Introduction to Stochastic Processes
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Lecture 44
The Elementary Renewal Theorem

So, now we have some understanding of how this function empty looks like but we have just define one process empty, based on our X size and Z case we can actually define more random processes right. So, let us see what all we can discuss, what we can we define. So okay, before we that some properties.

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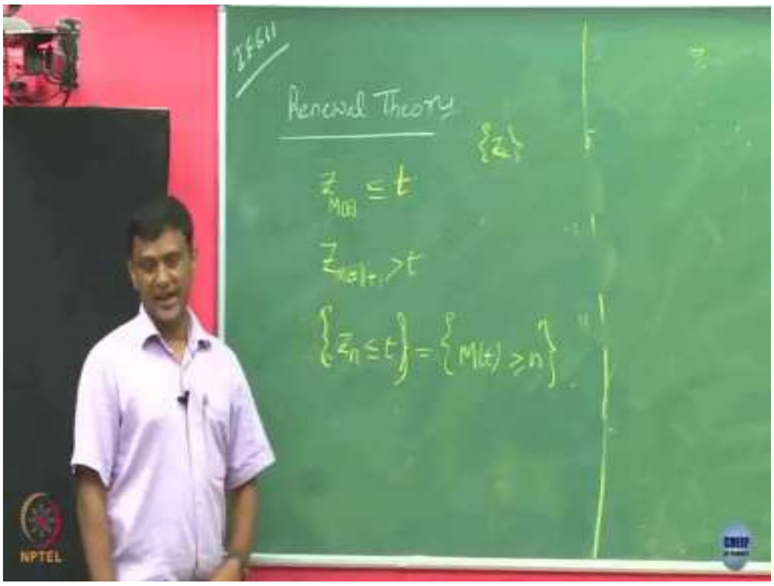

Let us take, Z M of t so, Z subscripted by M of t. So this M of t we know is a integer value, so what I am looking at is a renewal happening for the M t th time or M th renewal okay. So, given a t you know that M t is going to denote the number of renewal that has happened in the 0 t interval okay and if you look at Z of M t, so what is this, this basically now I am looking, so Z t is , Z k is a sequence of random variable right?

Now M t is a random variable, I can treat this as a random time and now I am asking the value of Z at this random time M of t right. Now, my claim is this quantity is going to be less than or equals to t, why is that? So, what we are doing is M of t by our definition told me included all the renewals that has happened before time t, right.

So, naturally any renewal that is happened before t must have happened before time t. So, because of that this property is natural. When you are going to look at M of t th renewal of your process that then M of t th renewal should have happened before time t. So, just go back and try to convince yourself all this properties.

Now, similarly what about M of t plus 1 greater than or equals to t or strictly greater than, it has to be greater than t right, why is that? Because by definition M of t has included all the renewals that has happened till time t, if we are looking at the one more renewal that better happened after time t right so, that is why this property holds. And one more property is that is we are going to exploit these two events.

Let may be this two sets, they are the same why is that? So we are saying that suppose let us say this event implies, let first say this event is a sub set of this. Suppose, Zn is less than or equals to t, what does this mean, the nth renewal is happened before time t, now if nth renewal is happened before time t then better be like M t should be larger than or at least equal to n, right.

Because, if we already know nth renewal has happened before t may be there could have been possibly more event, renewals are happened just before t, so because of that M t has to be at least n or more okay. So, we know that this event is included here but, what about the other case. Suppose, if you know that M of t is going to be greater than or equals to n that means in the interval 0 to t at least n renewals have happened okay.

Then, it must be the case that the nth renewals must have happened before time t, right, nth renewals must have happened before time t, so because of that this is also included in this, so, you can now it is clear that these are the same sets if not we had just go again and convince yourself okay. So, these are some property which we exploit in coming up with some interesting results.

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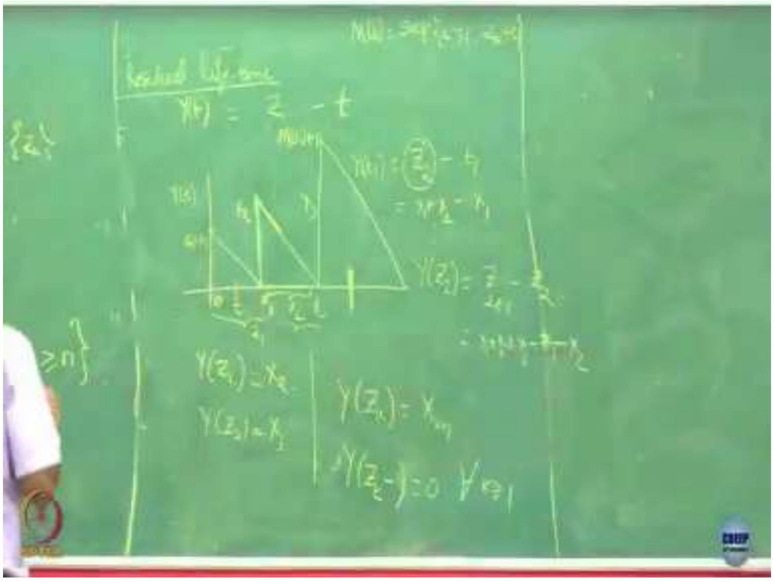

Okay now define an another process, I am call Residual Lifetime. So, I am going to define a process Y t now capital Y of t that is define for every t in this format Z of subscript Z of M t plus 1 minus t so, what is the actually capturing. So, before we try to understand what this is actually capturing let us try to draw how this looks like. Okay this is t and let us say what is this Y t, so at t equals to 0 let us say take M of t to be 0 so, at t equals to 0 what are this, it is Z 1 right minus t so, t is 0 that means at time 0 this is Z1 but Z1 we know same as X1 by definitions, Z1 is same as X1.

So, at time 0 so this quantity is Z1 or this is same as X1 okay and now let us increase t, okay as I increase t at some point the second renewal will also be included right and then as I increase this t, so as I increase this t, till some point this guy M of t is going to remain to 0 only after that it is going to increase to index 1 okay let us say for time being is it not going to increase.

So, in that case this guy is going to be Z1 and now there is a minus subtracting time here what is this going to happen as t increase, it will be decreasing, it will be decreasing till what time? Till, first arrival can happen right? And what is that? Can X1, at X1 arrival has happened so, because of that this guy suddenly becomes 1 now, it was 0 so far and now it is Z2 right and what is this now Z2 and in the Z2 you are what is this t, t is already X1 it has taken already 1.

So, what is this jump in this case, is it going to jump at this point, what is this jump? It is going to be the amount of X2. Once it jumps to X2 after that these guy is not immediately going to make 3 right, it remains at 1 till some point but these guy is continue to increase right, so it is going to pull it down and what this guy is going to pull down to? It is going get to 0, so this guy at this point till this point this is already X1.

Now let us say I take it up to X2 here that time this guy is already so, when I take it from okay let us say I increase it by amount X2 here so, then this guy becomes X1 plus X2 right but, X1 plus X2 is Z2 actually. So, this has made it becomes 0 so, this height here and this heights they are the same.

So, it is falling down to 0 at what rate, that unit negative rate right? So, you can just keep on plotting like this. So, now based on this what you can see now again coming back to my battery when I charged it, its battery is full whatever its inherent life was that time as a time process its life goes to 0, 0 so that is why we are going to at any time t you take an arbitrary time t this process is going to tell how much more time, how much time is residual time is left in that before that guy dies.

So, Y t the function, Y t at any time t gives me that time remaining before a next renewal happens or in the battery example case the time before the battery gets discharged. So, till this point you are clear, when X1, when t equals to X1 not there, so better to write this function. So, let us take t equals to X1, X1 so t equals to X1 that is X1 is basically Z1 right, so Z1 is definitely less than or equals to Z1, so k is included in that and M of t becomes 1 at this point.

So, at this point what is this Z of 1 plus 1, so this is exactly Z2, Z2 at this point suddenly at this point it has become X2 why because, this X is already X1 and this guy is so, okay Y of t1 is equals to Z2 minus t1 but, t1 we have taken to be X1, so this is going to be Z2 I know is X1 plus X2 minus X1 this is X2. But once, I come here till my t increases by amount X2 my k value of M of t is not going to change right.

So, till this time this guy is going to remains 2 but, this guy is increasing. So, it is going to pull it down right, but suddenly when you have increased till X2 then your t becomes X1 plus X2 which is Z2, so this will make your M of t, 2 it will make so, because of that now you become Z3 right. So, Z3 but, t is so at time Z2 you have this becomes Z2 plus 1 minus Z2 right but, this is X1 plus X2 plus X3 minus X1 minus X2.

So, this is suddenly here X3 jump and then it is going to, going to go down to 0, so like that so, basically this function is Y of t is going to capture so, if you just tell me what is the time, if at this time this function defines how much more life is remaining, that is why it is called residual lifetime.

So, if I am going to just take Y to be Zk here at time t I going to substitute Zk what we expect this value to be, so when I had so this duration is what basically Z1 right, Z1 and X1 are the same. At this point, this value happen to be X2 and so okay just let us write this Z1 and what happens Y equals to Z2 that is at this point, it has become X3, so in general based on this can I write Y of Zk is equals to X, k plus 1 and how what about Y Zk minus just before the kth renewal.

Student: (())(16:00)

Professor: So, what is happening just before the jumps it is 0 right, this going to be 0, this is going to be for all k greater than or equals to 1.

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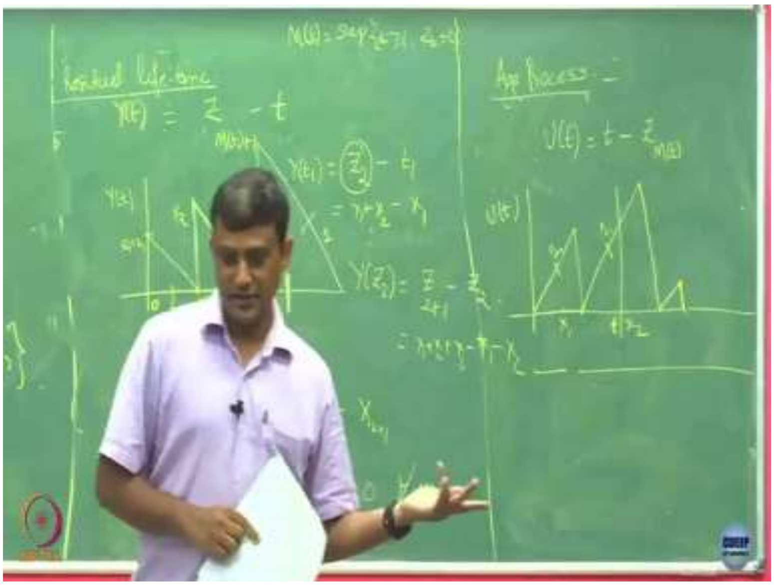

Okay, so now let us define another process, okay this is called Age Process. U t I am going to define as t minus Z of M of t. Okay so, I already argued that Z of M t is going to be less than or equals to t because, of this is this quantity is going to be positive or non-negative? It is going to be non-negative, right, because, Z of M t is going to be always less than or equals to t.

So, let us plot this, so at time t equals to 0, M of t let us take it as 0 and Z of 0 let us take it 0 because, Z of Z process is define for k greater than or equals to 1 right, so, for Z of 0 let us take it as 0. So, at time t equals to 0 what its value, it is going to be 0 and as I increase t what is happening? Right, this guy is not going to suddenly change it is going to remain there for some time but this guy t is increasing and it is increasing, increasing till what time?

At some point when t I take to be X1 suddenly this guy become Z1 right, at Z1 at t is also Z1 so, its falls to 0 what is this guy, this guy here distance X1, X1 and what is this value, U t here right, it has also increased by X t amount right, because, till when it is increasing this guy part was 0 so, it was like U t equals to t that was the kind of linearly increasing, this is also of the same height .

Okay now, it has fall on 0 at t equals to X1, now after that if I go t to just beyond X1 this guy is not going to suddenly change, it is going to still remain X1 but this guy t is now increasing from X1 so, what you expect it is increasing, till what point, X2 then, so this is with slope 1, slope 1 and so was here.

So, can you now see why it is called age process? So age always increases right, let us say, you have replaced your battery at this time and you take some time at this point so, what it basically tells you is so, basically if you just look at this time t and it has so happened that you have basically replaced your battery at this point your age has at this point is this much.

See like one so, the thing here because I am subtracting this amount, this is basically capturing that I have renewed, replace the battery, so even though I am looking at this time I am not counting age from this point, I am counting the age from the point where it has been replaced or recharged. So and that part is trying to get rid of the things still that it only capturing what the time since the battery has been replaced.

Okay so, these are this processes this age process, residual process they try to help us understand okay how frequently may be if I can characterized this processes for my system then they will tell us how frequently my component is going to fail even if I replace them and that will tell us how quickly I should take it for repair. So if I know this, this is going to fail with this kind of behavior I can come up with a criteria okay before this guy fails I will make sure that I will do some maintenance work.

So, that even that guy, so when things fail you are in loss right, may be like if you have a company or something if some component there fails, the machine is not working you are not basically earning money. So, what you want is your, you do not want your machine or your process to stop, so if you kind of analyse and anticipate when it is likely going to survive or what is the quality of my component then before it gets bad you can plan a schedule repair process and make it continue to work without interruptions.

Okay, this can be, next one can be this, these are realizations right like you charged it and it work for longer time for some reason and next time and it became bad and when you again charge it may be it become bad again okay. Okay, so, I have basically the process, the renewal process we have defined right it was IID process for the point n greater than or equals to 2.

So, to describe all the points after n greater than or equals to 2 I just have to specify one distribution and the first cycle could have different distributions so, if I just specify you the distribution of the first cycle so, I am going to interchange the use cycles and renewals now like when a renewal happens I am going to call it as one cycle completed.

Okay so, if I specify the distribution of the first cycle and the distribution of the second cycle that is enough right, that completely specifies my renewal process here because that is the only two things I know, I knew because these are independent process and for k n greater than or equals to 2 this is all IID.

But, now suppose I have this information can I compute the quantities of my interests, for example, can I compute the distribution of my Zis, can I compute my distribution of my Mt right and can I compute the expected value of my Mt. So, let us see how can we do that.

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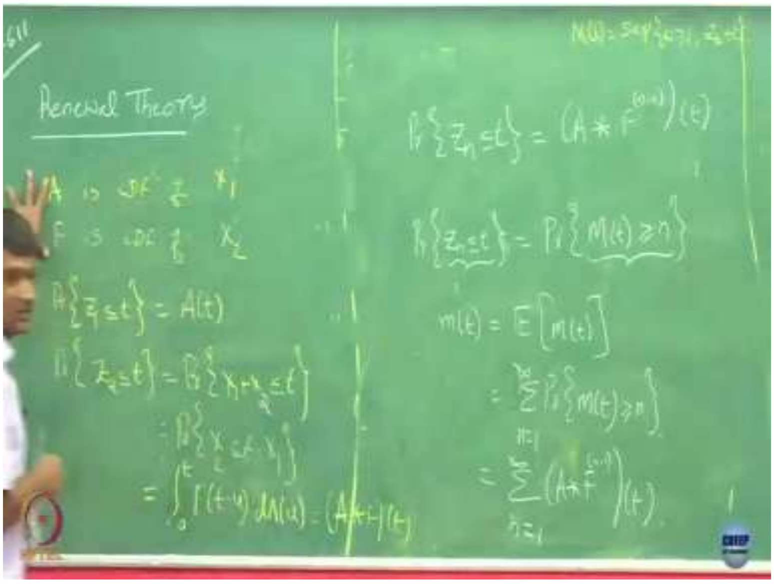

Suppose, A is CDF of X1 and F is CDF of X2, so if I give these two CDF s is it enough for to characterize my renewal process right. So, the first cycle is distributed as X1 and the subsequence are distributed as per X2. Now, if I am going to capture what is the probability that Z1 were equals to t what is this going to be so, that can be a computed, this is nothing but the CDF of X1 right because Z1 is same as X1. So, this is simply A of t okay.

Now, what about probability of Z2 greater than or equals to 2. So, Z2 now is X1 plus X2 t which I can write as X2 less than or equals to 2 X1. So, you guys are already dealt with such cases how to find out the distribution of this, so how you are going to find out the distribution of this by using double integrations right, you first let this X1 take the values as per distribution and for each possible values then you compute X2 upper bounded by that value as per the distribution of X2.

And you have when you do this you will end up something like integration of okay basically, will end up with something like 0 to t f of t minus u and dA u I am just writing this is just like simplification of this which you can basically, I can think it as a convolution of A and F process at time t.

So, this is then you can just like, just think of this as you can express it as a convolution of your distributions A and F here. Now, you can repeat this process for any n now, we can now think of the question what is the probability that Zn less than or equals to t. So, you can work out this I am just skipping the details it can be thought as a convolution of F and all this processes computed like this.

Yeah so this are basically convolution of A and n minus 1 times the F functions, convolution of this computed it A. Now, based on this now we are ready to kind of find the distribution of M of t why is that? I know that probability of Zn less than or equals to t is equals to probability that M of t greater than or equals to n, we have already shown these two events are equivalent right.

So, if these two events are the same then their probability be better the same. So, you see that now we have this, now how I am going to use this further information to find, so fine, I have this distribution so based on this I know just using A and F I will able to find the distribution of my process M of t.

Now, how to count the expected value, so, we know that this expected value can also be come found as probability that so, I know that my M of t is going to take integer value, integer values. So, this is going to probability that which is equals to n equals to 1 to infinity, it is correct? This also you know, right the expectation of a random variable I can express it like this.

So, here M of t is taking discrete value outcomes that is why I am just writing summation had it been a continuous value, I had to replace it by integration. Fine, so, then I know this, so now going back to this relation and this relation I have this is nothing but n equals to 1 of A into equals to this. So, you just see that just by knowing these two CDFs of a renewal process I should be able to compute all the statistics of a related process. These Zns, these M of ts and its mean values, okay.

Student: (())(30:55)

Professor: This is equivalent to computing it CDF, it is a discreet value, right, CDF should be enough for us so, just use this, you already have, what more you want.

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Okay fine, so let us conclude with the main result of this part just going to state it as a theorem. This is called Elementary Renewal Theorem. Given, lifetimes so it requires three hypothesis so, X2 almost surely here. So what it says is okay as usual we are going to take a mutually independent non-negative sequence of random variables okay. Such that there IID distributed for n greater than or equals to 2 further let us say the probability that each of the renewal times being finite is 1.

Yeah so all may x case are finite with probability 1 and I am also allowing but I am allowing this X k to take the value infinity okay so all I want is these expectation of X k to be any value but I want them to be strictly greater than 0 okay all these quantities for all k. If this happens so, notice that this expectation of k they are same for all k when k is greater than or equals to 2. Expectation of X1 could be different from expectation of X2 but, expectation of X2, X3 and all other are going to be same.

Now, it says that the mean number of renewals in the interval 0 t as t goes to infinity is 1 by expectation of X2, so what is M of t? The number of renewals in the intervals 0 t, when you are dividing it by t that means basically mean number of renewals in the interval 0 t okay and you are letting t go to infinity.

So, when you are going let that go to infinity that rate is going to be like 1 upon expectation of X2. Okay let us, try to understand why expectation of X1 is not coming into picture, why only expectation of X2 is coming into picture. Okay so, see that for k greater than or equals to 1 we are assuming that expectation of, sorry probabilities of x case less than infinity right. So, that means I am going to hit state J whatever state I start from in some finite time, right.

That is the why it is X of probability of X1 is less than infinity is so happens is probability1. So, at some point I am sure I am going to hit state J and after that I am looking at coming back to the state I, J again and again right. And now from state I to some state J I have done that in the first cycle within some finite time and after that I am looking at again coming back to J again and again right.

So, the first cycle has some finite time but here I am looking at t letting go to infinity what are the contribution that has happened from initially some state to coming to J for the first time that is going to be vanishing right, it will not contribute, but what contribute to the sum of the other terms. So this M of t is going to include all the renewals till time t right, the first renewal has happened sometime, it is going to take some portion.

But most of the other time what happened the other renewals have happened in the other time. So, the time contributes to whatever like this, the first time contribution to M of t is going to be very small right and now because of that, that does not going to affect expectation of X1 is not going to affect this.

But, why that then the thing is that the others is happening at this rate, fine. So, in M of t most of the renewals are like second, third right like other than first once and now, all of them are kind of identically distributed X2, X3 are all of them and now when you are going to divide it by t this is basically you are asking number of returns to state J over a period of time t.

So, in a way that has to be inversely proportional to so, this your basically on an average how many returns to state J are happening right. So, basically there okay if you are going to take a sometime t and then you are going to divide it t so number of renewals by t basically happening on so, how much time it took for one renewal to happen. So, the one renewal happening time you expect it to be inversely proportional to the expected time of that renewal itself right.

M of t is the number of renewals that are happened in the time t, so what here are happening is basically per unit time how many renewals are happening right so, this some number of M of t number of renewals are happened in the interval 0 t by dividing it by t so, then how many renewals are happening per the unit time? That is the question you are asking, right.

So, we are saying that, that number of renewals per unit time is 1 upon expectation of X2 right, if something is happening so, you are repeating so Xt is basically telling you at what time you are repeating right. So, if you are repeating very fast, so expectation of X2 is going to be very fast that means you are repeating very fast that means expectation of X2 is small.

Because, you are happening again and again very frequently, if this is small then you basically going to see lot of renewals in the given interval right, so that is why if this expectation is going to be small, you are going to see lot of renewals in a given duration. If this expectation is going to be large that means the renewals are happening after long-long time.

So, you do not see more number of renewals in a given duration 0 t so, that is why this number of renewals per unit time is going to be inversely related to the how frequently you are going to observe this renewals okay. And this is almost sure result because, this Mt are random quantities so, that is why this is random.

And now the last part of it says that, if instead of M t you look at this expectation that is again converging to the same limit. So, you what was our large number set, summation of xi by n it went to mean value, right, is it necessary that always the average of random quantities has to converge to some random variable, it can converge to a constant and that is exactly is happening.

So, this is exactly saying that the average number of renewals is a constant and that constant is given by 1 upon expectation of X2. Okay so, this quantity here is nothing but the expectation of this but as usual we cannot directly derive this quantity from this because, in general we cannot just take expectation of this quantity here because, expectation and limits cannot be interchanged it needs some careful analysis.

So, because of the last perfect time what we will do is, we will not going to the prove of this, the prove is actually interesting but in the last class we will just cover one more theorem called Renewal Reward Theorem and just stop it there, okay.