Introduction to Stochastic Process
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Lecture 20
Properties of Poisson Process (Part 2)

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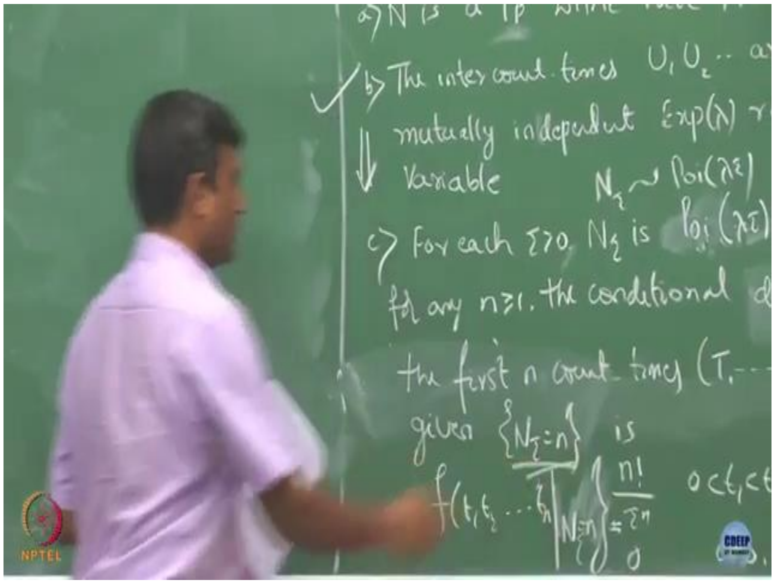

Okay, now let us try to prove B implies C, this is a bit involved, but let us see if we can argue this. What we want now is, suppose we assume this B then I want to show that for any tau, n tau is Poisson distributed with rate tau lambda and further the conditional joint distribution of my count times is expressed in this format, okay. So now let us first focus on what does this event means?

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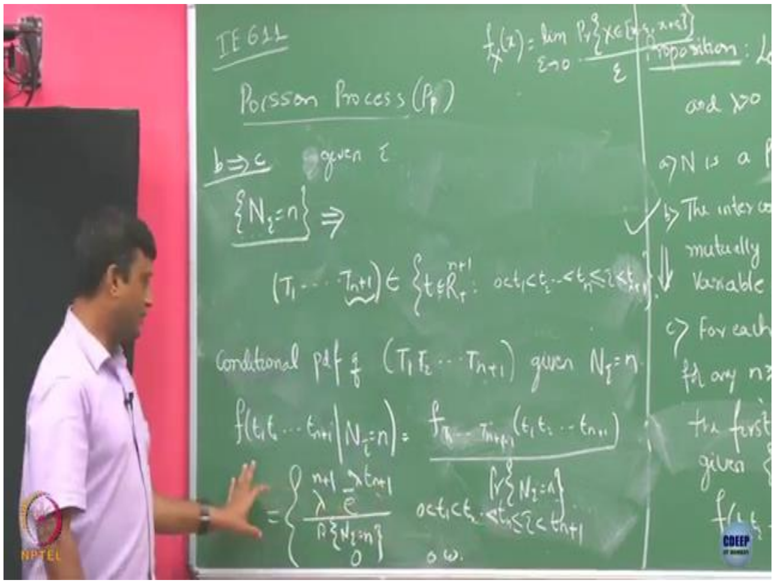

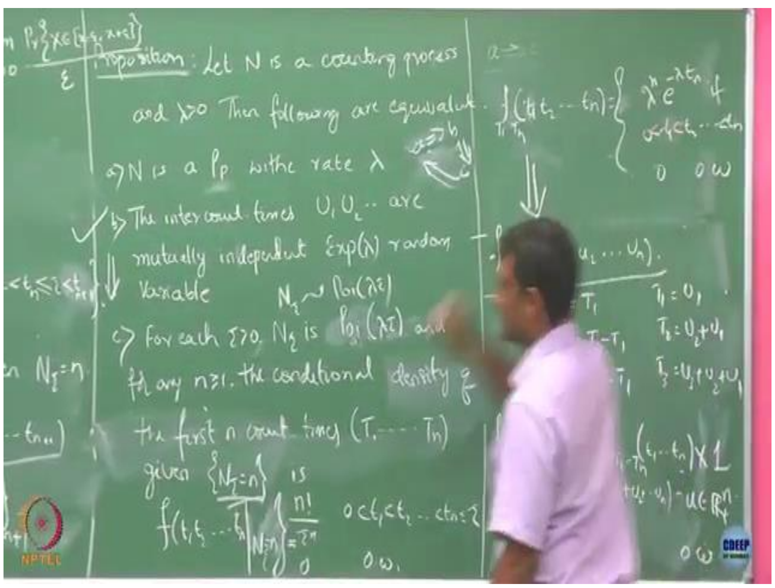

So, given tau, n tau equals to n means by the time tau, n counts are happened, right? That means n tau equals to n means exactly n count might have happened before time t and plus 1th count should have happened after tau.

Then that that is when I am going to call n tau equals to n right. So that is what I am writing here. So, that means like if I am going to look at my n of this random variable, T1 all the way up to Tn plus 1. It should be such that if you are going to look at the realization, T1, T2 all the way up to

Tn should that is the times of this counts should have happened before or at least by the time tau including tau and the T plus 1th count time should have happened after tau.

This happens, when this happens that is what we are going to call n tau equals to n right. If this T plus n has happened before tau that means n plus count is already happened before, before time tau right. So, that is why I have to express this completely also, have to include this Tn plus 1. Suppose I just say this and leave it like this. It is not clear that this is going to capture this because I have not told you whether the Tn plus 1th count has happened before tau or not.

If that has happened before tau then this is not correct. So, for this to make sure I have to explicitly said that the n plus 1th count happened after tau, okay. Now let us try to find out the conditional. So, how I am going to obtain this? I want to find conditional PDF of this set of random variables given n tau equal to n. So, how I am going to leverage already the things that I know, I know that.

If you just forget the conditional part the joint distribution of this n random variable I already computed it, right. I have this distribution, okay. But now I have this conditioning here, what does this conditioning will imply to me? So because of this conditioning, if you go back and recall our conditional density property, because of this conditional link, I will get an extra term which divides this PDF here, this probability is the one which divides your. So let me make that clear here.

So, if I want to, so I am just trying to use the definition of my conditional probability that we have defined earlier. So, because of this conditioning extra probability is showing up here in the denominator. Now, if you just apply this definition here. What I will have is this is nothing but. lambda to the power. Now n is going to be replaced by what? In this formula, for how to use that formula here, I have to replace n with n plus 1 and then what?

E to the power lambda times Tn is going to be replaced by Tn plus 1 right, because I am looking at Tn plus 1 here, and then this term is there and this is for the case where my t1 is greater than t2 and tn and then this is a equal to tau and this is equal to tn plus 1 here and it is going to be 0 otherwise. Let us see that, when I adopted this formula from here to this case, that ordering earlier by ordering did not include any tau here, right. It was just t1, t2 all the way up to tn arranged in strictly increasing order.

But now that I have this tau factor here, so this joint PDF is defined on this T such that these Ts are strictly increasing and such that this Tn plus 1 is strictly larger than tau, that is where I could use the PDF in this format, okay. This is because I have to ensure that I have been conditioned on the fact that there are n count still time tau, okay. Now, so we are there like we wanted to define this distribution, conditional distribution.

We have this conditional distribution but the only thing that has thrown extra is we have ended up with this tn plus 1 variable here. But whereas what we wanted to show did not include tn plus 1, right. It was only, it was only talking about joint distribution of n random variable here. Now how to derive? So I have this joint distribution, which is now of n plus 1 random variables.

Now, if I want to get a joint distribution of only in the first n variables here, how to get that? So, just integrate this function over what?

Student: tn plus 1

Professor: tn plus 1 and what is the possible values of tn plus 1?

Student: (())(09:25)

Professor: Tau to infinity, right? It could be any value. So, if you just do that integration, what you will end up is.

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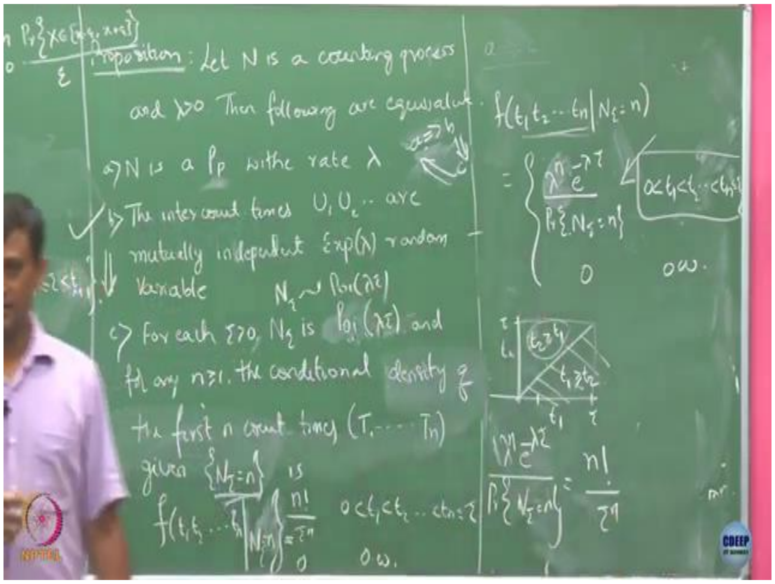

So if you just do this integration t1, t2 all the way up to tn given n tau equals to n So, after doing this integration, what you will end up is e to the power lambda, sorry, lambda to the power n is there anyway. And then tau divided by probability that n tau is equals to n and this is for all 0 t1, t2 al the way up to tn. And this is (())(10:38) equals to tau, this is going to be 0 otherwise.

So I have just integrated this function between tau to infinity. If we just write like, I am just ended up with this function. So, now if I just look into this function, does this function depends on any of these variables here? Earlier than I have this function at least is dependent on the variable tn plus 1, which was my argument. But now after doing this integrating out, it is only lambda to the power, e to the power lambda tau and the denominator also, this only my conditioned n tau equals to n.

Now what is this? I have a distribution, I have PDF and this is a constant, a PDF which takes only constant value. It should be, then what this is, it should be kind of uniform distribution, right. Like I have ended up with the distribution which is uniform and if it is a uniform distribution and also we know that a PDF should be such that, if you integrate it over its region, it should end up what, it should indicate what?

So if that is the case and if it is a uniform distribution or it takes a constant value throughout, what should be the value of that CDF itself at every point? The function itself, so okay. So let me refresh it. So, if I had a PDF which is independent of my argument. That means at every point it is going to say the same value, right. Like, for example, in this case, you give me any point, it is going to take the same value.

Further, if I going to integrate this over the entire region, it should equals to 1 that means necessarily that at every point this function is nothing but reciprocal of area, right, then only if I am going to integrate and if it is constant then I will end up with a 1 value. So, in that case, this value should whatever I got here should be reciprocal of the area of the region, where this ordering holds, you understand this?

So, let us imagine for a 2-dimensional case. I am letting tau each one of these to go up to so I have this t1, I have this tau and this is I am only letting it go till tau-tau because both t1 and t2 are random variables which can take utmost value tau, right. Now, what is the region here where t1 is going to be greater than t2? So, let us say I draw a 45 degree line, in all of this region t1 will be greater than t2 or t2 will be greater than t1?

T2 is going to be greater than, right. If I am going to fix this and if I go all the way up the t1 will be. So, I have this region right. So, I am just let us focus on this, so if you just focus on this. What is the area of this entire area? Tau square and now what is the range where t this ordering okay. In this case, I am interested in the t1 less than t2 right. So this, I mean I can ignore this part because that line itself do not have any area.

The case where t1 equals to t2 I will, what is the area of this region? That is going to be exactly half, right? Half t square by 2. In that case, it is going to be tau square by 2. So, similarly if you extend it to n dimension that area where this ordering holds if I am going to test it what, that will be any guess?

Student: tau raise to n divided by n factorial.

Professor: Tau raise to n divided by n factorial, why that should be?

Student: That is the result.

Professor: That is the result okay, fine let us take it as a result, okay. So then, what this, what you expect this value to be equal to? 1 by that quantity, right tau to the power n divided by. So, (I) we expect this to be equals to reciprocal of that is n factorial t to the power n okay and you see that that is what I have already mentioned here, but this is just like one step of this, right? Like, I have to show that for each tau this is a Poisson random variable.

What I have just showed is, this conditional distribution satisfies conditioned on this n tau equals to n is going to be this n factorial by tau to the power n. Now, how to show this, that for every tau get to that 1, n tau is a Poisson with rate lambda, lambda T. Whatever this relation I have you If I go ahead and manipulate it.

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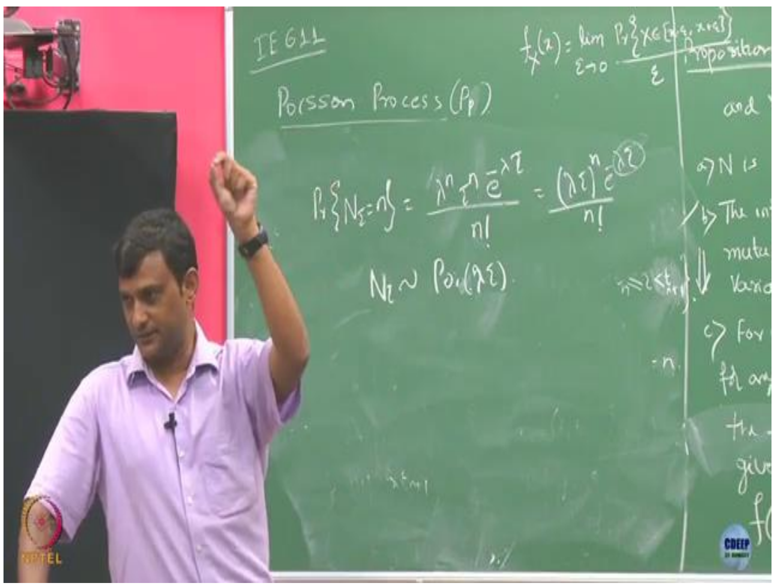

I am going to get probability that tau n tau equals to n is equals to what, lambda n tau n, e to the power lambda t, lambda tau divided by n factorial. So, what is this? So what is this the random variable then n tau? So, I am saying that the probability that this random variable n tau takes value n is expressed in terms of this, but now what this looks similar to or what this exactly is? Poisson with what? So n tau is Poisson. So, I could just write this as lambda tau to the power n and e to the power lambda tau divided by n square right. So, this is my rate and this is Poisson with rate lambda tau. And that is exactly our claim here.

So, in a way what this third point, c point is telling is, a Poisson process can also be thought of like an uniform random variables, a collection of independent uniform random variables where that n points occurs can be thought of like they are going to happen uniformly in the interval tau, that is each points I am going to draw, that is going to happen uniformly with parameter tau. But I have also this constraint that this points need not be arbitrary, they have to be in increasing order, right.

So, that is (tau) t1 has to be greater than t2 like this. So, because of this I have end up with this n factorial term here. Okay, now quickly argue why if this is true this implies that my process is Poisson, okay. So, now what I need to show? If I am going to assume this C property, I need to show that all the 3 properties of the, If I assume that some process is such that it is going to satisfy all it is satisfies point C, then I have to show that this also satisfies. This implies 3 properties of my Poisson process okay.

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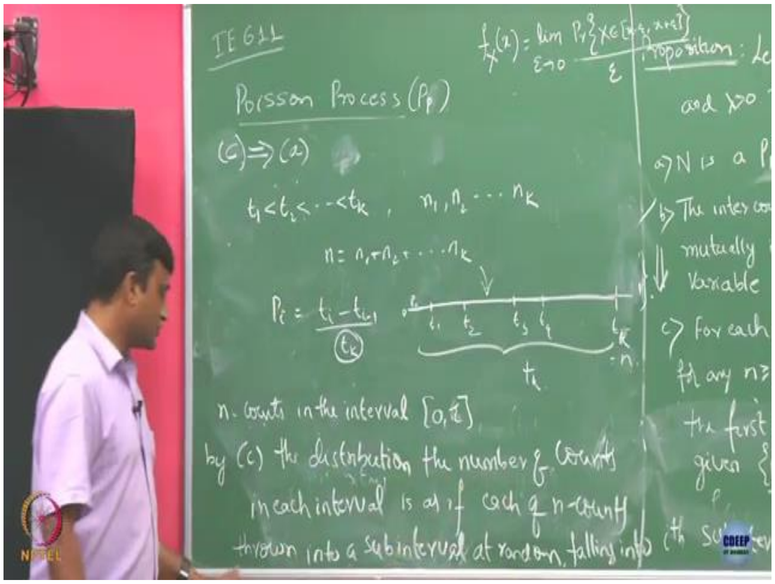

Now let us try to see C equals to a. Suppose I am given this let say t1, t2 and up to tk I am given this tk variables and also this n numbers n1, n2 all the way up to nk and then let us say I am going to define n to be n1 plus n2 all the way up to nk. So, think of this I have been given as the time, count times, t1, t2 all the way up to tk. Now I am going to define and also imagine that this n1, n2, nk are nothing but the number of items you want to put in the interval 0 to t1 is n1 and that between t1 to t2 as n2 like this.

So, I have total n items here, n1 which is the sum of all these items. Now I am going to define some Pi here, Pi is going to be defined as ti minus ti minus 1 divided by tk. So, what is this?

Suppose I have let us say t1 here, t2 here and t3 is bigger and t4 is smaller and I have some t5 here and this is like starting 0. So, p1 is going to be proportional to this interval, p2 is going to be proportional to this interval and p3 is going to be proportional to this, like this.

So, notice that (ti) tk is constant, that is the total length, let us say this is tk. So, this is tk and now for every Pi I am just going to take the length of the interval. So, I have basically defined probability Pi that is (corres) that is proportional to the length of the intervals. Pi I will just defined now. So, I have just, I will say what, where I am going to use it like Pi is I have defined to be a number which is defined like this.

You can think this as like a probability term here. So, if you just add it over all Is, it is going to add up to 1, right between 1 to k here. You have to convince yourself like if I am going to use assume this c holds and I think that, okay just let me write this. So, by c I am assuming c holds right, by c the distribution of.

So what we are saying is if because of this uniform distribution as a interpretation we have from this C point we can assume that, if you are going to define this Pis to be probabilities proportional to the interval n and if I have n numbers, let us say given to me and if I throw this n on items on this interval probability that it falls in this interval is going to be proportional to the length of this interval, okay.

So, that is what we are going to now use. So, the distribution, the number of counts in each interval is as if each of the n counts thrown into the sub-interval at random falling into the ith interval is going to be Pi.

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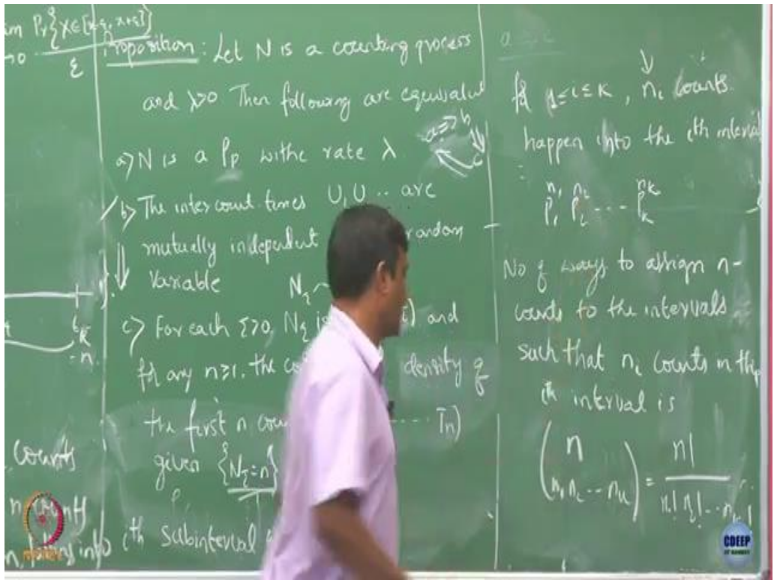

So now, so if we say that we have N items and ni items are going to fall in the ith interval. So, what it is tis probability is going to be? Like if I have n objects and I throw them n1 of them go into the first interval, n2 of them go into the second interval and nk of them go into the kth interval.

What is the probability is going to be? Like I am choosing each one of the object independently and I am just throwing, every time I throw the probability that it is going to fall in ith interval is pi which is independent of the other. So, then what is the probability that ni counts go into the ith interval is? It is going to be p1 into n1, p2 into n2 into pk into nk. So, this is for a given n1, n2 all the way up to nk, right?

But when I have n total counts, in how many ways I can partition them into n1, n2 all the way up to nk such that their sum is equal to n? I have a multinomial distributions, right. In this, so the number of ways such that ni counts in the ith interval is simply n choose n1, n2 all the way up to nk which is defined as n factorial divided by n1 factorial n2 factorial all the way up to nk factorial, right.

So, I have n things in different, like where the count is happening I am just like throwing them, right? Like I am just interested in how many of them go into the ith interval. And then I am just looking at the probabilities in this fashion. So, for a given n1, n2 like this, this is a probability, but right now I do not know how this n and total n gets split across this, right? So, I have this multinomial distribution. Now finally how to explore this now.

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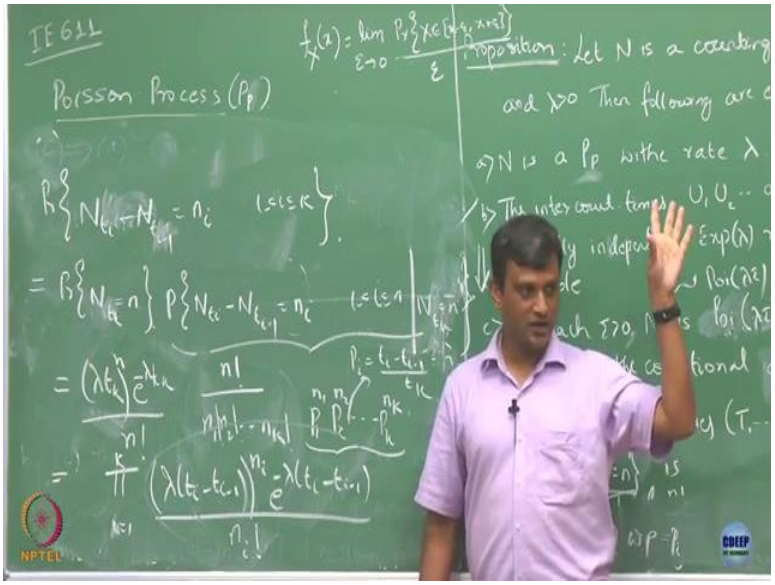

Now I have this, so suppose let us say N ti minus N ti minus 1 is equals to ni. Let us say for I equals 1 to n. So, let us compute this probability. I want to compute that, okay in the interval between t1 and let say ti and ti plus 1 or between ti and ti minus 1. So there should be i minus 1 there are ni objects, okay. This can write it as N ti, N tk equals to n total number of objects and probability that N ti minus N ti minus 1 equals to ni between I and k given condition that N tk equals to n.

Now, coming back to this c third property again here, from the third property that point c here, we already know that this distribution is what probability that Nt equals to n. This is a Poisson distributed with rate lambda tk, right? That is the length of the interval here. So, this is going to be lambda tk and how many counts till then? There are n counts e to the power lambda tk divided by n factorial. And what is this? So, I am basically asking there are ni counts between in this interval and there are n such intervals, right?

So, how I am going to do? I first going to look for a possible realization and then when I know that possible realization, I have this is the probability associated with that. So, I will consider it all such possibilities, right? From there are these many ways of choosing, these many ways of partitioning this n into n1, n2 all the way up to nk. When I have this kind of partition, what is the associated probability? Associated probability is p1 to the power n1, p to the power nk and all the way up to p to the power k, nk.

So, if you just replace this guy by its n factorial divided by n1, n2 all the way up to nk factorial and further if you replace each of the probabilities as we defined earlier in terms of their intervals, that is just you will just replace this pi by what? Ti minus ti minus 1 divided by tk. If you just replace what we are going to get, t equals to I k equals to 1 (sorry) I equals to 1 to k. So, you just verify that, I am just directly writing this, okay.

So, what is this now? I am finally able to express this probability here as the product of k (prob) terms here. What is each time here? If you just focus, let us take one particular i here, I is ranging from 1 to k, right? Let us take one particular k. So, if you just look into that, what is this is going to be? So, that is going to be a Poisson with rate lambda ti minus ti minus 1. That is a lambda, that is a random variable with Poisson distributed with rate.

And this is nothing but the probability that, that random variable taking what value? Ni value and that is exactly this value, right? For particular I, but now what I am able to write it as? It is now able to write it as all this as a product of all this k values. So, that means what we have done is this probability is nothing but, so these are like in the increments, right? I am able to express these increments as the product probabilities, each probability term here corresponds to a Poisson random variable.

So, that means I am already showing that the third property of the Poisson process that this independents increment. So, each of this random variable corresponds to Poisson random variable with rate which is going to be proportional to the length of the interval. Further we have shown that through this product that these increments are independent because each one of them I am able to write it as it product fashion, right?

So, is it clear that like by expressing this, I have been able to show both second and third property that is independent increments and that each of these random variable is Poisson distributed with the rate corresponding to lambda times the length of the (())(35:29). And the first property that counting process that is obvious, right? Like we are doing a counting and you can just verify the definition of counting process on this.

So, with this, we are able to show that c implies the 3 properties of a Poisson process. So, hence C also implies, if C holds then it is clear that my process is a Poisson process. So, any of this characterization you can check and that all implies the same Poisson process. So, okay let us stop here, if you have any doubts on this you can ask now.