Game Theory

Prof. K. S. Mallikarjuna Rao
Department of Industrial Engineering & Operations Research
Indian Institute of Technology – Bombay

Lecture - 01
Combinatorial Games: Introduction and Examples

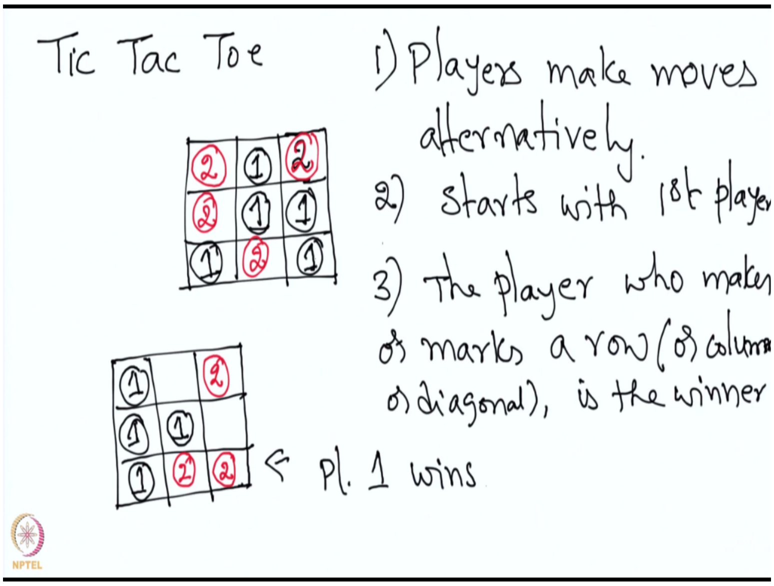
Game theory is a mathematical discipline which models conflicting behavior between agents. The subject originated during the World War and the poly mathematician Von Neumann is considered to be the founder of this subject. The subject has some earlier results by Emile Borel and other people but Von Neumann is considered as the mathematician who created this branch of the subject.

Von Neumann's book with economist Morgenstern, '*Games and Economic Behavior*', is the first book which laid the foundations of this subject. So, in this subject, there are two or more people who make their decisions simultaneously and accordingly, each player gets a benefit or payoff. Now, the objective of each player is to decide how to choose their decision depending on others' choices as well.

Of course, the major problem here is that when a player is making his decision, he does not know what the other players are choosing. These games are known as economic games, classical games, etc. There is another class of games known as combinatorial games. Combinatorial games are the games that we have played as a kid. The examples are Tic-Tac-Toe, Chess, etc.

The theory of combinatorial games was developed by John Conway, Richard Guy and Elwyn Berlekamp. In fact, they have written a very famous book called '*Winning Ways*', currently with 4 volumes. Now, we will concentrate on combinatorial games. We will walk through some examples. Through these examples we will introduce the terminology used in this theory. We start with Tic-Tac-Toe.

(Refer Slide Time: 03:03)

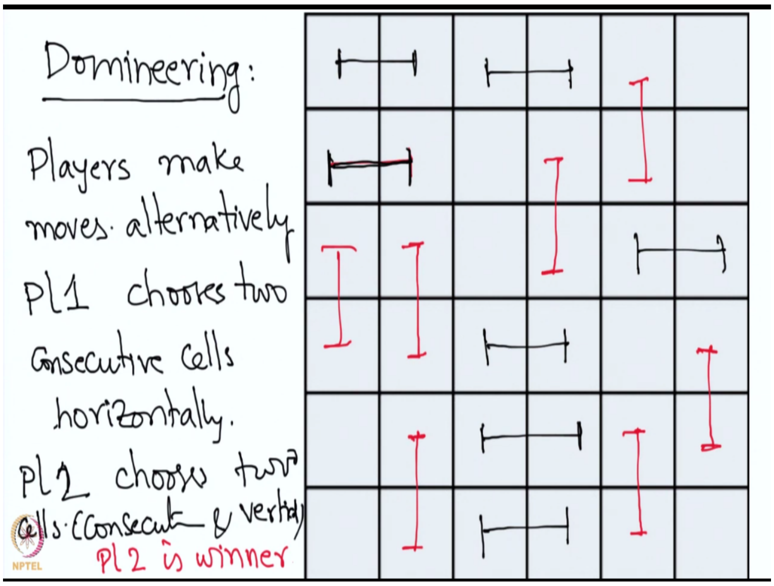


Tic-Tac-Toe is a game played by 2 players on a 3 x 3 grid. Players make alternative moves. The game starts with the first player marking one of the cells with his marker. Next, the second player puts her marker in one of the empty cells. This continues until a player marks a row, column or diagonal, thereby winning the game, or no empty cell remaining without a player winning, resulting in a draw. One of the instances of this game played can be seen in the slide above.

In this game, we have an option of draw but in most common combinatorial games, we always assume that the game ends either in a win or a loss. Even though we have used this to illustrate the ideas of combinatorial games, in the classical sense, Tic-Tac-Toe is not a combinatorial game.

Now, we will see another game called ‘Domineering’.

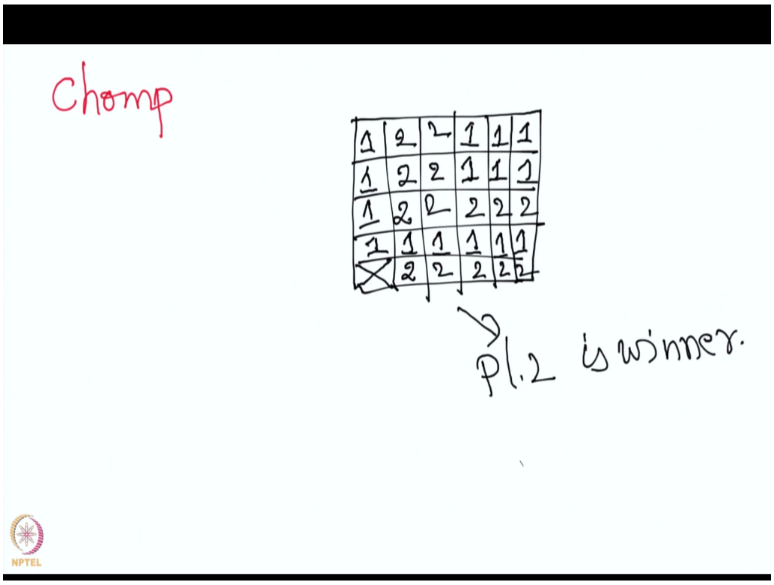
(Refer Slide Time: 07:42)



The game of *Domineering* consists of a square *n x n* collection of cells where the players make moves alternatively. Player 1 chooses two consecutive cells horizontally and places his marker. Player 2 then chooses two cells, consecutively and vertically. Refer to the above slide for one such instance of the game, to get a better idea of how the moves are made. Here, Player 1 makes moves horizontally, in black and Player 2 makes moves vertically, in red. They keep alternatively making their moves until one of them runs out of a valid move, thus losing the game. Once the game ends, whoever makes the last move is the winner.

Note that, in the above instance of the game of Domineering, Player 1 has no more moves to make. Player 2 is the person to make the last move and hence, Player 2 is the winner. Typically, in these combinatorial games, the last person to make the move is the winner.
We call this *normal* play. There is another version where the last person to make a move loses. These games are known as *Misère* games. Let us explain another game which is known as the game of *Chomp*.

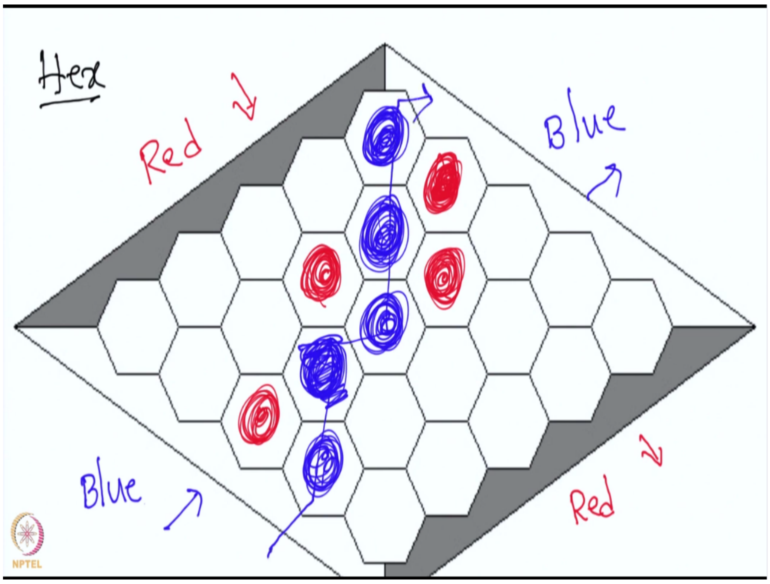
(Refer Slide Time: 12:17)



This game is again played on a rectangular collection of cells. Each cell is considered as a piece of chocolate and the bottom left cell is considered poisonous. Players make moves alternatively choosing a particular cell. If a player chooses a cell, she not only takes that particular cell, she also takes all the cells above and to the right of it. The game ends when the bottom left cell which is poisonous, is taken by a player. The person to take this cell loses the game, with the other player winning. Refer to the slide above for one such instance of the game of Chomp. In this instance, Player 1 ends up having to take the poisonous cell as it is the only one left in the end, thereby losing the game.

Now, we will look at another game, known as *Hex*.

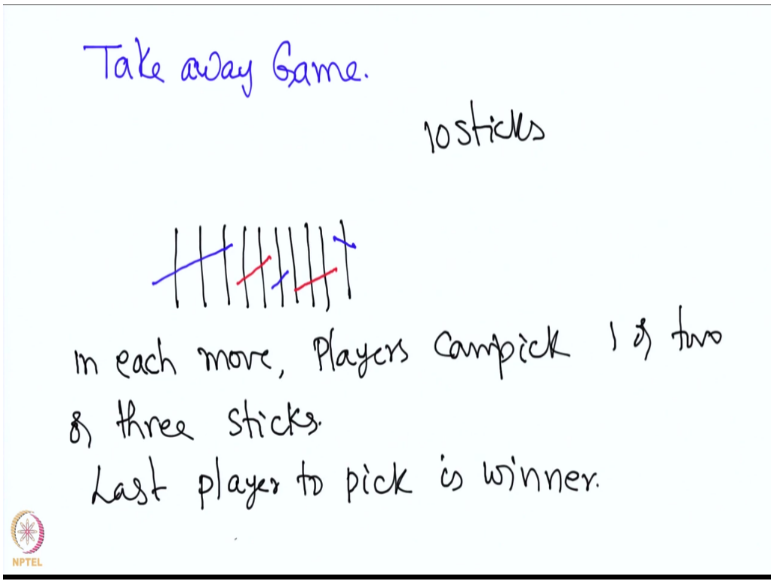
(Refer Slide Time: 14:12)



The game of *Hex* is played between two players, the blue player(Player 1) and the red player(Player 2). The goal of this game is to make a path from one side to the other, as can be seen in the slide above. The blue player wishes to mark the hexagonal cells such that he makes a path from his side to the other side. Similarly, the red player would like to make a path from her side to the other side. They make their moves alternatively, where in each move, the players pick one of the hexagon cells and mark them with their respective color. In the above instance of the game, the blue player wins the game.

Next, we see one more game, known as the Take Away game.

(Refer Slide Time: 16:49)

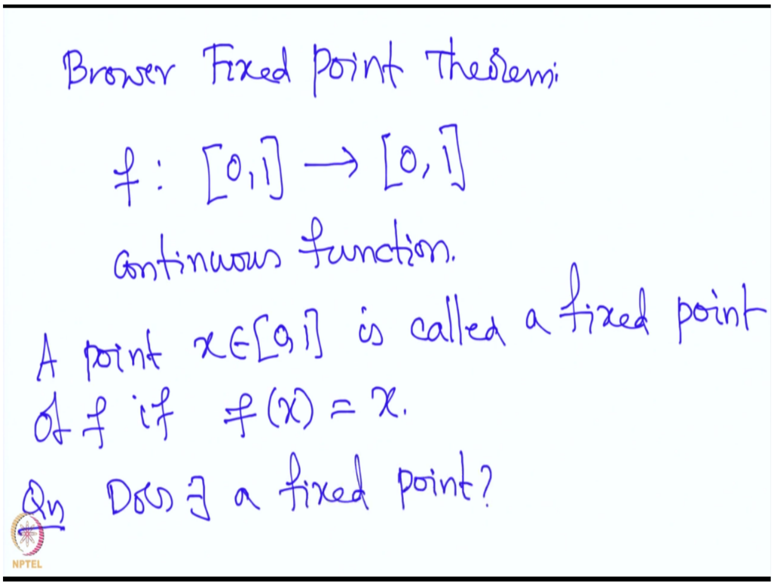


In the *Take Away game*, there is a set of sticks. Two players alternatively make moves and in each move, pick one,two or three sticks. Again, the last player to pick is the winner. In the above instance, we have 10 sticks and as we can see the blue player(Player 1) has picked last and won the game.

The whole idea of these combinatorial games is to understand such games and see whether any player has a winning strategy(strategy by which she can force a win) or not. For example, in this instance of Take away game with 10 sticks, can we decisively say that Player 1 always has a winning strategy? Or is it true that Player 2 can enforce a win? We will discuss about some of these games and understand their winning strategies, later in this course.

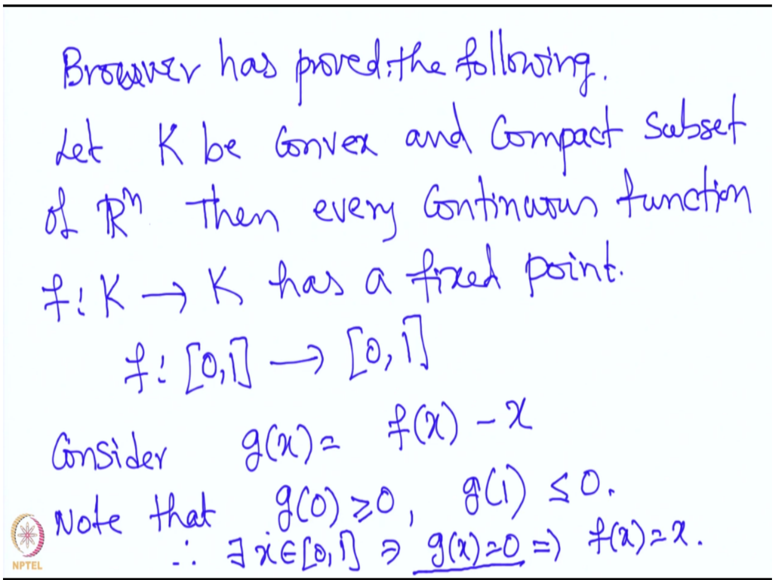
The following is one very interesting application of combinatorial games. Using the game of Hex, we can prove the *Brouwer's Fixed-Point theorem*.

(Refer Slide Time: 19:53)



Definition: Let $f:[0,1]\to[0,1]$ be a continuous function. A point $x\in[0,1]$ is called a *fixed point* of f if $f(x)=x$.

(Refer Slide Time: 21:00)



Theorem(Brouwer's Fixed Point Theorem). *Let K be a convex and compact subset of $\mathbb{R}^n$. Then, every continuous function $f:K\to K$ has a fixed point in K.*

In this course, we will see how the game of Hex can be used to prove this fixed point theorem. Let us briefly look at the one-dimensional case.

Proof(one-dimensional case): Let $f:[0,1] \rightarrow [0,1]$. Consider $g(x)=f(x)-x$. Note that, $g(0) \geq 0$ and $g(1) \leq 0$. Therefore, $\exists x \in [0,1]$ such that $g(x)=0 \Rightarrow f(x)=x$.

Here, the main thing is the guarantee of existence of a zero of the function *g*. This is left for the reader as an exercise.