

Game Theory
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Lecture - 36
Cooperative Games: The Core

In the previous lecture, we have seen that an imputation is an allocation of payoffs to the individuals in such a way that the allocation satisfies individual and collective rationality.

Domination of imputation

An imputation x is said to dominate an imputation y if \exists a coalition C such that

$$\sum_{i \in C} x_i \leq v(C)$$

and $x_i > y_i \quad \forall i \in C.$

1) An imputation need not dominate another.

$$x = (150, 150, 0) \quad y = (0, 150, 150)$$



2) relation of domination is not transitive.

$$(0, 180, 120) \text{ dominates } (150, 150, 0)$$

$$(150, 150, 0) \text{ dominates } (90, 0, 210)$$

$$(90, 0, 210) \text{ dominates } (0, 180, 120)$$

3) It is +ble that every imputation is dominated by some other imputation. (Exercise).



So, now that we have recalled the definition of imputation and all. Now, we will go to introduce what is called the core.

Core

Given TU game (N, v) , where N is the set of players and v is the worth of coalitions. Core is the set of all imputations which are coalitionally rational, i.e.,

$$\text{Core}(N, v) = \left\{ x \in \mathcal{R}^n \mid \sum_{i \in N} x_i = v(N), \sum_{i \in C} x_i \geq v(C) \forall C \subseteq N \right\}$$

Consequences of the definition of Core

A coalition C can improve on an allocation $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ if

$$v(C) > \sum_{i \in C} x_i$$

This implies C can improve on x if \exists some allocation y s.t. y is feasible for C and the players in C get strictly higher payoffs than x .



From above we can say that,

An allocation x is said to be in Core of (N, v) iff x is feasible for N and no coalition can improve upon it, i.e.,

$$\sum_{i \in N} x_i = v(N)$$
$$\sum_{i \in C} x_i \geq v(C) \forall C \subseteq N$$

An allocation x is said to be not in core, if there exists some coalition C such that all players in C would do strictly better in some other allocation.

Examples

Example 1: Divide the dollar game

Divide the dollar game:

ver 1

$$v\{1,2,3\} = 300$$
$$v\{\{1,2\}\} = v\{\{2,3\}\} = v\{\{1,3\}\} = 0$$
$$v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$$

ver 2

$$v(\{1,2,3\}) = 300 = v(\{1,2\})$$
$$v(\{1,3\}) = v(\{2,3\}) = v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$$



Divide the dollar game (Version 1)

Core (N, v)

$$\left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid \begin{array}{l} x_1 + x_2 + x_3 = 300, \\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \end{array} \right\}$$

Version 2.

$$\left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid \begin{array}{l} x_1 + x_2 = 300, \quad x_1 \geq 0, x_2 \geq 0 \\ x_3 = 0 \end{array} \right\}$$



The core in above two versions of divide the dollar game is turn out to be infinite set.

Example 2:

Ex $N = \{1, 2, 3\}$

$v(1) = v(2) = v(3) = 0,$

$v(12) = 0.25 \quad v(13) = 0.5 \quad v(23) = 0.75$

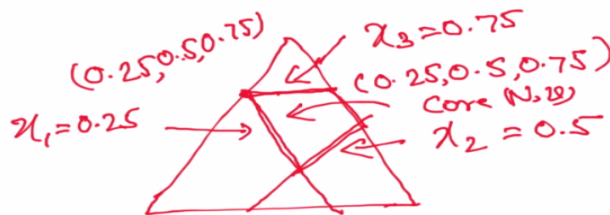
$v(123) = 1$

$x \in \text{Core}(N, v) \iff$

$x_1, x_2, x_3 \geq 0, \quad x_1 + x_2 \geq 0.25, \quad x_1 + x_3 \geq 0.5, \quad x_2 + x_3 \geq 0.75$
 $x_1 + x_2 + x_3 = 1$



$\Rightarrow x_1 \leq 0.25, \quad x_2 \leq 0.5, \quad x_3 \leq 0.75$




The core of this game is the middle region of the above figure. All the points in this region constitutes core.

Example 3:


Pl. 1 has a house which she values
₹ 1 million and wishes to sell
2 potential buyers Pl. 2 & 3.
Valuation ₹ 2 million.
have with them ₹ 2 million each
Suppose Pl. 1 sells to Pl. 2 at p where
 $1 \leq p \leq 2$.

utility of Pl. 1 = p
utility of Pl. 2 = $(2-p) + 2 = 4-p$



Pl. 3 = 2
value thing if Pl. 1 sells to Pl. 3
 $v(1) = 1$ $v(2) = 2$ $v(3) = 2$
 $v(1,2) = v(1,3) = v(2,3) = 4$
 $v(1,2,3) = 6$

$x \in \text{Core}(N, v)$ iff
 $x_1 \geq 1, x_2 \geq 2, x_3 \geq 2$
 $x_1 + x_2 \geq 4$ $x_2 + x_3 \geq 4$ $x_1 + x_3 \geq 4$
 $x_1 + x_2 + x_3 = 6$ $(2, 2, 2) \leftarrow \text{Core.}$



$(2, 2, 2)$ is unique core of the above game.

Example 4:

Glove Market

5 Suppliers of gloves

First two players can each supply one left glove
and the other three can supply one right glove each

$$N_L = \{1, 2\}, \quad N_R = \{3, 4, 5\}$$

worth of each coalition is the number of matched
pairs that it can assemble

$$C = \{1, 3\} \quad v(C) = 1$$



$$N_L = \{1, 2\}, \quad N_R = \{3, 4\}$$

$$\text{Core} = \left\{ \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \right\}$$

$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)$ is the core of above game. So with these examples, we conclude this lecture.