

Time value of money-Concepts and Calculations
Prof. Bikash Mohanty
Department of Chemical Engineering
Indian Institute of Technology, Roorkee

Lecture - 13
Multiple Cash Flow-1 and 2

Welcome to the lecture series on Time value of money-Concepts and Calculations. In this lecture we will deal multiple cash flows part one and part two. The cash flow is the amount of fund that is flowing in and out of the company. If a company is consistently generating more cash than it is using, the company will be able to increase its dividend, reduce debt and acquire another company. The annual cash flow of a company is the net profit it gets plus the depreciation charges for that year.

In general cash flow of a project runs for the life time of the project. It is necessary to convert these to equivalent values either by discounting future cash flow values or compounding earlier cash flow values. In the present unit use a present worth or future worth of different cash flow patterns dealing with the equal and unequal cash flow patterns and continuous cash flow patterns will be demonstrated.

(Refer Slide Time: 02:00)

How to construct a cash flow diagram ?

- The horizontal (time) axis is marked off in equal increments, one per period, up to the duration of the project.
- Receipts are represented by arrows directed upward.
- Disbursements are represented by arrows directed downward. The arrow length is approximately proportional to the magnitude of the cash flow.
- Two or more transfers in the same period are placed end to end, and these may be combined.
- Expenses incurred before $t = 0$ are called sunk costs. Sunk costs are not relevant to the problem unless they have tax consequences in an after-tax analysis.

For example, consider a mechanical device that will cost Rs.20,000 when purchased. Maintenance will cost Rs.1000 each year. The device will generate revenues of Rs.5000 each year for five years, after which the salvage value is expected to be Rs.7000. The cash flow diagram is shown in (a), and a simplified version is shown (b).

Now, let us see how to construct a cash flow diagram. The horizontal time axis is marked up in equal increments 1 per period up to the duration of the project. That means, duration of the project is from here to at this point. Receipts are represented by arrows

directed upward. This is the receipt 5,000, this is the receipt 5,000, they have marked arrow upward. This is a receipt, this is the receipt and there are two receipts here. Disbursements are represented by arrows directed downward. This is the disbursement 20,000. The arrow length is approximately proportional to the magnitude of the cash flow. Two or more transfers in the same period are placed end to end and this may be combined afterwards. Like, here 5,000 is the receipt and 1,000 is the disbursement. So, they are put end to end. Expenses incurred before t is equal to 0 are called sunk cost. Sunk cost are not relevant to the problem unless they have tax consequences in an after tax analysis.

Now, based on this let us plot a cash flow diagram. Now for this we are taking an example. For example, consider a mechanical device that will cost 20,000 rupees. Now, this is 20,000 rupees at disbursement because you have to spend 20,000 rupees. So, this 20,000 is put downward, when purchased. Maintenance will cost rupees 1,000 each year. So, maintenance is another disbursement. So, maintenance are put like this each year. The device will generate revenue of 5,000 each year for 5 years. So, this is revenue that is receipts so 5,000 is put like this. After which this salvage value is expected to be 7,000 rupees. So, after the useful life of the equipment it can be sold and this salvage value is 7,000. So, this 7,000 is put above the 5,000. The cash flow diagram is shown in a, and its simplified version is shown in b.

Now, this is the simplified version, this disbursement here it is 20,000 from 5,000, 1,000 deducted. So, it is 4,000, 4,000, 4,000, 4,000 and 5,000 plus 7,000 it is 12,000 minus 1,000 so it is 11,000. These standard cash flows are single payment cash flows, uniform series cash flows and gradient series cash flows. Let us explain these three types of cash flows. Single payment cash flow; a single payment cash flow can occur at the beginning of the time line, designated has t equal to 0, at the end of the time line designated t is equal to N or at any time in between.

(Refer Slide Time: 05:20)

The standard cash flows are single payment cash flow, uniform series cash flow, and gradient series cash flow.

Types of Cash Flow

Single payment cash flow	A single payment cash flow can occur at the beginning of the time line (designated as $t = 0$), at the end of the time line (designated as $t = N$), or at any time in between.
Uniform series cash flow	The uniform series cash flow, illustrated in Fig. 1, consists of a series of equal transactions starting at $t = 1$ and ending at $t = N$. The symbol A (representing an annual amount) is typically given to the magnitude of each individual cash flow.
Gradient series cash flow	The gradient series cash flow, illustrated in Fig. 2, starts with a cash flow (typically given the symbol G) at $t = 2$ and increases by G each year until $t = N$, at which time the final cash flow is $(N - 1)G$. The value of the gradient at $t = 1$ is zero.

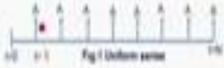


Fig 1 Uniform series

Note: Flows do not begin at the beginning of a year (i.e., the year 1 cash flow is at $t = 1$, not $t = 0$).

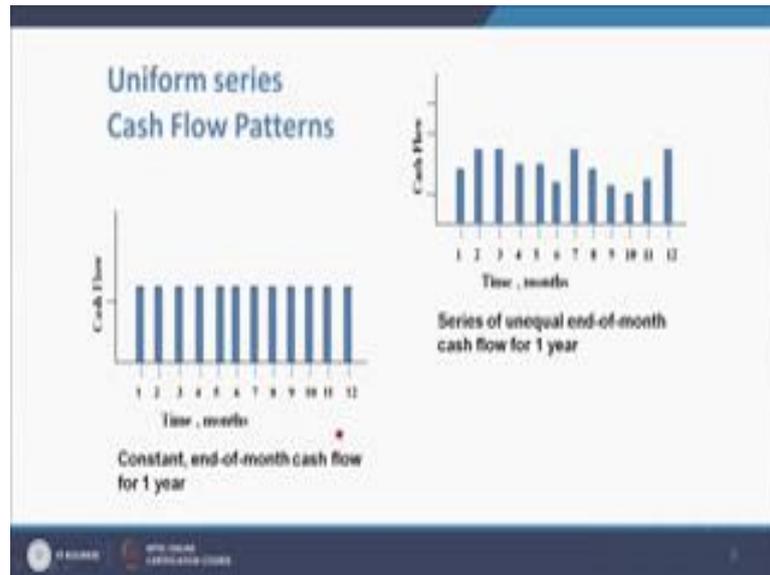


Fig 2 Gradient series cash flow

Here we see that, at equal to 1 there is a cash flow, at t equal to 2 also there is a cash flow. So, this is a uniform series cash flow, but this is not a single payment cash flow. In a single payment cash flow, any one of these places there be a cash flow. A uniform series cash flow, this is a uniform series cash flow illustrated in figure 1 consists of series of equal transactions starting at t is equal to 1, there is a transaction, t is equal to 2, there is a transaction, t is equal to 3, there is a transaction and so on so forth up to t is equal to N . And ending at t equal to N the symbol A , representing an annual amount is typically given to the magnitude of each individual cash flow.

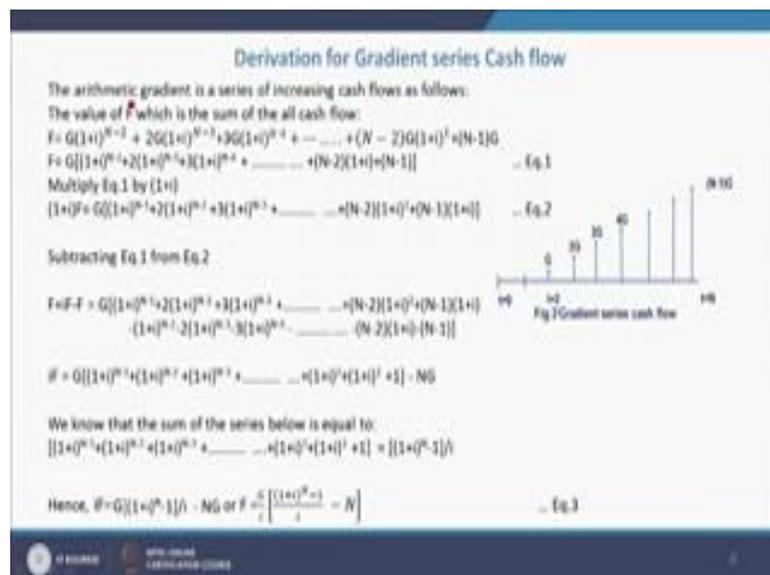
So, we are representing here the magnitude of the cash flow as A . Now gradient series cash flow; the gradient series cash flow is from here. The gradient series cash flow illustrated in figure 2 starts with a cash flow typically given the symbolic g at t equal to 2. So, at t equal to 2, there is a cash flow g and increases by g each year until t equal to N . So, this cash flow increases up to N at which time the final cash flow is N minus 1, in brackets into g . The value of the gradient at t equal to 1 is 0. Here t equal to 1 is 0. Uniform series cash flow patterns

(Refer Slide Time: 07:06)



This type of pattern a cash flow is called constant end of the month cash flow for a year. So, this is a cash flow is shown every month or end of the month for a duration of 12 month. Here series of unequal end of the month cash flow for 1 year. Here we will see that the cash flow is not uniform though it is end of the month cash flow for a period of 1 year.

(Refer Slide Time: 07:31)



Now, derivation for gradient series cash flow; uniform cash flow derivations we have already seen because they are basically annuities. So, in the section of annuity we have

seen the derivative. Here we will derive the derivation for gradient series cash flow. Now the arithmetic gradient is a series of increasing cash flow as follows: The value of F which is the sum of the all cash flow is equal to $G \frac{1+i^N-1}{i}$. Now if I am finding the final value and this is $t = 2$, then here is the final value at t equal to N . Then if I find out the value of G at this point then, it will be G into $1+i$ to the power $N-2$. The second t equal to 3 , $2G$ it will be $2G$ into $1+i$ to the power $N-3$ and so on so forth up to $N-1$ G because here there is no compounding period and that is why it will be $N-1$ G and here this would be compounded up to this time period.

This would be compounded up to this, this would be compounded up to this and this would be compounded up to this. So, you can write down this series for future value. Now here we can take G common. So, this is the series we get. Now multiplying equation 1, this is equation 1 multiplying equation by $1+i$ this is $1+i$ F equal to G , this is $N-2$. So, it becomes $N-1$ when you multiply with $1+i$ and so on so forth up to here $N-1$ into $1+i$.

Now, subtracting equation 1 from equation 2; so this is equation 1, up to this and this is the value of equation 2 up to this then, we are taking G common. So, this is iF because this F and this F cancels out iF is equal to G into this whole series minus N G . We know that, the sum of the series below is equal to the summation of this series is equal to n brackets $1+i$ to the power $N-1$ divided by i . So, when we substitute the value of series by this then, it becomes iF is equal to G into $1+i$ to the power $N-1$ divided by i minus N G or F is equal to G by i in brackets $1+i$ to the power $N-1$ divided by i minus N . So, this is equation series. So, this gives you the future value of a gradient series of cash flow.

(Refer Slide Time: 10:37)

Multiplying Eq.3 with single payment present value factor gives:

$$P = \frac{G}{i} \left[\frac{(1+i)^N - 1}{i} - N \right] \left[\frac{1}{(1+i)^N} \right] + G \left[\frac{(1+i)^N - 1}{i^2(1+i)^N} \right] \quad \dots \text{Eq.4}$$

Or $(P/G, i, N) = \left[\frac{(1+i)^N - 1}{i^2(1+i)^N} \right] \quad \dots \text{Eq.5}$

Eq.5 is the arithmetic gradient present worth factor. Multiplying Eq.5 by sinking fund factor, we get:

$$A = \frac{G}{i} \left[\frac{(1+i)^N - 1}{i} - N \right] \left[\frac{i}{(1+i)^N - 1} \right] + G \left[\frac{(1+i)^N - 1}{i(1+i)^N - 1} \right]$$

$$(A/G, i, N) = \left[\frac{1}{i} - \frac{N}{(1+i)^N - 1} \right] \quad \dots \text{Eq.6}$$

Eq.6 is the arithmetic gradient uniform series factor

Now, if you want to find out the present value of this gradient cash flow then, P is equal to G by i into one plus i to the power N minus 1 divided by i minus N brackets close into 1 by 1 plus i to the power N. Now this multiplied by P is the F that is final value and hence, we can calculate P is equal to G this 1 plus i to the power N minus i N minus 1 divided by i square 1 plus i whole to the power N. What it has been done it has been solved i N here divided by i then this is multiplied this i multiplied i square and this is 1 plus i to the power N.

So, you can write down the factor P by G i N is equal to this. Equation 5 is the arithmetic gradient present worth factor, multiplying equation 5 by sinking fund factor we get. Now if you want to find out the value of A, that is the annuity then, we can multiply this by the sinking fund factor, this is sinking fund factor to give you this value. So, we can write down the arithmetic gradient uniform series factor as A by G i N is equal to 1 by i minus N 1 plus i to the power N minus 1. So, these equations will be used to solve the numerical.

(Refer Slide Time: 11:57)

Example-1: Assume that you will have no need for money during the next two years, and any money you receive will immediately go into your account and earn a 10% effective annual interest rate. Which of the following options - would be more desirable to you?

(a) Receives Rs. 100 now

(b) Receives Rs. 110 for one year

(c) Receives Rs. 121 in two years

Solution:
None of the options is superior under the situation given. If one chooses the first option, he will immediately place Rs.100 into a 10% account, and in two years the account will have grown to Rs.121. In fact, the account will contain Rs.121 at the end of two years regardless of which option you choose. Therefore, these alternatives are said to be equal.

Now, let us start taking numerical. Assume that you will have no need for money during the next 2 years and any money you receive will immediately go in to your account and earn a 10 percent effective annual interest rate. Which of the following options would be more desirable for you? Now one option is that, at t is equal to 0, you get 100 rupees, receive rupees 100 down or we receive 110 rupees in 1 year; that means, at the end of the first year you get 110 rupees or second is receive 121 in 2 years.

So, at the end of the second year, you get 121. So, which one of this offers you will take. The solution none of the option is superior under the situation given. If one chooses the first option, he will immediately place rupees 1 into a 10 percent account and in 20 years the account will have grown to rupees 121. In fact, the account will contain 121 at the end of the 2 years, regardless of which options you chose. Therefore, these alternatives are set to be equal.

(Refer Slide Time: 13:20)

Description of problem	Problem Type
Calculate Present Worth of Annual Cash flow with annual compounding when annual interest rate and cash flow at the end of the year is given	A
Calculate Future Worth of Annual Cash flow with annual compounding when annual interest rate and cash flow at the end of the year is given	B
Calculate Present Worth and Future worth of Cash flow with compounding other than annual when nominal interest rate and cash flow at the end of the period is given	C
Compare two cash flow patterns (A & B) in terms of their Present worth as well as future worth with compounding other than annually when nominal interest rate and cash flow for both patterns (A & B) at the end of the period is given	D
Find the present worth and future worth of an arithmetic gradient cash flow	E

Now, you see the multiple cash flow problem matrixes. So, we have five type of problems. One is calculate present worth of annual cash flow with annual compounding, when annual interest rate and cash flow at the end of the year is given and will call this type of problem; problem type A. Calculate future worth of annual cash flow with annual compounding when annual interest rate and cash flow at the end of the year is given, such type of problem will be called problem type B.

Calculate present worth and future worth of cash flow, with compounding other than annual, when nominal interest rate and cash flow at the end of the period is given will call this type of problem type C. Compare two cash flow patterns A and B in terms of their present worth as well as future worth with compounding other than annually, when nominal interest rate and cash flow for both patterns A and B at the end of the period is given. We will call this type of problem; problem D. Find the present worth and future worth of an arithmetic gradient cash flow will call it problem type E.

(Refer Slide Time: 14:41)

Problem Type - A

Example- 2: A cash flow consisting of Rs 5000 per year is received in one discrete amount at the end of each year for 10 years. Interest will be 9 % per year compounded annually. Determine the present worth of the cash flow.

Solution:

Given: R = Rs.5000; i = 0.09 ; N = 10 years

Present Worth(P) = $\frac{R * [(1 + i)^N - 1]}{i * (1 + i)^N}$ Equal Cash Flow annually

$P = 5000 * [(1.09)^{10} - 1] / [0.09 * (1.09)^{10}] = 5000 * [2.36736 - 1] / [0.09 * 2.36736] =$
Rs.32088.25

he present worth of the uniform cash flow is Rs.32088.25

Now, take another example, example 2. A cash flow consisting of rupees 5,000 per year is received in one discrete amount at the end of each year for 10 years. Interest will be 9 percent per year compounded annually determining the present worth of the cash flow. This is equal cash flow annually. This is problem type A. So, given is R is equal to or say A is equal to 5,000 here we are representing A with R, i equal to 0.09 N equal to 10 years because this is annuity problem. So, present worth will be R in the brackets 1 plus i to the power N minus 1 divided by i into 1 plus i to the power of n.

So, this is the formula for finding all the present worth. When you put values into this formulas R is 5,000. This is 1 plus i is 1.09 to the power 10 because it is for 10 years divided by i 0.09 and 1 plus i is 1.09 to the power 10. This comes out to be 32,088.25; that means, the present worth of the equal cash flow annually for 10 years will be equal to 32088.25.

Let us take another example. An unequal end of the year cash flow consisting of 5,000, 9,000 and 12,000 at the end of first year, second year and third year respectively has been received. Interest rate it is 10.5 percent per year compounding annually. Determine the present worth of the total amount of this start of the first year.

(Refer Slide Time: 16:30)

Problem Type- 1

Example-3: An unequal end of the year cash flow consisting of Rs 5000 , Rs.9000 and Rs.12,000 at the end of 1st , 2nd and 3rd year respectively has been received. Interest rate is 10.5 % per year compounded annually. Determine the present worth of the total amount at the start of the first year.

Solution:

Present worth of the money Rs.5000 invested at the end of 1st year $=5000/(1+0.105)^1 =Rs.4524.886$
Present worth of the money Rs.9000 invested at the end of 2nd year $=9000/(1+0.105)^2 =Rs.7370.856$
Present worth of the money Rs.12000 invested at the end of 3rd year $=12000/(1+0.105)^3=Rs.8893.944$
Thus the present worth of the unequal end of the year cash flow is = Rs.20789.686

Present worth

4524.886
7370.856
8893.944
20789.686

10000
9000
12000

1st year
2nd year
3rd year

Unequal Cash Flow annually

Now such type of problem which is a unequal end of the year cash flow cannot be solved by formula and hence it has to be solved by first principle. Now if you see the time line here at the end of first year, we have invested 5,000 rupees, at the end of second year we have invested 9,000 rupees and end of the third year we have invested 12,000 rupees.

Now, these 12,000 rupees has to be brought to its present value. This 9,000 has to be brought to its present value and this 5,000 has to be brought to its present value and if the 5,000 is brought to the present value, it is 4,524.886. If now, 9,000 are brought to its present value, it is 7,370.856 and if it is 12,000 is brought to the present value, it is 8,893.994 and when we add all these present values it is 20,789.686.

Now, how this 5,000 will be brought to its present value? Here we will see. Present worth of the money 5,000 invested at the end of the first year is equal to 5,000 divided by 1 plus the interest rate. This is 10.5 so expressed in ratio it is 0.105 to the power 1 is equal to 4,525.886 and the present value of this will be 9,000 divided by 1 plus i to the power 2, which comes out to be 7,370.856 and this is the present value of this is 12,000 divided by 1 plus i and i is 0.105 and to the power 3 because 1 year, 2 year and 3 year and that is why to the power of 3. So, it is 8,893.944. So, when we sum it up, it becomes 20,789.686.

So, this is how the present worth of a unequal end of the year cash flow can be calculated from the first principle. Example 4; an graduate plans to buy a home he has been advised

that his monthly house and property tax payment should not exceed 40 percent of his disposable monthly income after researching the market.

(Refer Slide Time: 19:04)

Problem Type - A

Example-4: An graduate plans to buy a home. He has been advised that his monthly house and property tax payment should not exceed 40% of his disposable monthly income. After researching the market, he determines that he can obtain a 30-year home loan for 7% annual interest per year, compounded monthly. His monthly property tax payment will be approximately Rs.200. What is the maximum amount he can pay for a house if his disposable monthly income is Rs.3000?

(A) Rs.117890.4
 (B) Rs.145790.2
 (C) Rs.150334.1
 (D) Rs.186709.3

He determines that he can obtain a 30 year home loan for 7 percent annual interest per year compounded monthly. His monthly property tax, will be approximately rupees 200. What is the maximum amount he can pay for a house, if his disposable monthly income is 3,000 rupees?

(Refer Slide Time: 19:43)

Problem Type - A

Solution

Cash flow diagram:
 The amount available for monthly house instalment payments, A is:
 $= \text{Rs.}3000(0.4) - \text{Rs.}200 = \text{Rs.}1000$

The present worth of this annuity of Rs.1000 for 360 months = $A(P/A, 7\%, 360)$
 $= A \left[\frac{1 + r/m)^{mN} - 1}{r/m} \right] / [r/m(1 + r/m)^{mN}]$

Where $r=7\%$; $m=12$, $r/m=0.07/12=0.0058333$
 $mN=30\text{years} \times 12\text{ months/year} = 360\text{ months}$

$P = A \left[\frac{1 + r/m)^{mN} - 1}{r/m} \right] / [r/m(1 + r/m)^{mN}] = 1000 \left[\frac{[1 + 0.0058333]^{360} - 1}{0.0058333} \right] / [0.0058333(1 + 0.0058333)^{360}]$
 $= 1000 \left[\frac{7.116497}{0.047337882} \right] = \text{Rs.}150334.1$

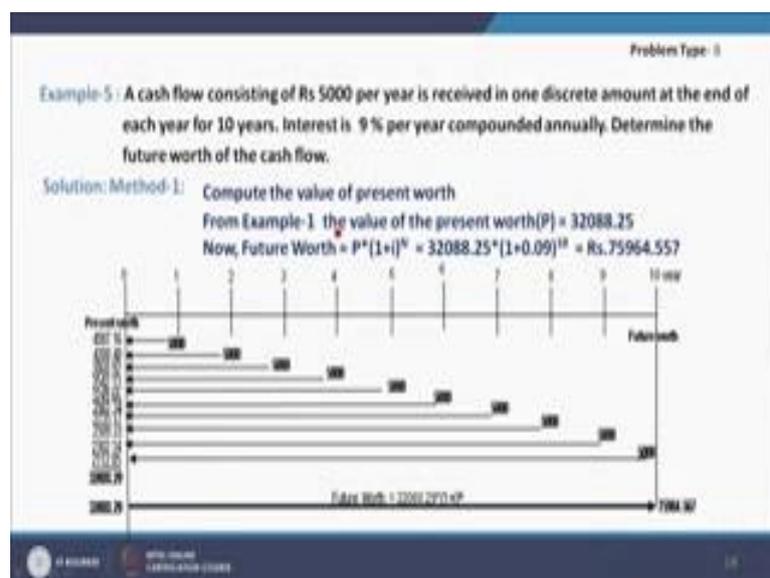
The maximum amount he can pay for the house is Rs.150334.1

Let us see the solution. Here in the time line, from t equal to 1, the person is able to pay rupees A amount A at t equal to 2, also t equal to 3, t equal to 4, up to t equal to 360, is able to pay rupees A that is amount A. Now, let us see how much you can pay, which is equal to amount A. So, the calculation is here, the amount available for monthly house installment payment A is rupees 3,000 into 0.4, this is 40 percent of 3,000 minus the rent which is paying 200. So, this comes out to be 1,000. So, at the max he can pay rupees 1,000 per month up to 360 months.

So, the present worth of this annuity of 1,000 for 360 month is equal to A into P by A 7 percent and 360 and the formula for this is this. This is a discrete compounding formula, where r is equal to 7 percent m equal to 12. So, r by m is equal to 0.07 divided by 12 equal to 0.0058333 m into N is 30 years into 12 months, which comes out to be 360 months, when you put this value here. So, the present worth is equal to 150334.1. So, the maximum amount he can pay for the house is, 150334.1. So, when if he sinks this P amount here, he will get this P amount here, due to this annuity and hence he can purchase the house at t equal to 0, whose maximum price will be rupees 150334.1.

Now, take another problem the cash flow consisting of rupees 5,000 per year is received in one discrete amount at the end of the each year for 10 years; that means, each year end he is paying 5,000 rupees for 10 years. Interest rate is 9 percent per year compounded annually determine the future worth of the cash flow.

(Refer Slide Time: 22:04)



Now, the same problem for present worth has been solved in example 1. So, the method 1, we are taking the present worth from the answer of example 1, which comes out to be 32,088.25 and this present worth is converted into future worth, by multiplying with $1 + i$ to the power N factor. So, this comes out to be 75,954.557. Now this is a method through which we can find out the future value.

(Refer Slide Time: 22:41)

Problem Type- 1

Method-2:

Future worth of payment Rs.5000 at the end of 1st year = $5000(1+i)^9 = 5000(1.09)^9 = 10859.4664$
 Future worth of payment Rs.5000 at the end of 2nd year = $5000(1+i)^8 = 5000(1.09)^8 = 9962.81$
 Future worth of payment Rs.5000 at the end of 3rd year = $5000(1+i)^7 = 5000(1.09)^7 = 9140.196$
 Future worth of payment Rs.5000 at the end of 4th year = $5000(1+i)^6 = 5000(1.09)^6 = 8385.501$
 Future worth of payment Rs.5000 at the end of 5th year = $5000(1+i)^5 = 5000(1.09)^5 = 7693.12$
 Future worth of payment Rs.5000 at the end of 6th year = $5000(1+i)^4 = 5000(1.09)^4 = 7057.908$
 Future worth of payment Rs.5000 at the end of 7th year = $5000(1+i)^3 = 5000(1.09)^3 = 6475.145$
 Future worth of payment Rs.5000 at the end of 8th year = $5000(1+i)^2 = 5000(1.09)^2 = 5940.5$
 Future worth of payment Rs.5000 at the end of 9th year = $5000(1+i)^1 = 5000(1.09)^1 = 5450.00$
 Future worth of payment Rs.5000 at the end of 10th year = $5000(1+i)^0 = 5000(1.09)^0 = 5000.00$
 75964.65

Future worth of the cash flow is Rs.75964.65

Method-3:

$R = F \cdot i / [(1+i)^N - 1]$
 or $F = R / i / [(1+i)^N - 1] = Rs.75964.65$

Equal Cash Flow annually

Let us see the second method. The second method is from first principle. So, we will find out the future worth of each payment, which is done at the end of a year. So, the future of the payment 5,000 at the end of the first year is equal to 5,000 into $1 + i$ to the power 9 because this will earn interest for 9 years only. So, this comes out to be 10,859.4664.

Similarly, the future worth for the 5,000 at the end of second year; this 5,000 into $1 + i$ to the power 8 is equal to 5,000, $1 + 0.09$ to the power 8 comes out to be 9,962.81. So, in the similar way we find out the future worth of all the payments up to tenth year. In the tenth year, it will not earn any interest because it is paid at the end of tenth year and at the end of tenth year we are finding out the future worth and that is why it is 5,000 only. So, when we add up this comes out to be 75,964.65.

Third method we can directly use our formula, to find out the future worth of the annuity R . Here, we are using R symbol for annuity. So, future R future worth of the annuity R , if you use this formula it comes out to be 75,964.65.

(Refer Slide Time: 24:09)

Problem Type - I

Example 6: An unequal end of the year cash flow consisting of Rs 5000 , Rs.9000 and Rs.12,000 at the end of 1st , 2nd and 3rd year respectively has been received. Interest rate is 10.5 % per year compounded annually. Determine the Future worth of the total amount at the end of the 3rd year

Solution:

Future worth of the money Rs.5000 invested at the end of 1st year = $5000 \times (1+0.105)^2 = \text{Rs.}6105.125$
Future worth of the money Rs.9000 invested at the end of 2nd year = $9000 \times (1+0.105)^1 = \text{Rs.}9945$
Future worth of the money Rs.12000 invested at the end of 3rd year = $\text{Rs.}12000$
So Future worth of the unequal end of the year cash flow = $\text{Rs.}28050.13$

Now, let take unequal cash flow annually. An unequal end of the year cash flow consisting of rupees 5,000, 9,000 and 12,000 at the end of first year, second year, third year respectively has been received. Interest rate of 10.5 per year compounded annually. Determine the future worth of the total amount at the end of the third year. So, any unequal cash flow has to be solved from the first principles. So, here this is the time line at the end of the first year, 5,000 is paid, at the end of the third year 5,000 is paid and at the rate of the third year 12,000 is paid. So, all these future worth has to be calculated for these amounts.

So, the future worth of this 5,000 which is paid at the end of first year is 6,105.125. Future worth of this 9,000, which is paid at with the end of second year is 9,945 and the future worth of this 12,000 paid at the end of third year is 12,000. So, the few with this can be calculated like this. Future worth of money of 5,000 invested at the end of first year, is 5,000 into 1 plus 0.105 to the power 2 because this will earn interest for the first year and this second year. So, 2 years, it will earn interest. It will not earn interest for this period of time. So, this is the earned interest for this period of time and for this period time which is 2 years.

So, 1 plus 0.105 is basically the interest rate. So, this comes out to be 6,105.125. Similarly the money which is 9,000 it will earn only interest for 1 year. So, this is 9,000 divided by sorry 9,000 into this is should be 9,000 into 1 plus 0.1015 to the power 1

comes out to be 9,945 and this money 12,000 will not an any interest, it is at the end of the third year and my future worth is also calculated at the end of third year. So, it is 12,000. So, when i add them up it comes to be 28,050.13.

(Refer Slide Time: 26:39)

Problem Type: C

Example 7: A constant end of the month Cash flows with compounding monthly when nominal interest rate is 10% is given below. Find the present worth and future worth of the cash flow.

End of the month, t	Constant monthly cash flow, \$t	End of the month, t	Constant monthly cash flow, \$t
1	10000	16	15000
2	10000	17	15000
3	10000	18	15000
4	10000	19	15000
5	10000	20	15000
6	10000	21	15000
7	10000	22	15000
8	10000	23	15000
9	10000	24	15000
10	10000		
11	10000		
12	10000		
13	10000		
14	10000		
15	10000		
		Total of cash flow	300000

Now, let us take another example. A constant end of the month cash flow with compounding monthly when nominal interest rate is 10 percent is given below. Find the present worth and the future worth. Now if you see this cash flow, this is being flowing from month 1 to month 24, that is 2 years and the total value of this cash flow is 316000. Now as the payments are given at different time periods, we can find out the present worth and future worth of this cash flow.

(Refer Slide Time: 27:09)

Problem Type: C

Solution:
 Annual interest rate $r = 10\%$; $r/m = 0.10/12$; $N = 12 \times 2 = 24$ (J varies from $J = 1, 24$); Investment at the end of the month for 2 years is given:
 Present worth factor for the first month $= 1/(1+r/m) = 1/(1+0.10/12)^1 = 0.991736$
 Present worth for the 1st end the month investment (Cash Flow A) $= 15000 \times 0.991736 = 14876.03$
 Future worth factor for the first month $= (1+r/m)^N = (1+0.10/12)^{24} = 1.210305$
 Future worth for the 1st end the month investment (Cash Flow A) $= 15000 \times 1.210305 = 18154.58$
 Present worth factor for 24th month $= 1/(1+0.10/12)^{24} = 0.81941$
 Present worth for the 24th end the month investment (Cash Flow A) $= 15000 \times 0.81941 = 12291.14$
 Future worth factor for 24th month $= (1+r/m)^N = (1+0.10/12)^{24} = 1$
 Future worth for the 24th end the month investment (Cash Flow A) $= 15000 \times 1.0 = 15000$

End of the Month, J	PW factor	FW factor	Present worth	Future Worth
1	0.991736	1.210305	14876.03	18154.58
2	0.983539	1.200303	14753.09	18004.54
...
24	0.81941	1	12291.14	15000

Now, the annual interest rate is 10 percent. So, r by m is equal to 10 divided by 12, N is 12 into 2, 12 is the months per year and 2 is the years. So, it is 24. So, my J varies from 1 to 24. Investment at the end of the month for 2 years is given. So, the present worth factor for the first month is this, 1 divided by 1 plus r to the power m to the power J and when we put J equal to 1, this comes up to be 0.991734. So, here for the end of the month, this is 1 the present worth factor is this, for this and future worth factor is this. Future worth factor is 1 plus r divided by m to the power N minus J . So, N is 24 and J is 1 for the first month. So, it comes out to be 1.210305. So, when I multiplied this with the value that is, 15,000 then it converts into present worth 14876.03 and 15,000 into this comes out to be 18,154.58.

Similarly, for the month 2 I can calculate the values and I can calculate the PW factor and FW factors and then find present worth and the future worth and for month 24, I can calculate also the present worth factor and future worth. Factor present worth factor is this future worth factor will be 1 obviously, because at the end of the second year that is at the end of the 24th month, I am finding out the future worth this present worth becomes 12,291 and future worth becomes 15,000. Now here I have noted the future worth factor, present worth factor, present worth and future worth.

(Refer Slide Time: 29:14)

Problem Type: C

End of the Month, t	PW Factor	FW Factor	Present worth	Future Worth
17	0.86842	1.059812	13026.3	15897.18
18	0.861243	1.051053	12918.65	15765.8
19	0.854325	1.042367	12811.84	15635.5
20	0.84767	1.033752	12706	15506.28
21	0.840666	1.025209	12600.99	15378.13
22	0.833323	1.016736	12496.85	15251.04
23	0.826238	1.008333	12393.57	15125
24	0.81941	1	12291.14	15000
			Total Present worth	Total Future worth
			325062.8	396703.7

Method-2:
 Given: $m=12$, $N=2$, $r=10\%$
 $P = A \left[\frac{1 + (r/m)^{mN} - 1}{(r/m)(1 + r/m)^{mN}} \right] = 1500 \left[\frac{1 + (0.1/12)^{24} - 1}{(0.1/12)(1 + 0.1/12)^{24}} \right] = \text{Rs. } 325064.12$
 $F = P \cdot (1 + r/m)^{mN} = \text{Rs. } 396705.32$

And when we add them up, so the present worth is 325062.8 and the future worth is 396703.7 as here the uniform payments have been made. So, it can be calculated the present worth and future worth can be calculated is in formula. So, if you see here given m equal to 12 and equal to 2, r is equal to 10 percent and r by m we have to calculate here. So, r by m is 0.1 divided by 12. So, when you put them into this formula, the values it becomes 325064.12. So, little bit of change in rupees about 2 rupees or so. Here because there will be errors in this calculations and rounding of errors and that is why this 2 rupees difference has come and the future value we can calculate from this is a present value is available. So, this comes 396705, here also 2 rupees difference is there due to the rounding off.

(Refer Slide Time: 30:28)

Problem Type: 0

Example 8 Compare two cash flows one constant end of the year Cash flows (A) and other unequal end of the year cash flow(B) with compounding other than annually when nominal interest rate is 12% and cash flows at the end of the year are given below using Present worth method

End of the month, j	Constant monthly Cash Flow(A), \$j	Unequal monthly Cash Flow(B), \$j	End of the month, j	Constant monthly Cash Flow(A), \$j	Unequal monthly Cash Flow(B), \$j
1	15000	8000	16	15000	12500
2	15000	12000	17	15000	16000
3	15000	16000	18	15000	20000
4	15000	20000	19	15000	17000
5	15000	7000	20	15000	13000
6	15000	13000	21	15000	16500
7	15000	17000	22	15000	16200
8	15000	9000	23	15000	17500
9	15000	16000	24	15000	14000
10	15000	13000			
11	15000	20000			
12	15000	15000			
13	15000	16000			
14	15000	9000			
15	15000	16000			
Total of cash flow				360000	360000

Now, take another problem compare two cash flows. One constant end of the year, cash flow A and the other unequal end of the year cash flow P, with compounding other than annually when nominal interest rate is 12 percent and cash flow at the end of the year are given below using present worth factor. So, there are two cash flows are given and one is the constant end of the year and other is unequal end of the year. And the summation of both the cash flows is 360000. Though the summation of both the cash flows is 360000 that present value worth will be different.

(Refer Slide Time: 31:07)

Problem Type: 0

Solution

Annual interest rate $r = 12\%$; $r/m = 0.12/12$; $N = 12 \times 2 = 24$ (j varies from $j=1, 24$); Investment at the end of the month for 2 years is given:

Present worth factor for the first month $= 1/(1+r/m) = 1/(1+0.12/12)^1 = 0.99009901$

Present worth for the 1st end of the month investment (Cash Flow A) $= 15000 \times 0.99009901 = 14851.48515$

Present worth for the 1st end of the month investment (Cash flow B) $= 8000 \times 0.99009901 = 7920.79208$

Present worth Factor for 24th month $= 1/(1+0.12/12)^{24} = 0.787566127$

Present worth for the 24th end of the month investment (Cash Flow A) $= 15000 \times 0.787566127 = 11813.4919$

Present worth for the 24th end of the month investment (Cash flow B) $= 14000 \times 0.787566127 = 11025.92578$

End of the Month, j	Present worth factor	Present worth of end of month equal amount cash flow A	Present worth of end of month unequal amount cash flow B
1	0.99009901	14851.48515	7920.792079
2	0.980296049	14704.44074	11763.55259
...
24	0.787566127	11813.49191	11025.92578

So, will see that; so here again we are calculating because this is a discretely compounding problem. Annual interest rate is 12 percent r by m is 0.12 divided by 12, N is 12 into 2, 12 is for 12 months per year and J varies from 1 to 24. So, present worth factor for the first month is 0.9909901 and similarly we have calculated the present worth at the end of the month, equal amount. So, cash flow this is for equal cash flow. So, this comes out to 14,851 and present worth for the unequal cash flow, comes out to be 7920.79207.

So, here the factor present worth factor is this. So, the present worth of the first end of the month investment cash flow is 15,000 into this factor comes out to be 14,851 where this 14,851 is let in here. Now present worth of the first end of the month investment for cash flow B; this is 8,000 invested into the same factor comes out to be 7,920.79208 this is written here. Now for twenty 24 months, we can see here the factor is 0.787566127 this and the present worth of the cash flow is this.

(Refer Slide Time: 32:55)

End of the Month, J	PW factor	PW of end of month equal amount cash flow A	PW of end of month unequal amount cash flow B
1	0.9909901	14851.40313	7820.79219
2	0.9820904	14704.44014	7740.52719
3	0.9732911	14558.45222	7660.26219
4	0.9644924	14414.79517	7580.00000
5	0.9556944	14271.90531	7500.00000
6	0.9468972	14130.47032	7420.00000
7	0.9381005	13990.17982	7340.00000
8	0.9293042	13850.72434	7260.00014
9	0.9205082	13713.00736	7180.43719
10	0.9117125	13576.84232	7100.73041
11	0.9029171	13442.13374	7020.88294
12	0.8941220	13311.78636	6940.89478
13	0.8853270	13183.69589	6860.76719
14	0.8765320	13057.76784	6780.50011
15	0.8677370	12933.99872	6700.09555
16	0.8589420	12812.38513	6620.55379
17	0.8501470	12692.92371	6540.87526
18	0.8413520	12575.61114	6460.96145
19	0.8325570	12460.35493	6380.80379
20	0.8237620	12347.15166	6300.40279
21	0.8149670	12235.99884	6220.75979
22	0.8061720	12126.89497	6140.87526
23	0.7973770	12019.83854	6060.75071
24	0.7885820	11914.82813	5980.38614
Total Present worth		318650.8089	317121.1192
(PW) of cash flow		318650.8089	317121.1192

Conclusion: Though the sum of the cash flows A and B are same (Rs. 3,60,000) their present worth are different. Indicating that more amount has been paid through cash flow A than cash flow B as the present worth of the cash flow A is Rs. 318650.8089 where as for cash flow B it is Rs. 317121.1192

And present worth of cash flow B is this. Similarly, we have calculated all the cash flows present worth and then when we add them up for cash flow A, the present worth is 318650 and for cash flow B, this is 317121.1192. So, what conclusion we make, though the sum up of the cash flow A and B are same; that is 360000 their present worth are different. Indicating that more amount has been paid through cash flow A than cash flow B (Refer Time: 30:31) I am paying more in cash flow A than cash flow B as the present

worth of the cash flow A, is 3,18,650.8089. Whereas, that of the cash flow B it is 317121.1192. So, this is the conclusion withdraw out of it.

Let us take an example, example 9. Compare two cash flows, one constant end of the year cash flow A and the other unequal end of the year cash flow B, with compounding other than annually when annual interest rate is 12 percent.

(Refer Slide Time: 34:14)

Problem Type - 0

Example-9 : Compare two cash flows one constant end of the year Cash flows (A) and other unequal end of the year cash flow(B) with compounding other than annually when annual interest rate is 12% and cash flows at the end of the year are given below using future worth method

End of the month	Constant monthly Cash Flow (₹)	Unequal monthly Cash Flow (₹)	End of the month	Constant monthly Cash Flow (₹)	Unequal monthly Cash Flow (₹)
1	10000	8000	16	10000	12000
2	10000	5000	17	10000	10000
3	10000	9000	18	10000	20000
4	10000	20000	19	10000	15000
5	10000	7000	20	10000	10000
6	10000	10000	21	10000	16000
7	10000	15000	22	10000	10200
8	10000	9000	23	10000	17000
9	10000	10000	24	10000	14000
10	10000	10000			
11	10000	20000			
12	10000	10000			
13	10000	10000			
14	10000	5000			
15	10000	10000			
Total of cash flow	300000	300000			

And cash flow at the end of the year is given below is using future worth method. Now here, we find the cash flow is for about 2 years and every month there is a cash flow. Now for constant monthly cash flow it is 15,000 and for unequal it is varying

(Refer Slide Time: 34:32)

Solution Problem Type: □

Annual interest rate $r = 12\%$; $r/m = 0.12/12$; $N = 12 \times 2 = 24$; Investment at the end of the month for 2 years is given

Future worth factor for the first month $= (1+r/m)^{N-J} = (1+0.12/12)^{24-1} = 1.257163018$

Future worth for the 1st end the month investment (Cash flow A) $= 15000 \times 1.257163018 = 18857.4452$

Future worth for the 1st end the month investment (Cash flow B) $= 8000 \times 1.257163018 = 10057.3041$

Future worth Factor for 24th month $= (1+r/m)^{N-J} = (1+0.12/12)^{24-24} = 1$

Future worth for the 24th end the month investment (Cash flow A) $= 15000 \times 1.0 = 15000$

Future worth for the 24th end the month investment (Cash flow B) $= 14000 \times 1.0 = 14000$

End of the Month, J	Future-worth factor	Future worth of end of month equal amount cash flow A	Future worth of end of month unequal amount cash flow B
1	1.257163018	18857.44528	10057.30415
2	1.24471586	18670.7379	14936.59032
...
24	1	15000	14000

Let us see the solution. Now the annual interest rate is 12 percent r by m is 0.12 divided by 12, N is 12 into to 2; that is for 2 years and 12 month's per year which comes out to 24. Investment at the end of the month for 2 years is given. Future, worth factor for the first month 1 plus r divided by 8 into the power N minus J ; so N is 24 and for the first month it is J is 1. So, it is 24 minus 1 and the future worth factor comes out to be 1.25163018 and this factor is this.

So, the future worth for the cash flow A, will be 15,000 into this factor comes out to be 18,857.4452 this value and for the cash flow B this will be, 8,000 into this factor which comes out to be 10,057.30415. Similarly for the second month can be calculated and for the 24th month the future worth factor will be 1. So, this will be 15,000 and this will be 14,000 and if you calculate like this.

(Refer Slide Time: 35:51)

End of the Month, t	Cash flow	PW of end of month equal amount cash flow A	PW of end of month unequal amount cash flow B
1	1,25,74,9018	185,748,138	100,57,384,01
2	1,24,47,1588	186,72,7179	149,16,590,32
3	1,23,19,3938	188,8,8,913	2,18,0,09,899
4	1,21,91,6288	18,30,8,906	3,44,01,808
5	1,20,63,8638	18,12,8,8436	9,60,81,71,20
6	1,19,36,0988	17,94,2,1,214	17,54,9,91,708
7	1,18,08,3338	17,76,5,6,647	26,13,1,75,11
8	1,16,80,5688	17,58,8,7,967	33,39,49,713
9	1,15,52,8038	17,40,1,4,31	40,75,50,328
10	1,14,25,0388	17,22,4,1,11	48,11,1,64,77
11	1,12,97,2738	17,04,7,1,39,21	55,47,83,211
12	1,11,69,5088	16,86,1,7,49	62,83,67,543
13	1,10,41,7438	16,68,5,2,52	70,19,49,855
14	1,09,13,9788	16,50,8,3,38	77,55,30,009
15	1,07,86,2138	16,32,1,7,929	84,91,1,8,81
16	1,06,58,4488	16,14,4,8,058	92,27,9,881
17	1,05,30,6838	15,96,7,8,048	100,63,9,71,69
18	1,04,02,9188	15,78,1,8,078	109,00,0,7,77

End of the Month, t	Cash flow	PW of end of month equal amount cash flow A	PW of end of month unequal amount cash flow B
18	1,02,75,1538	15,60,4,9,175	117,36,1,7,891
19	1,01,47,3888	15,42,7,9,015	125,72,0,0,15
20	1,00,19,6238	15,24,1,8,859	134,08,0,0,15
21	98,91,8588	15,06,4,8,703	142,44,0,0,15
22	97,64,0938	14,88,7,8,547	150,80,0,0,15
23	96,36,3288	14,70,1,8,391	159,16,0,0,15
24	95,08,5638	14,52,4,8,235	167,52,0,0,15

Total Future worth of cash flow	404601.9728	402659.6728
--	--------------------	--------------------

Conclusion: Though the sum of the cash flows A and B are same (Rs. 3,60,000) their future worth are different. Indicating that more amount has been paid through cash flow A than cash flow B as the future worth of the cash flow A is Rs. 404601.97 where as for cash flow B it is Rs. 402659.67

Problem Type - 1

So, you can fill up in this table and we find that, this future worth of this is 404601.9728, this is 402659.6728. So, what conclusion we derive out of it is that, though the sum of the cash flows A and B are the same that is 63,60,000 their future worth's are different, indicating that more money has been paid through cash flow A than the cash flow B, as the future of worth of the cash flow A is 4,04,601.97. Where as that of the cash flow B is 402659.67.

(Refer Slide Time: 36:44)

Problem Type - 1

Example 10: Compare two cash flows one constant end of the year Cash flows (A) and other unequal end of the year cash flow(B) with compounding other than annually when nominal interest rate is 12% and cash flows at the end of the year are given below using future worth method

End of the Month, t	Constant monthly Cash Flow(A), Rs.	Unequal monthly Cash Flow(B), Rs.	End of the Month, t	Constant monthly Cash Flow(A), Rs.	Unequal monthly Cash Flow(B), Rs.
1	1000	800	16	1000	1200
2	1000	1300	17	1000	1600
3	1000	1800	18	1000	2000
4	1000	2000	19	1000	1700
5	1000	700	20	1000	1800
6	1000	1700	21	1000	1600
7	1000	2700	22	1000	1000
8	1000	900	23	1000	1700
9	1000	700	24	1000	1600
10	1000	1300			
11	1000	2000			
12	1000	1700			
13	1000	700			
14	1000	900			
15	1000	1600			
			Total cash flow	36000	31100

Problem Type - 1

Now, let us take another example. Example 10 compare two cash flows, one constant end of the year and which is called cash flow A and the other unequal end of the year which was called cash flow B, with compounding other than annually, when nominal interest rate is 12 percent and cash flows at the end of the year are given below. End of the months are given below, basically using future worth method. So, here we have two cash flows. The summation of these cash flows is different. One is 360000 rupees another is 361 rupees 556.

(Refer Slide Time: 37:23)

Problem Type - 0

Solution :

Annual interest rate $r = 12\%$; $r/m = 0.12/12$; $N = 12 * 2 = 24$; investment at the end of the month for 2 years is given

Future worth factor for the first month $= (1+r/m)^{N-J} = (1+0.12/12)^{24-1} = 1.257163018$

Future worth for the 1st end the month investment (Cash Flow A) $= 15000 * 1.257163018 = 18857.4453$

Future worth for the 1st end the month investment (Cash flow B) $= 8000 * 1.257163018 = 10057.3042$

Future worth Factor for 24th month $= (1+r/m)^{N-N} = (1+0.12/12)^{24-24} = 1$

Future worth for the 24th end the month investment (Cash Flow A) $= 15000 * 1.0 = 15000$

Future worth for the 24th end the month investment (Cash flow B) $= 14000 * 1.0 = 14000$

End of the Month, J	Future worth factor	Future worth of end of month equal amount cash flow A	Future worth of end of month unequal amount cash flow B
1	1.257163018	18857.44528	10057.30415
2	1.24471586	18670.7379	16803.66411
...
24	1	15000	14000

Now, if we calculate it, the annual rate of interest r is 12 percent, r by m is 0.12 divided by 12, N is 12 into 2 that is 24; investment at the end of the month for 2 years is given future worth factor of the month is 1 plus r divided by m to the power N minus J . So, for the first month J is equal to 1. So, this comes out to be 1.257163018, this is the factor, future worth factor. So, this future worth factor is written here.

Now, the amount is future worth amount is 15,000 for cash flow A 15,000 into this factor which comes out to be 18,857.4453. So, the amount is written here and future worth for the cash flow B is, this is cash flow B is 8,000 into this factor comes out to be 10,057.3042. Similarly, we filled up all the future worth's at 24 month, the future worth factor is 1 that is, why the values remain same. It is 15,000 here and this is 14,000 here. The one way fill up this table, the future worth of both the cash flows are same. Whereas

the sum of both the cash flows was not same, but the future worth of the cash flows is the same conclusion.

(Refer Slide Time: 38:49)

End of the Month, t	Cash Flow	FW of end of month equal amount cash Flow A	FW of end of month unequal amount cash Flow B
1	1,25,716,801.8	185,744,519	100,73,384.13
2	1,24,471,588	186,70,717.9	100,9,644.11
3	1,23,219,396	188,8,291.1	22,183,054.93
4	1,21,950,004	18,302,850.6	244,03,805.8
5	1,20,680,612	18,121,834.36	90,60,817.28
6	1,19,411,220	17,942,712.14	1,98,9,33,933
7	1,18,141,828	17,766,586.7	20,113,17,533
8	1,16,872,436	17,593,457.7	11,139,49,713
9	1,15,603,044	17,424,324.01	18,75,52,429
10	1,14,333,652	17,259,187.3	14,813,16,777
11	1,13,064,260	17,097,047.23	28,612,31,021
12	1,11,794,868	16,937,903.74	12,395,07,533
13	1,10,525,476	16,781,757.1	17,790,48,355
14	1,09,256,084	16,629,607.88	16,819,93,079
15	1,07,986,692	16,481,455.79	19,686,13,811
16	1,06,717,300	16,337,301.4	13,515,70,813
17	1,05,447,908	16,197,145.18	20,792,17,169
18	1,04,178,516	16,060,987.26	26,538,02,777
Total Future worth of cash flow		404601.9728	404601.9213

Conclusion: Though the sum of the cash flows A and B are different their future worth are almost same (difference in paisa). Indicating that same amount has been paid through cash flow A and cash flow B as the future worth of the cash flow A and B are Rs. 404601.97 where as that for cash flow B is Rs. 404601.92.

Problem Type: 

Though the some of the cash flows A and B are different their future worth are almost same only difference in paisa, indicating that same amount has been paid through the cash flow A and cash flow B as their future worth of the cash flow A and B are 4,04,601.92. Whereas the cash flow B is, it is only 404601.92. So, this is the conclusion you make. Though that primary summation of the cash flows are different, but the future worth is same; that means, so we are investing same amount of money in both the cash flows.

Now, this is the last question. Example number 11; a bonus package pays an employee rupees 2,000 at the end of the first year 2,600 at the end of the second year and so on. For the first 9 years of employment. What is the present worth of the bonus package at 8 percent interest?

(Refer Slide Time: 40:06)

Example 11: A bonus package pays an employee Rs.2000 at the end of the first year, Rs.2600 at the end of the second year, and so on, for the first nine years of employment. What is the present worth of the bonus package at 8% interest?

Solution:
 The problem can be divided into two parts an annuity of Rs.2000 up to ten years and an arithmetic gradient series annuity having G=Rs.600. The present worth of the bonus package is:

$$P = Rs.2000(P/A, 8\%, 10) + Rs.600(P/G, 8\%, 10)$$

$$P = Rs.2000 \frac{[1 + (1.08)^{10} - 1]}{0.08(1.08)^{10}} + Rs.600 \frac{[(1.08)^{10} - (1.08) - 1]}{0.08^2(1.08)^{10}}$$

$$P = 2000 * 6.71008 + 600 * 25.9768314 = Rs.29006.26$$


Note: Flows do not begin at the beginning of a year (i.e., the year 1 cash flow is at t=1, not t=0).



So, this is can be converted into to two series, one series which is put at t equal to 1, 2,000 t equal to 2,000, t equal to 3,200 so on so forth up to t equal to N 2,000. And then 2,600 minus 2,000 its start from t equal to 2,600 and then it grows up to N minus 1G. So, this can be converted into so solution of the problem can be divided into two parts and annuity of 2,000 up to 10 years and then arithmetic gradient series annuity having G is equal to rupees 600. So, the present worth of the package is P equal to rupees 2,000 into this factor P by A 8 percent 10 and then plus rupees 600 to the factor P by G 8 percent 10.

So, here we have written the values of these factors. So, P by A 8 percent comma 10 factor is 1 plus i to the power N minus 1 divided by i into 1 plus i to the power of N. This is the summation of annuity of 2,000 rupees per year up to N periods and this is a arithmetic gradient series present worth. So, when we solve this problem we get P is equal to rupees 29,006.26.

Thank you.