \leq 0$$

$$0 \leq x_2, x_3, x_4 \leq 1$$

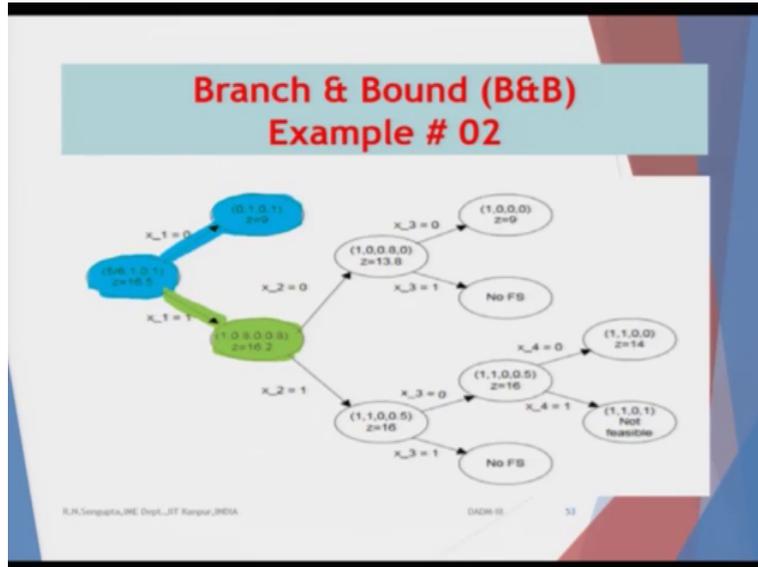
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Branch & Bound (B&B) Example # 02

▶ Branch # 02 ($x_1 = 0$ or $x_1 = 1$)
and we consider $x_1 = 1$

The optimal solution is $Z=16.2$
and $x_1 = 1, x_2 = 0.8, x_3 = 0, x_4 = 0.8$

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So now is the branch 2, so actually it should be the second part of the branch 1 but I am mentioning as branch 2. So branch 2 was actually 2 routes which I am just highlighting mention again x_1 is 0, x_1 is 1 and we now consider x_1 as 1. So, now as x_1 is 1 the problem formulation objective function is like this $9x_1$ plus $5x_2$ plus $6x_3$ plus $4x_4$ is the objective function. Now if you see in the 9 value is coming because x_1 is 1 so obviously 9 into x_1 is 9 technically $5x_2$ is there which is fine, $x_3 = 0$ technically that we should have written the whole problem formulation but I am writing it in the more much more condensed form in order to highlight for where we are changing, why we are changing.

So it is $6x_3$ now x_3 can definitely be 0 or 1 because now we are trying to relax it, relax in the sense not from the point of view of linear programming relaxation, relaxation means in which direction you are going whether it is possible or not. Then the last part of the objective function is $4x_4$ and what are the constraints, there are 4 constraints the constraints go like this I should basically highlight it, so the constraints are $3x_1 + 3x_2 + 5x_3 + 2x_4$ is less than equal to 10. So $6x_1 + 6x_2 + 1x_3 + 1x_4$ it goes on to the right hand side $10 - 6 = 4$ that why it is 4, I will just highlight it, I will erase it later.

And the other values of $3x_2$ which is the 2nd term in the 1st constraint plus $5x_3 + 2x_4$ remains is less than equal to 4. The 2nd constraint let me check or let us check $x_3 + x_4$ is less than equal to 1, so it remains $x_3 + x_4$ is less than equal to 1. The 3rd constraint was minus $x_1 + x_3$ is less than equal to 0, so now x_1 is basically 1. So what if becomes? So, if I take minus 1 to the right hand side so it becomes x_3 is less than equal to 1, so x_3 is less than equal to 1 which is the 3rd constraint. And finally the 4th constraint, what is was let us check, minus $x_2 + x_4$ is less than equal to 0, so it will remain as it is minus $x_2 + x_4$ is less than equal to 0. So now here x_1 is 1 it is fixed so it is not a decision variable the other decision variables are x_2, x_3, x_4 they are binary in the actual problem formulation we will relax it, do not worry So, it is x_2, x_3, x_4 can be 0 and 1.

Now we will relax the problem, that means make it a linear programming, objective function remains the same the 1st constraint same with respect to what we have just discussed, the 1st constraint remains the same, the 2nd constraint remain the same, 3rd constraint remain the same 3rd constraint means x_3 is less than equal to 1. 4th constraint remains the same which is minus $x_2 + x_4$ is less than equal to 1. Now we relax it and make the variables as continuous between 0 and 1 so they are as follows which I will highlight, so it is x_2, x_3, x_4 are all between 0 and 1 both inclusive. So we will basically try to solve the problem for a linear programming where x_1 is fixed at 1, already there. So it is out of the problem and now it is, I am using the word but it would not make sense much in trying to highlight because we would not be able to draw the diagram.

So, we are now basically solving a problem in the 3 dimensional cases where it is x_2, x_3 and x_4 . This is the 2nd branch in the bottom part which I will just highlight. The optimum solution

comes out to be Z is equal to 16 point 2, x_1 is 1 already x_2 is 0.8 which is in the linear case it is fine not in the binary cases it is not fine. x_3 is 0 which is fine, x_4 is 0.8 which is not okay because in the binary cases it should be 0 and 1. So, let me check where we are in the micro and the macro scale considering the overall branch and bound diagram. So where we are that I said I will highlight using green color and which other areas we have already visited we will highlight it by blue color. So 1st we will highlight the blue part, we have already visited this, we have already visited this because this is the area from where we started. So now the green part where we came actually, this is green and this is the area where we are.

So, that value Z as highlighted is 16.2 and the corresponding values of x_1 to x_4 are 1, 0.8, 0, 0.8 and we will basically consider accordingly. Now interestingly note down here there are 2 values of x which are not binary, so technically their branches would go into 2 stages, why 2 stages I am not going to highlight it anything I will just mark it. So, if you consider we would basically consider in this sequence x_2 branched out into 0 and 1 which I have done here then later on x_4 as also would be branched out, so we will technically we should consider it and even if it is not marked we will come that later on as we solve the problem this will become clear. So, we are in this green region in the micro level, and the macro level graph is given and we will basically solve it. So this is a feasible point and we will compare that with the other feasible point and check which is the optimum one.

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Branch & Bound (B&B)
Example # 02

► Branch # 03 ($x_2 = 0$ or $x_2 = 1$) and we consider $x_2 = 0$

s.t.:

~~$9x_1 + 5x_2 + 6x_3 + 4x_4$~~
 $Max Z = 9 + 6x_3$
 $5x_3 \leq 4$
 x_3 is binary

R.N.Sempeta, JRE Dept., JIT Karur, JREDA DASH 10 14

**Branch & Bound (B&B)
Example # 02**

► Branch # 03 ($x_2 = 0$ or $x_2 = 1$) and we consider $x_2 = 0$

$Max Z = 9 + 6x_3$

s.t.: $6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$
 $x_3 + x_4 \leq 1$
 $5x_3 \leq 4$
 x_3 is binary

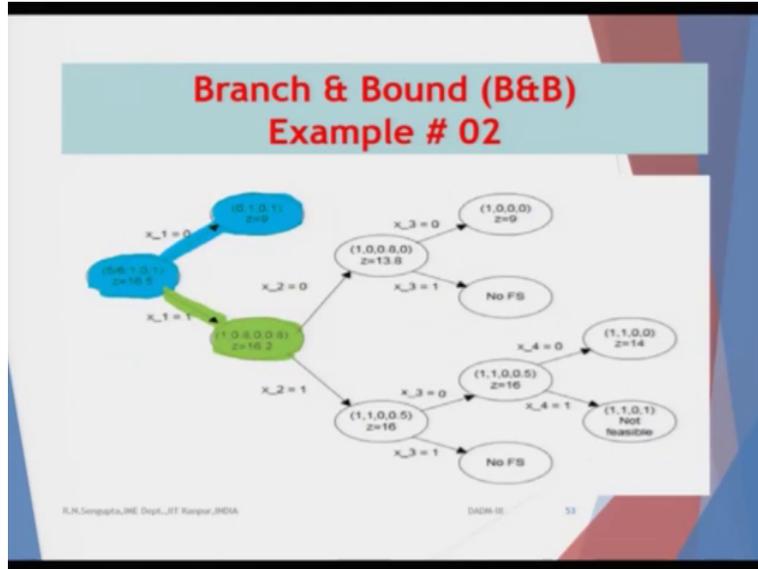
S.K. Sengupta, IIT Kanpur, Dept. of Mechanical Engineering

Now we take the branches based on the fact that x_2 point, which is basically 0.8 can be either 0 and 1 so we will basically do it like follows. The branch 3 which you are going to consider, so this was basically branch 1, this was branch 2, branch 3, branch 4 we will go but technically they could have been mentioned as branch 1a, 1b, 2a, 2b and so and so forth but I just wanted to highlight it accordingly. So, we are in the branch 3 stage, so the branch 3 basically is that we have gone into x_2 is equal to 0 and if x_2 is equal to 0 and the problem formulations are like x_1 is 1 already, x_2 is now 0, x_3 is 0, x_4 is what you have to consider is basically 0 and 1, so this is the problem. So now we want to maximize and the maximization problem let me write down the actual objective function because that what I said I will be doing at each and every stage. 9, 5, 6, 4 so let me go in this way.

I am writing the objective function, so x_1 is 1, why it is 1, because x_1 is 1 in this branch with the 2nd one which I mentioned. So this remains as 9 fine, x_2 is 1, 0 so it vanishes which is fine it is not there. x_3 is 0 but we will consider that we are trying to branch and x_4 is the branches we are going to take which is not there so this will vanish. Now the corresponding values are and obviously so this need the objective function which is fine. And the constraints I will highlight the constraints and then proceed further on. So we are here, I will first highlight the constraints one at a time. So constraints are, just highlight it. So this is (1), So this is 6 (6 minus), so 6 goes to the right hand side 10 minus 6 is 4, x_2 is 0 vanishes, x_3 comes, x_4 vanishes, so this is $5x_3$ is less than 4. What is the second constraint $x_3 + x_4$ is less than 1, so x_3 plus x_4 is less than 1, so x_3 is a

binary one, so it is equal to 1 so obviously it would not be considered so x_4 is not there it would not be.

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So, based on that we proceed and write the overall problem formulation and then we will go into the next branch accordingly. So, here where we are, we are trying to basically go into the branches accordingly and consider x_2 as 0 or 1 and then solve the problems. So we will proceed step by step, so our macro picture is this which we will solve, which we have already solved the blue part, we are in the green part, we have already solved and then as we proceed this greens becomes blues and we will basically see proceeds which is favorable, which are feasible which are not feasible. And with this I will end the, last but one lecture in the 10th week and we will consider the 50th lecture in the 10th week and consider the detail solution in further more analysis for this binary problem using the branch and bound method. Have a nice day and thank you very much.