Game Theory
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Lecture - 39
Cooperative Games: The Nucleolus

In the previous lecture, we proved that Shapley value is the unique allocation rule satisfying certain axioms. Now, in this lecture we will discuss another notion of a solution to cooperative games which is known as nucleolus. The concept of nucleolus depends on ”excess of a coalition”.

Excess of coalition

Let $S \subseteq N$ be any coalition and x be imputation. Excess of a coalition is given by

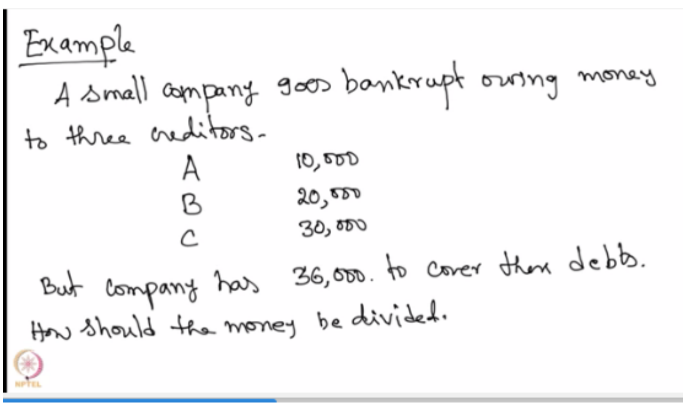
$$e(x, S) = v(S) - \sum_{i \in S} x_i$$

Which measures the dissatisfaction with an offered imputation x in a coalition S. There are total of $2^n - 1$ excess functions.
Let $m = 2^n - 1$. Consider a vector $e(x) = (e_1(x), e_2(x), \ldots, e_m(x))$, where $e_i(x) = e(x, S_i)\ \forall i = 1, \ldots, m$. $S_1, \ldots, S_m$ are indexed in such a way that $e_1(x) \geq e_2(x) \geq \cdots \geq e_m(x)$. here, for a particular imputation x and for all possible coalition (i.e., $2^n - 1$), we have ordered excess functions.

For $x, y \in \mathcal{R}^n$, x is said to be lexicographically smaller than y if there $\exists$ k, $1 \leq k \leq m$ such that $e_i(x) = e_i(y)\ \forall i < k$ and $e_k(x) = e_k(y)$. The ordering is called lexicographic because it mimics alphabetical ordering used to arrange words in a dictionary.

The Nucleolus

The lexicographic minimum is called the nucleolus of the game.
First, we will look at an example.



$N = \{A, B, C\}$

$v(\phi) = 0$, $v(A,B,C) = 36$

$v(A) = 0$ $v(B) = 0$

$v(C) = 6$, $v(AB) = 6$ $v(AC) = 16$

$v(BC) = 26$

$x = (x_1, x_2, x_3)$ efficient allocation

$x_1 + x_2 + x_3 = 36$

S	$v(S)$	$e(x,S)$	$(6,12,18)$	$(5,12,19)$
A	0	$-x_1$	-6	-5
B	0	$-x_2$	-12	-12
C	6	$6-x_3$	-12	-13
AB	6	$6-x_1-x_2$	-12	-11
AC	16	$16-x_1-x_3$	-8	-8
BC	26	$26-x_2-x_3$	-4	-5

The excess function for every coalition is calculated above.

To reach nucleolus, we need calculate excess function for different allocation vectors. And with little work, we can get nucleolus to be $(5, 10.5, 20.5)$. That is, Company should give 5000, 10500 and 20500 to A,B and C respectively. Also, Shapley value for this example, which can be calculated easily, is given by (6,11,19).

Existence of Nucleolus

Before this solution concept, we have studied two other solution concept, Core and Shapley value. We know that when the core exists, it can be multiple but on the other hand Shapley value always exists and it is unique. Now we will see existence of Nucleolus and if exists it is unique or multiple. This leads to following theorem,

Theorem 1. *There exists an unique nucleolus for each cooperative game* (N, v).

Proof. Existence of Nucleolus: Recall excess function,

$$\begin{aligned} e(x) =& (e_1(x), \ldots, e_m(x)) \\ e_1(x) =& \max_{i=1,\ldots,m} e(x, S_i) \\ e_2(x) =& \min_{j=1,\ldots,m} \left(\max_{i \neq j} e(x, S_i) \right) \\ e_3(x) =& \min_{j,k=1,\ldots,m} \left(\max_{i \neq j \neq k} e(x, S_i) \right) \\ & \vdots \end{aligned}$$

Now, all these functions $e_1(x), \ldots, e_m(x)$ are continuous and set of imputations are compact set and we know a function on compact attains maxima or minima. Thus, this implies excess functions $e_1(x), \ldots, e_m(x)$ will attain a minima. Now, if $e_1(x)$ attain unique minimum, then this minimizer is going to be nucleolus. But if it attains multiple minima then we will move to $e_2(x)$ and look for unique minima and if it also attains multiple minima then we will move to $e_3(x)$ and so on. This implies existence of nucleolus.

One can easily follow the proof of uniqueness of nucleolus from the paper by Schmeidler or a book by Narahari.

□

Remark: Computing nucleolus is harder than shapley value.