

Data Analysis and Decision Making-III
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Lecture No. 37

Welcome back my dear friends, a very good morning, good afternoon, good evening to all of you wherever you are in this part of the globe and this is the DADM 3 which is data analyzing and decision making 3 course under the NPTEL MOOC series, and as you know this course total duration is spread over 12 weeks, the total contact hour is 30 hours which when broken down into number of lectures is 60 because each lecture is for half an hour and in each week you have 5 lectures and after each week you have an assignment, so if you can see we are in 37th lecture which is eighth week, so we have already completed 7 assignments and in totality you will have 12 assignments after that you will take the final examination.

So if you remember we were discussing about the DA method I told you about the background what are the DMUs what are the input bundles output bundles and how many DMUs are there, why this DA method is utilized considering there would be many subjectivity the concept also being brought into the picture and in this problem formulation if you remember I said that we will try to consider the inputs and outputs and then try to formulate an efficiency score and then try and then find out their ranking. But in doing so I also gave you 3 scenarios under which you will try to do that, one is the decreasing return to scale, next will be increasing return to the scale another would be the constant return to scale.

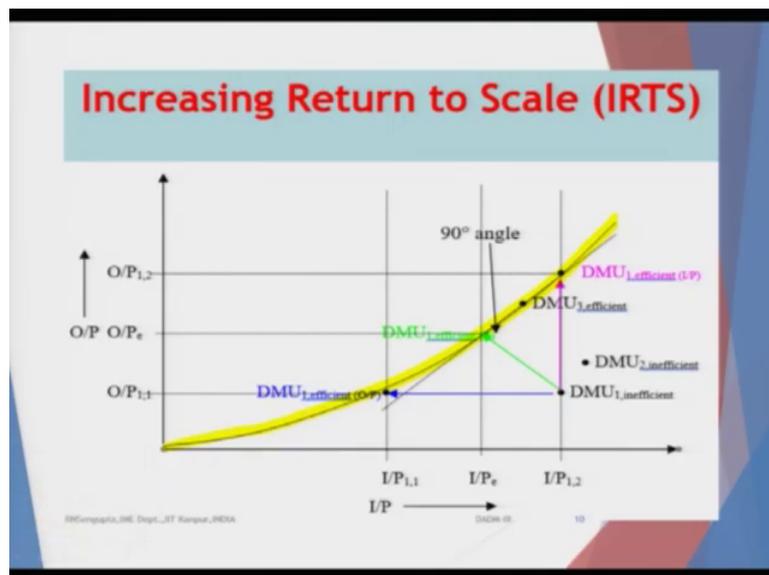
In the decreasing return to scale, increasing return to scale and constant return to scale, per unit increase in the input would basically bring down your profit per unit in the case of decreasing it will increase per unit for each additional inputs your output will increase per unit stage by stage and in the constant return to scale it will remain constant so which means the rate of change of the functional form in decreasing would be increasing under decreasing rate and so in the case of the increasing return to scale it will be increasing but increasing at an increasing rate and in the constant return to scale it will be increasing at a constant rate so obviously if the first derivative properties are known to you, you can basically talk about the second derivative properties for the decreasing, increasing and the constant return to scale.

Now in the decreasing return to scale I give two graphs when in one case the x axis had the bundle of inputs y axis has the bundle of outputs and there it was shown that any DMU trying to move to the if it is an inefficient trying to move to the efficient frontier would basically

move vertically up considering inputs are fixed and you are trying to increase the output you can move horizontally to the left such that the outputs are fixed by inputs are being decreased and another thing would be it can move in a tangential direction with respect to the curve which is the efficient frontier such that it will decrease input and increase in output in some proportions depending on which direction it is moving and how it can basically touch the efficient frontier considering the shortest distance would be the tangent point.

Now I am coming to the increasing return to the scale, so the analysis would go in the same logic, first we will consider in the x-axis the input bundle of goods then in the y-axis I will consider the output bundle goods and then reverse the discussion where the x-axis would be the output bundle and the y axis would be the input bundle, try to analyze that and then in the similar way go for the constant return to scale.

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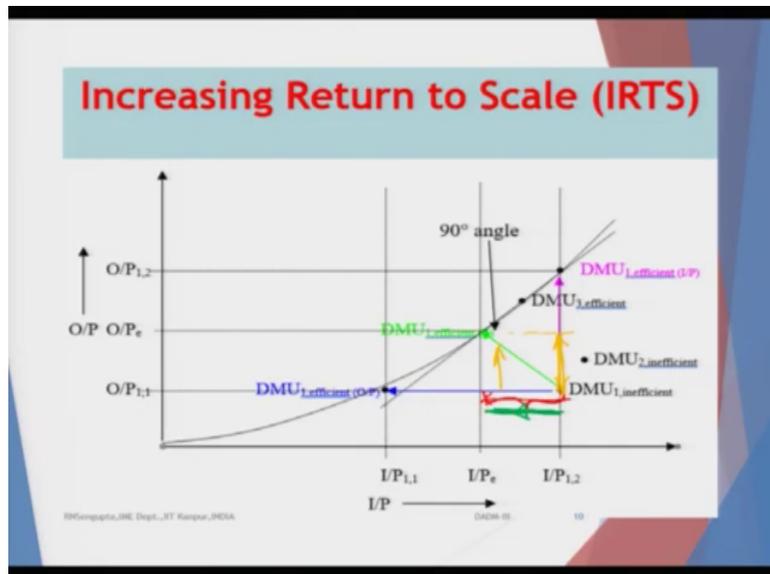
So considering this diagram I would be using the highlighter in order to specify it, so here you have the graph and your inputs are given in the x-axis and outputs are given in the y axis, the suffix E where the pointer is pointing whether output or (I/P) (04:34) to the input we will consider as efficient the subscript being for the efficient one and any other subscript like 1, 1, 1, 2 are would basically denote the position. So if the DMU 1 is inefficient similarly DMU 2 is also inefficient but we are not going to draw the movements of DMU 2, only we will highlight that for DMU 1 and basically that can be done for DMU 2 also.

So in the DMU 1 if this is inefficient then it can basically reach the efficient frontier considering the increasing return to scale it can move in three ways, one is vertically up which means the input bundle remain same, output increases and this vertical up is the pink

line, so it becomes the pink colored, colour scheme is there in order to make you understand so the pink one DMU 1 would be the efficient one considering is an input oriented, another thing which I which I did not discussed in the last lecture was basically the subscript I slash PUG basically trying to denote that is the input oriented model. So input oriented model I will come to that why and it is very logical, so in the input oriented model what you are doing is that inputs are technically fixed you are increasing the output at the same level of the input.

Now if the DMU 1 moves onto the horizontal left point which is the blue line it comes to the efficient frontier which is this black point so this is the blue one which is DMU 1 suffix inefficient and this is blue in colour.

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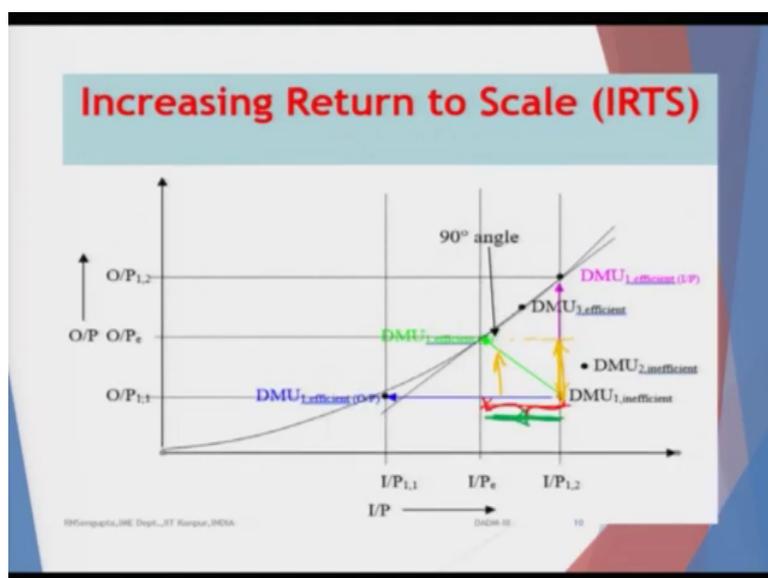
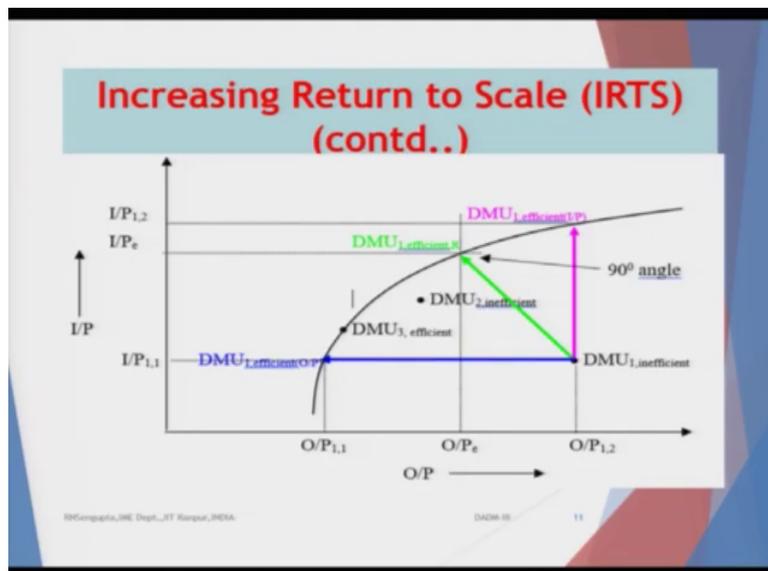
Let me show you it is blue in colour, so see this is blue in colour so it means that the output oriented model where the output is basically fixed and you are trying to basically decrease the input and if it moves radially obviously the radial movement may not have been drawn very absolutely perfectly but you will get the information that if it moves tangentially and it is tangent to this point when a 90 degree is as it being denoted here, so obviously what it would mean that the inputs have decrease by this amount and obviously the outputs were here so they have so input is decreasing and so here in this direction the input is decreasing and in this direction the output is basically increasing.

So output was here OP 1 to this so your output is increasing, so you most basically have both of them as in the decreasing return to sale you had input decreasing output increasing in some proportions depending in which direction it is moving and the tangential green one, here also

the DMU so this is increasing here this is what I meant this jump is the increase of the input and this decrease, (ohh sorry) here it should be the green line so this is the decrease in input.

So now obviously one point which is I will to highlight once I finish the decreasing and then do consider the increasing return to scales the last diagram, so I will come to that by considering all the 4 diagrams at one go. So in this case again I am repeating vertically movement up is input same output increasing, horizontal left input decreasing output fixed and radial movement in some tangential direction such there is tangent at the curve would basically means some proportion will decrease in input and proportion increase in the output. So the word proportion which I am using that will come out very easily through a diagram or using very simple trigonometry I am going to come to that.

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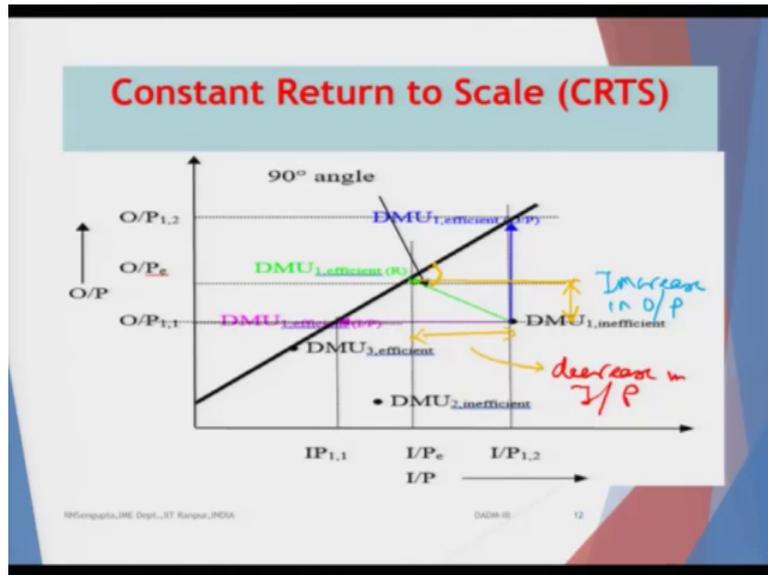
Now when I consider the increasing return to scales so considering the input is along the y-axis, output along the x-axis, and you have DMU 3 which is efficient so we are not going to consider that, DMU 2 is inefficient it can also move but I am not going to draw diagrams for that I have only consider DMU 1 which is the black dot here DMU 1 is inefficient it can move either vertically up, horizontally left and radially, so the radial point is basically it is going to touch the curve in such a way it is 90 degrees, so in that case when I consider the increasing return to scale it will mean this.

So here it would mean that I am able to increase the inputs and increase the outputs also, similarly this curve would also imply that for the same level of input, so I am trying to the points would have been on this side, my mistake. So basically they are here so any moment in the direction where they move would basically imply that the inputs and outputs are increasing or decreasing proportional to the case for the increasing return to scale.

would be the proportional decrease in the input and using the concept of triangle and congruency you can find out what would be the ratios and they can be basically find it out.

So if A B are such that C dash is in between middle point then the proportional concept can be found out very easily, it can be done for other ratios also you can do that so this you can do for the decreasing one and the increasing one return to scale also.

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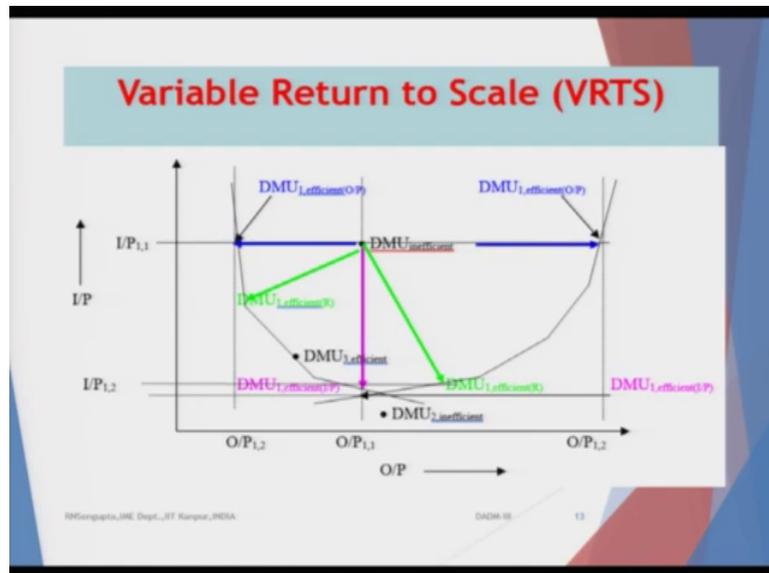


Now coming to the constant return to scale in this case you have only one diagram because the constant return to scale the diagram would be such whether you consider the inputs along the x-axis or outputs along the x-axis the line would be a straight line considering the rate of change of functional form is happening at the constant rate so d by dx increases at a constant rate so if I have any DMU which is inefficient so it can move vertically up the blue line it can move horizontally left which is the pink line.

So vertically moving up basically means again I am repeating sorry for that the inputs remains fixed output increases horizontal movement around to the left means output remains fixed input decreases and then if I consider any movement in the tangential direction so this angle is actually 90 degrees, so obviously it would move in this direction 90 degrees which would mean that there is proportional increase in the output and proportional decrease in the input, so this moves in this consider this is a 90 degrees for the time being, I do not want to clutter it too much.

So this is the increase in the output and this sit he decrease in the input, so this is decrease in input and this is increase in output and similarly when you have the output along the x-axis and input along the y-axis you can basically formulate the problem in the similar way.

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And in the case if you have the variable return to scale, if we can again done where we have different input along the x-axis or input along the y axis I have only take input along the y axis, then you can have different direction of movement depending upon the variable returns, it can either move vertically up or down depending on whether you have been able to draw the output along the y-axis.

So here our output is along the x-axis so the curve is in other way round downward and then you will have obviously the vertical lines left or right which is blue in colour would remain same in the case output is along the y-axis, but the pink line would be moving vertically up not down as in this diagram and the radial movement would be happening on to the north west corner method or on to the right corner method depending upon which direction you are moving considering that you want to basically proportional decrease the input and proportional increase the output and there obviously you can formulate the same find out the same way that depending upon the input oriented model or the output oriented model or both can be change, you can basically find out in what proportions the input and output would change to make it efficient the DMU is efficient.

Now having said that we have been talking about the drawing the diagrams but till now in the first 38 in the first class of the eight week and this is the second class of the eight week I have been telling about that we can formulate in (opt) optimization problem we can solve it but I

have not come into that so here what I am going to do now. Now there would be two different ways to handle the problem one is the output oriented model where your main concern is to do with the output and another input oriented model where your main concern will be something to do with input. So how you look at the picture would basically change the formulation of the problem but the general cracks of the matter how you solve it remain the same.

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**O/P Oriented Optimization Models
(for K DMUs)**

$$\max \left\{ \frac{\sum_{j=1}^N v_{jk} y_{jk}}{\sum_{i=1}^M u_{ik} x_{ik}} \right\}$$

s.t.:

$$\frac{\sum_{j=1}^N v_{j1} y_{j1}}{\sum_{i=1}^M u_{i1} x_{i1}} \leq 1$$

$$\vdots$$

$$\frac{\sum_{j=1}^N v_{jK} y_{jK}}{\sum_{i=1}^M u_{iK} x_{iK}} \leq 1$$

$$u_{ik}, v_{jk} \geq 0, i = 1, \dots, M; j = 1, \dots, N; k = 1, \dots, K$$

Now if you remember we have taken x and u are the corresponding say are corresponding decision variable and the weights for the input or input bundles and the suffix where I and K, I is from 1 to M and k small k is going to 1 to capital K, while the corresponding decision variables and the weights for the output bundles are y and mu so again their suffix are j and k, j changes from 1 to N capital N which is the number of output while M I am again repeating is the number of inputs while (j) k changes from 1 to capital K where K is the number of DMUs.

Now if you see this ratio what does the ratio means, the ratio means that I am trying to basically find out a maximization of a ratio where the ratio actually means the efficiency, so in the denominator you have basically in the numerator and denominator in the top part of the numerator you will basically have the total bundle of all the outputs that means what is coming out because that you are summing up by multiplying the decision variables with the weights and in the denominator, so numerator basically has the total bundle of output and the denominator you have the total bundle of the inputs.

So what you are trying to do here is you are trying to find out what is the ratio considering you want to maximize it, so higher the ratio is and obviously in the highest case it would be 1 and then in that case the efficiency of that DMU considering that k is equal to 1, 2, 3, 4 till capital K is 1 that means its actual efficiency is 1. Now you want to maximize the efficiency and you will do it for each and every k values, k is equal to 1, 2, 3, 4 till capital K , but in that case when you are trying to maximize the efficiency of the ratio you will have some constraints, these constraints are actually the constraints corresponding to all the rest of the DMU, so say for example if I am trying to maximize the efficiency of the first DMU I will basically keep the constraints of the first along with the second to the K th one.

When I am trying to basically maximize efficiency of the second I will keep the constraint corresponding to the first, second, third till the K th one, when I try to basically maximize the third one I will keep the constraint corresponding to first, second, third till the K th one till the last optimization problem which you all have is basically trying to maximize the ratio corresponding to the capital K th one and the constraints would be corresponding to the first, second, third till the capital K th one.

Which means that if you have 10 DMUs, I will have basically 10 such optimization problems, now the optimization problems look like it is basically a ratio which means it is nonlinear, well I agree with that but we will slowly convert this problem from a nonlinear one to linear one, so first let us pay attention to the constraints.

Now what are the constraints? Now if I am trying to maximize the efficiency which would mean that I would try to increase the efficiency of any K th one any arbitrary K th one, k can be 1, 2, 3, 4 till the capital K , corresponding to the fact that the ratios of all the rest 1 to capital K are all less than equal to 0, less than equal to 1 sorry. Which means that anyone can increase but the constraining fact would be none can be more than 1, so obviously maximum value of each of the efficiency for each of this capital K number DMUs would be maximum would be equal to 1 or in the practical sense it would be less than equal to 1.

So what I will have is for k is equal to 1 which is the first DMU, I will have the bundle of outputs divided by the bundle of inputs is less than equal to 1, similarly I have second constraint where in the numerator I will have the summation of the bundle of output for the second DMU divided by the bundle of inputs for the second DMU also equal to less than equal to 1. The third constraint would be in the numerator in the ratio the top part or the numerator would basically be the bundle of all the outputs corresponding to the third DMU

and in the denominator you will have basically the bundle of inputs for the third DMU and we will continue doing that till we reach the Kth one where the Kth one is shown here, where in the numerator we have the bundle of the outputs corresponding to the capital Kth one.

So say for example I said that if there are 10 DMUs it will be corresponding to the tenth DMU, bundle of outputs and in the denominator you will basically have the summation of the all the bundle of inputs for the Kth or the tenth DMU and all them would be less than equal to 1. And in this case obviously it will mean that all the values of weights u_{ik} , v_{jk} and obviously the variables which we are considering would definitely be greater than equal to 0.

Now this is the problem formulation and we have such K number of DMUs where K is 10, now if you see the problem is nonlinear one, maximization of a ratio and the constraints are also nonlinear because there are ratio less than equal to 1. Now what I will do is that for the output oriented model I will basically check as per the concept I will basically (()) (23:16) it a little bit and convert into a very simple linear programming. So here it happens, now watch carefully the maximization was a ratio now the maximization would be only a one term.

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**O/P Oriented Optimization Models
(for K DMUs) (contd..)**

$$\max \left\{ \sum_{j=1}^N v_{jk} y_{jk} \right\}$$

s.t.:

$$\sum_{j=1}^N v_{j1} y_{j1} - \sum_{i=1}^M u_{i1} x_{i1} \leq 0$$

$$\vdots$$

$$\sum_{j=1}^N v_{jk} y_{jk} - \sum_{i=1}^M u_{ik} x_{ik} \leq 0$$

$$\sum_{i=1}^M u_{ik} x_{ik} = 1$$

$$u_{ik}, v_{jk} \geq 0, i = 1, \dots, M; j = 1, \dots, N; k = 1, \dots, K$$

So I am going to basically convert the denominator which is there in the objective function and keep it normalize and fixed at 1, if I keep it at 1, maximizing the ratio will also mean that I am basically keeping the denominator fixed at 1 I can maximize the numerator also.

So they would basically mean the same thing, in the sense the constraints would also change accordingly, so what I have I am basically maximizing the numerator which is the bundle of a outputs, but as I am doing that I am converting each and every constraints along with the fact

that I am adding the extra k plus K plus l th constrain which is corresponding to the denominator which is being removed from the objective function and being brought into the constraint, so where it is let me check.

So this had in this problem actually this was the ratio divided by a summation of i is equal to 1 to capital M $u_{ik} x_{ik}$, so this was a bundle of inputs so this part is moved here, that means I am added the K th plus 1 constraint. As I am basically forcing the denominator to be 1 hence I force it to be 1, and try to basically maximize the numerator which I am doing.

Now as I am doing it the constraint will also be change in a very simple way, the initial constraint for the first one was what, the bundle of output for the first DMU divided by the bundle of inputs for the first DMU was less than equal to 1, so when I basically take the input on to the right hand side and then again bring it back to the left hand side the equation now becomes like this for each and every constraint.

The bundle of outputs, the bundle of inputs so there is the minus sign here and less than equal to 0 because I am considering the efficiency less than equal to 1, similarly in the second constraint it will be the bundle of outputs minus the bundle of inputs is less than equal to 0, similarly for the third one bundle of the outputs minus the bundle of inputs is less than equal to 0, I continue till I reach the K th one which is the bundle of outputs for the K th DMU which is tenth one minus the bundle of inputs for the K th one is less than equal to 0.

And this one which I have already discussed the denominator from the objective function has now come into the constraint, such that for now for the optimization problem I have converted that from a ratio one to a maximization of a linear function this is a linear function the constraints are already converted into a linear one because the ratios of less than equal to 1 has been converted into some part minus some part, bundle of outputs minus bundle of inputs is less than equal to 0, for each and every of this K th or the tenth DMU the ten number of DMUs and the eleventh one constraint is basically the bundle of inputs which is coming from the objective function.

So if we have 10 such DMUs initially we have the maximization of the ratios of 10 DMUs corresponding to the fact for each of them the constraints were the same so that means there were 10 such constraints, once it is converted now we have an objective function which is a maximization of the linear function, for all 10 there are different objective functions and obviously it was different in the initial case also, now in the second conversion we have maximization of the bundle of outputs for each and every DMU, the constraints 10 in number

are exactly the same, bundle of output minus bundle of inputs is less than equal to 0 and each of the tenth objective functions now we have the eleventh one which are different, the first one would be corresponding to the bundle of inputs for the first one, the eleventh one for the second objective function will be the bundle of inputs for the second DMU so and henceforth till the last one which is the tenth (opt) optimization problem, the eleventh constraint would basically be the constraint corresponding to the fact that it is the bundle of inputs for the tenth one is equal to 1.

So in each of these they are 1 and once you see the problem it is a very simple optimization problem which can be solved using the concept of linear programming and you can get the answer using the corner point solution where you basically finalize the tab low then considering and obviously if they are less than type you will add the or greater than type depending on the minus and plus sign whether they are you can basically add the slack and surplus and also basically use the concept of artificial variable if required, I am just giving you the background and then we solve the problem and get the values of the decision variables as required. So with this I will end this the second lecture in the eighth week and continue more discussion about the input oriented model for the DA and then go into the other concept of integer programming. Have a nice day and thank you very much.