**Basic Concepts in Modal Logic**
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**Lecture No – 15**
**Axiomatic Modal Logic: Some Proofs**

Welcome back in continuation to the last lecture where we where we discussed the Syntax of Modal Logic, we introduced various kinds of translations etcetera how, the sentence like it is possible that P can be it can be the case or it might be the case etcetera those sentences involves these modal operators how it can be translated appropriately to the language of modal logic etcetera is the one which we discussed in the last class.

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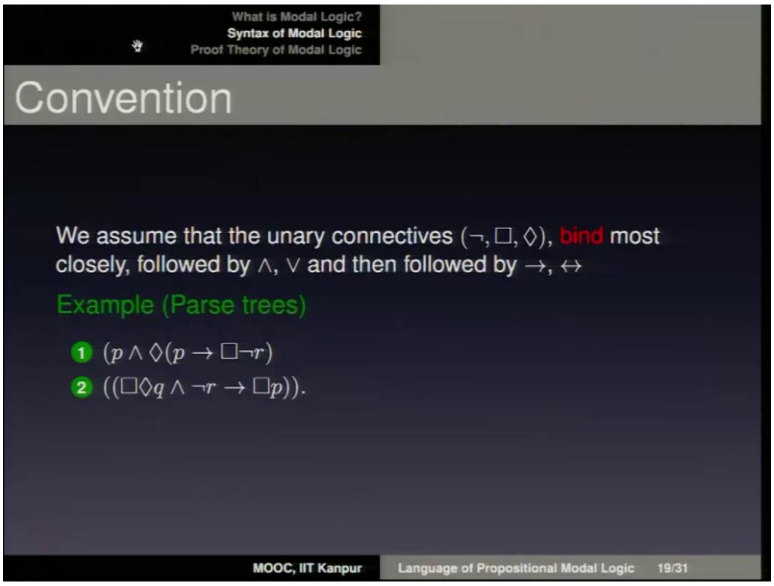


And we have also seen that any 2 given Modal Logic formulas, well formed formulas they do not have the same syntactical structure if you draw the tree diagram for the given formula, they always look different; if they are same they are considered to be identical to each other. So, in continuation to what we have seen in the last class there is a convention just like in the case of prepositional logic, we follow the same convention even in the case of Modal Logic as well. So, basically we are discussing Normal Modal

Logic which is considered to be extension of Classical Logic with 2 operators, it is possible that P and it is necessary that P is the case.

So, now for example, if you have given a formula which does not have any parenthesis, again in the case of Classical Logic we followed this convention that first you give preference to negation and then followed by the conjunction, dis-junction, implication and by implication is the one which we this is the one that we have used. And the preferences least preferences that we have given that is by implication will service, the major connective in your given formula, in the case of prepositional value. In the same way in case of prepositional logic, the first preference will be given to negation followed by that necessity and possibility, which bind closely followed by other things like conjunction and then followed by the we have the implication if and only it is like this. So, the convention that you follow is like this.

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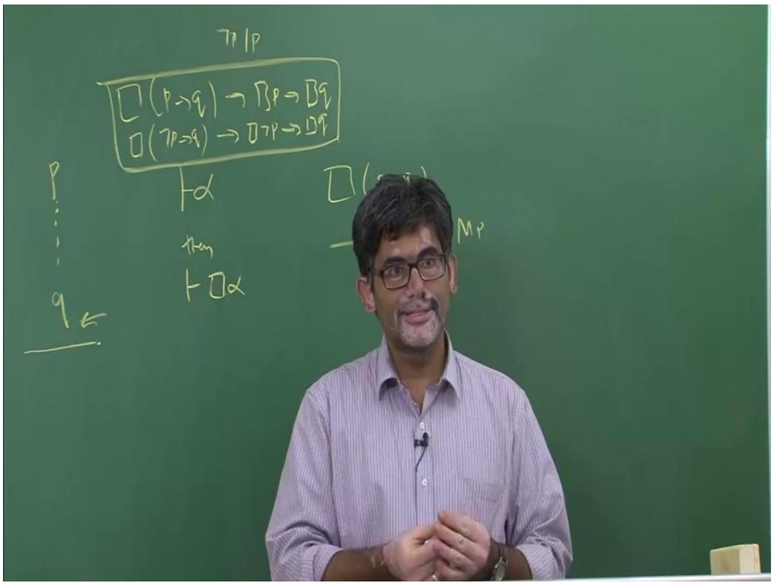


So, this is in the case of Classical Logic and this is the case for Modal Logic. So, first preference will be given to negation followed by the conjunction, dis-junction, and implication and by implication. So, in this case we have negation, necessity, possibility, and of course, all the other things and r implies if and only if. For example, if we have a formula like this sorry necessity of P q and R. So, we do not have any punctuation which is there in this particular kind of thing. So, this formula can be read in 2 different ways just as in the case of Classical Logic it can be read as necessity of P implies q . r, r the other than only is that necessity of P implies q . r, r. So, these 2 formulas have different syntactic structure. So, now, how to put there thesis for this one suppose it is not given to us.

So, now, we follow the conviction we have listed on the right hand side of the board. So, the first preference will be given to the necessity operator, necessity followed by this thing and then we have only these connectives which are like with the Prepositional Logic implication and r. So, in our preference first preference should be given to r in this case also. So, we put bracket here like this and then we close it like this. So, now, this sentence needs to be read like this necessity of P implies q r r. So, in that way you if you parenthesis is not given we follow some kind of convention and using the convention you will come to read the formulas in a better way. So, here is there are some more examples that are here necessity of, possibility of q and not r implies necessity of P. If you use this parsing kind of tree then what how you right it is in this way, first you take care of the negation and modal operators etcetera and then followed by the you take care of all the logical connectives exactly in the same order as in the case of Prepositional Logic.

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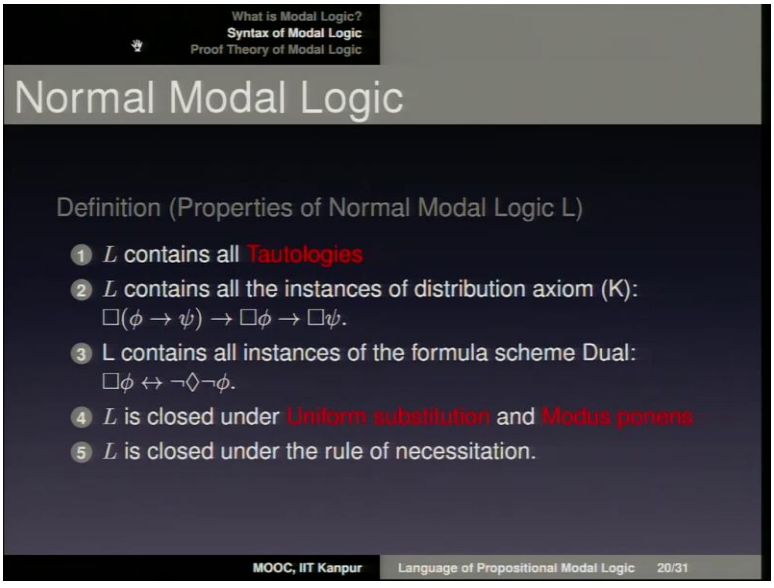

So, now what do you mean by saying that something Modal Logic's are considered to be Normal Modal Logic?

A Normal Modal Logic by definition has these properties L should contain all the tautologies all the tautologies of Prepositional Logic are intact they are not disturbing them because Modal Logic is considered to be extension of Classical Logic. Classical Prepositional Logic if you extend the Classical Prepositional Logic we have Normal Preposition Logic, Normal Modal Preposition Logic.

So, L contains all the instances of distribution axioms which is characteristics axiom for a particular Logical System is K, it is necessary that 5 in plus 5 implies it is necessary that 5 in case implies it is necessary that 5 implies it is necessary that psi and L should contain all the instances of the instances scheme such as duals for example, necessity of phi can be defined in terms of possibility in this sense necessity of phi if and only it is impossible that not phi is the case in the same way phi can be defined as it is not necessary that not phi and given a Logical System it has to be closed under Uniform substitution and it has to be closed under even Modus Ponens. Modus Ponens it is necessary that P implies q and then we have necessity of P and then we have necessity of q . So, this is Modus Pones proof and then we also have another rule which states like this if alpha is already a theorem or a valid formula then all the valid formulas obviously, have to be necessarily true then we have this thing necessity of alpha from alpha.

So, this rule of (Refer Time: 07:02) should not be used loosely in the sense that if suppose alpha is said to be contingent kind of sentence or for example, alpha stands for this chalk piece is yellow in color that does not imply that this is necessarily it is necessarily true that this chalk piece is in yellow color it apply only to theorems which are already proved ones or they are considered to be valid formulas any thought all the tautologies views are to be necessarily true. So, other thing is that L is closed under rule of necessity that is what we have been discussing here if any formula you have alpha then the necessity of alpha should also be you can also be inferred from that thing that alpha is the (Refer Time: 07:46).

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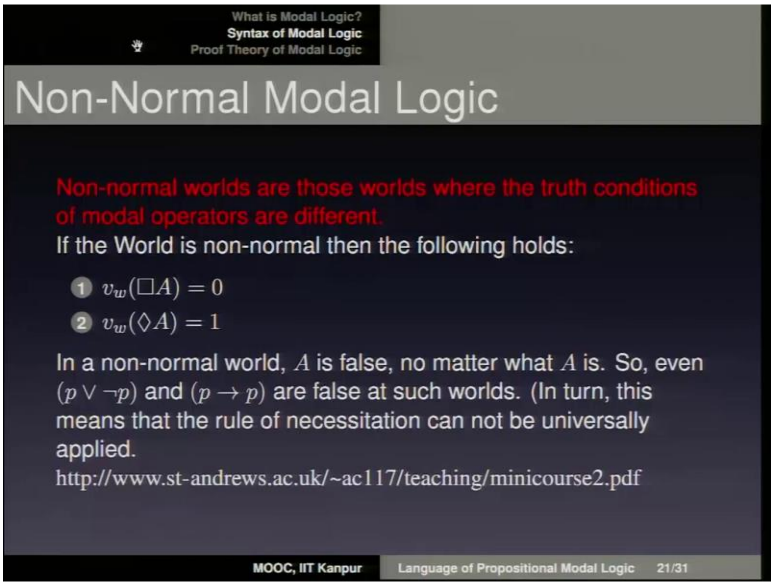


So, in the case of Non-Normal Modal Logic's in particular they are the worlds it consists of the worlds, possible worlds where the truth conditions of modal operators are different where is in the case of Normal Modal Logic suppose, if necessity of A is true then obviously, possibility of A is going to be true everywhere, but in the case of Non-Normal Modal Logical Systems that behave in a slightly different way the necessity operator is going may be false, but the possibility operator it is possible that A in a given world W can be true so, that means, in a Non-Normal world this is what Lewis says talking about particularly when he has come up with 5 logical systems S1 to S5 he is of the view that first two systems S1 to S3. In fact, they capture stick implication in a much more better way than the other kinds of systems that he has come up with they are S4 and S5.

So, in a Non-Normal world A is considered to be false no matter what A is. So, even this P are not P which is considered to be tautology and even P implies P are also going to be false at those such kind of worlds. So, this means that the rule of Necessitation cannot be universally applied in Non-Normal Modal Logic's. So, that is the main difference between Normal Modal Logical Systems and Non-Normal Modal Logical Systems.

In our course we will be focusing our attention on Normal Modal Logical System where the universe the necessity operator derives in a uniform manner or any logic that follows the law of distribution, distribution axiom that is K and which is closed under Modus Ponens can be called as Normal Modal Logic's.

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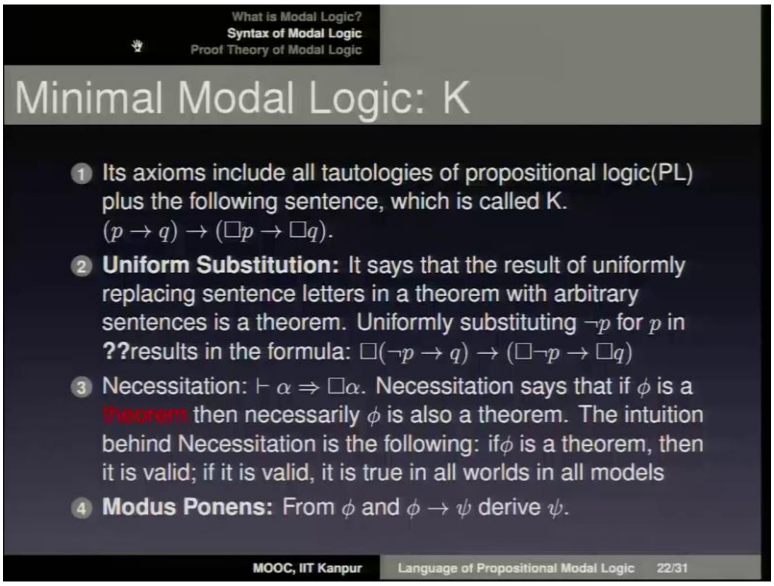


So, what is considered to be the Minimal Modal Logic's under Normal Modal Prepositional Logic. So, it is due to (Refer Time: 10:03) every logical system is named after some logician or the other it is seems to be the case that this logic the name as name was due to this logician (Refer Time: 10:17). So, it is axioms includes all the tautologies of prepositional logic which are intact plus which have distribution axiom necessity of P implies q implies necessity of P plus necessity of q and we have Uniform substitution rules for example, if you have necessity of P implies q for example, if you got P how necessity of P implies necessity of q .

Now if you uniformly substitute let us say not P for P. So, this means wherever P occurs you substitute it not P. So, now, this formula becomes not P plus q implies which is necessary that not P implies necessity of q . So, these are Uniform substitution of this one the only thing is that we replace P with not P uniformly in this formula any given formula if we substitute for any prepositional variable you substitute any other kind of variable then obviously, it is considered to be Uniform substitution and we have a Necessitation rule which is said to be Uniform substitution and we have a Necessitation rule which we have already discussed and we have Modus Pones rule this is k is a system that includes all these things.

So, in the next class we will be talking more about the Axiomatic Systems and we will be seeing some of the theorems some of the theorems within the given Modal Logical Systems and we are also going to see how this theorems can be proved by using this axioms, Uniform substitution and Necessitation and Modus Pones rules. Suppose, if you say alpha is a theorem in K if and only there is a sequence of formulas of which alpha is considered to be that last such kind of step in the sequence the last step is considered to be the theorem last step of your proof is considered to be the theorem. So, that final step has to be either theorem or it has to be axiom or else it is derived from derived by means of one of the rules from formulas appearing higher up in the sequence. So, we have something like P and q .

So, the idea here is that your journey starts from the truth and your journey also ends with the truth the last step of your theorem the last step of your proof is considered to be a theorem and every step that that is we are substantially taking into consideration from P all the way down to q are either theorems or you arrive at those steps that means, of uniform substitution or you apply some kind of rules that Modus Pones etcetera.

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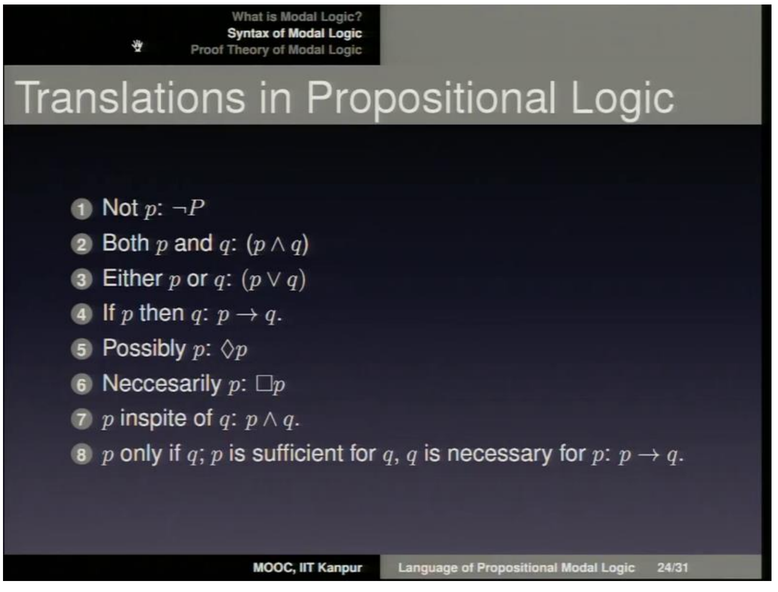

So, let us consider a little bit of translation concerning Normal Prepositional Modal Logic. Normal Modal Logic these are like this all the things translations which are there in the prepositional logic are like this not P is considered to be not P both P and q are considered to be P and q either P or P.

P or q sometimes it can be used in exclude reasons and include reasons unless until I stake it specifically it is used in the inclusive reasons that is P r q or it can be in the case of both of them coffee or tea that given the choice I will take both of these coffee as well as tea also, but if you say exclusive r one excludes the other possibility fruit salads or ice creams if you take fruit salad you are not supposed to take ice cream and of you take ice cream then you are not supposed to take the fruit salad that is exclusive r not both of them we will be consumed.

So, possibility of P is simply calculated as diamond P, necessity of P, box P and P in spite of q it is written in terms of P and q , P only if q and P is sufficient condition for q r q is necessary for P all this are translated as P implies q if we say that P is necessary condition for q it is translated to q implies P.

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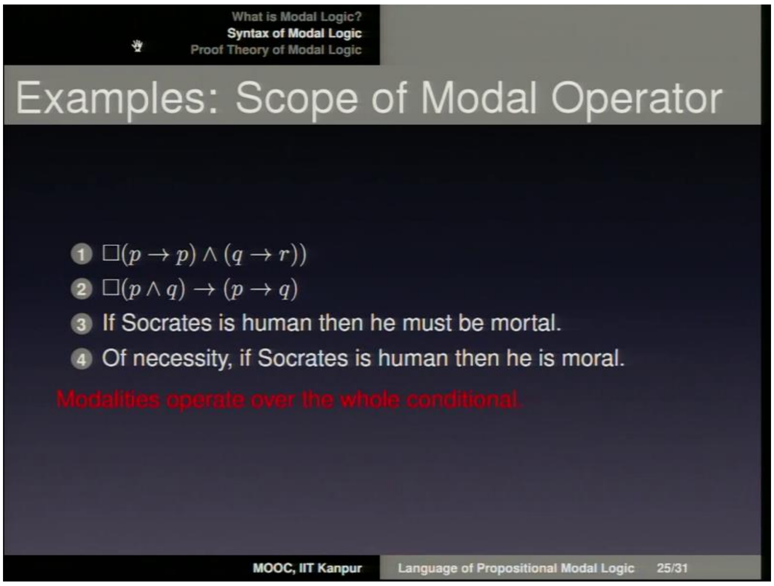

We need to know little bit about the Scope of Modal Operator. So, we have this problem right from the antibiotic as to whether the morality applies to the whole conditionals or part of the conditional. So, in general what we are trying to say that the modality applies to the whole conditionals it is scope is big rather than it is having narrow scope. So, in this case necessity of P implies P and q implies R. In this case the necessity operator operates on the whole conditional that P implies P and q implies r that is operated by this necessity operator. q in the second case is like this necessity of P and q implies P and q . So, this necessity applies to P and q .

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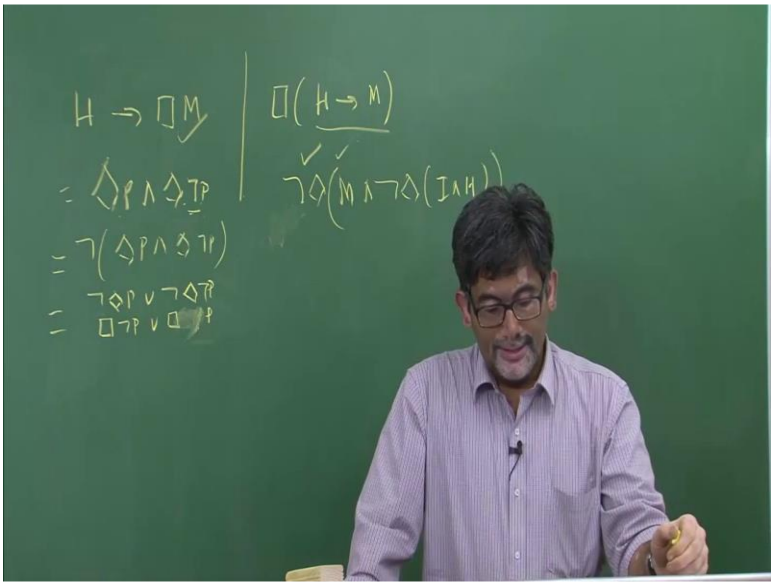

Now, consider this interesting example third example if Socrates is human then he is he must be mortal. So, usually we translate it of like this. So, there are 2 translations which are possible for the second thing if Socrates is human then he must be mortal assuming that H stands for human and then M stands for mortal you can write like this X and like this, but we skip it like convenience.

So, the second translation is this one necessity of H implies M. So, most of the cases for your (Refer Time: 16:19) is that this is the correct translation over the other one suppose if you are in a confusion that you have two such kind of translation the idea is that if you go with the simply literal translation this seems to be the appropriate kind of translation

Socrates is human he must be mortal here it this says that if Socrates is human then he has to be mortal. So, this necessity operator operates over the whole condition. So, most of the cases this is considered to be the correct translation rather than this one this is having narrow scope this wider scope.

In the same way forth example of necessity if Socrates is human then he is mortal this is not moral it is mortal. So, the moralities operate over the whole condition rather than the part of the condition.

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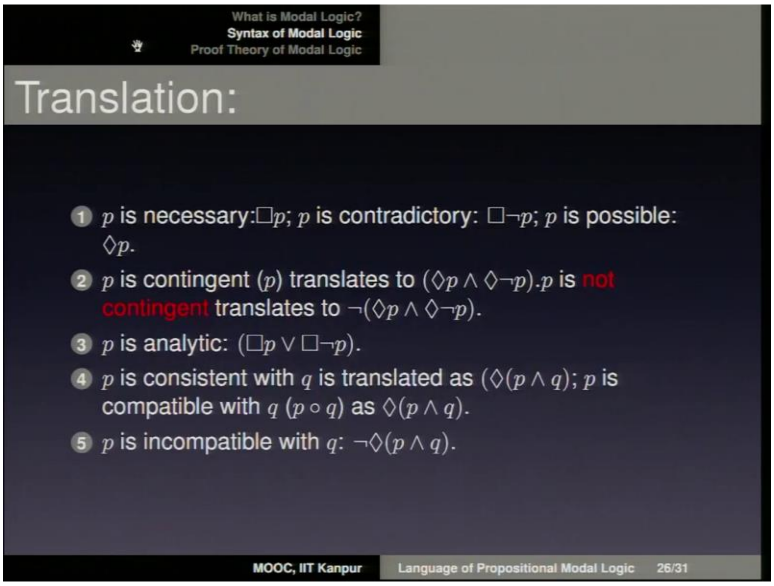


So, now there are other kinds of things you will be you will be coming across while translating the English language sentences appropriately into the language of Modal Logic. So, as you have seen the case of prepositional logic there are 3 kinds of sentences in the one hand we got tautologies those sentences that are always considered to be true and contradistinctions those sentences that are always false and there are sentences which are always true sometimes false it is raining outside, sometimes it is true sometimes it is false today it is true tomorrow it may be false.

So, now, how to translate these sentences appropriately into the Language of Modal Logic P is necessary is simply translated to necessity of P, box P is contradictory not P has to be necessary and P is possible is simply translated as diamond P now P is contingent like you know anything which is related to things which are related to future I shall be in my native place on so and so date. So, that is a future contingent sentence. So, that is translated as possibility of P when you say possibility of P it is also possible that it is not P also.

So, when you book your ticket and then obviously, you will be thinking you will be hoping that you will be in your native place or some other thing might happen some other commitment you may not be in your native place (Refer Time: 18:53) the train is canceled or some other thing happens then you will not be in your native place that means, of you say that something is possible that P that always be the case that I mean there is also something which is not P also.

So, if you do not make this distinction of possibility of P, necessity of P and P then in Classical Logic this possibility of P and possibility of not P is simply viewed as this because it does not make any distinction between necessity and something which is considered to be actual there is no such kind of difference. So, this is interpreted as this one. So, this is considered to be contradiction, but in Modal Logic this need not have to be contradiction it is possible that P and of course, it is also possible that not P is also the case.

So, P is not contingent, not contingent means you need to put one negation here. So, something which is contingent is this one something which is not contingent is this one necessity of P necessity of not P of course, you know if you want to simply it a little bit then it will become negation of possibility of P negation of conjunction is r and this will become not P is P. So, again it further simplifies to this thing necessity of not P r necessity of not P, the negation of negation of this one second negation of this and negation of possibility of not P, so this is not.

So, not P as it is. So, this is negation of not P. So, this becomes this one negation of sorry necessity of P. So, then the sentence is said to be analytic then either P has to be necessity not P has to be necessity. So, P is analytic is to has to be in this way and P is consistent with q in logic we use phrase consistency, but in the case of natural sciences etcetera similar kind of word that is compatibility. So, P is consistent to q means P and q has to be I mean P has to be positive. So, it is possible that P and q stands for P is consistent with q , but P is consistent with q . P is compatible with q just you want to take say that this is different from the logical consistency and we have used a different symbol here.

So, that is P is O q it is same as it is possible that P and q and P is incompatible with q means it is not possible that P and q is the case. So, usually compatibility etcetera all these things we use it because the physical necessity. So, when we discuss about more about possible worlds then we are going to see things which are true in all possible worlds main logical necessity and metaphysical necessity I mean things which are true in only some necessity with respect to laws etcetera Nomic necessity are some other kinds of necessity usually we have different kinds of necessity. So, here is some of the examples.

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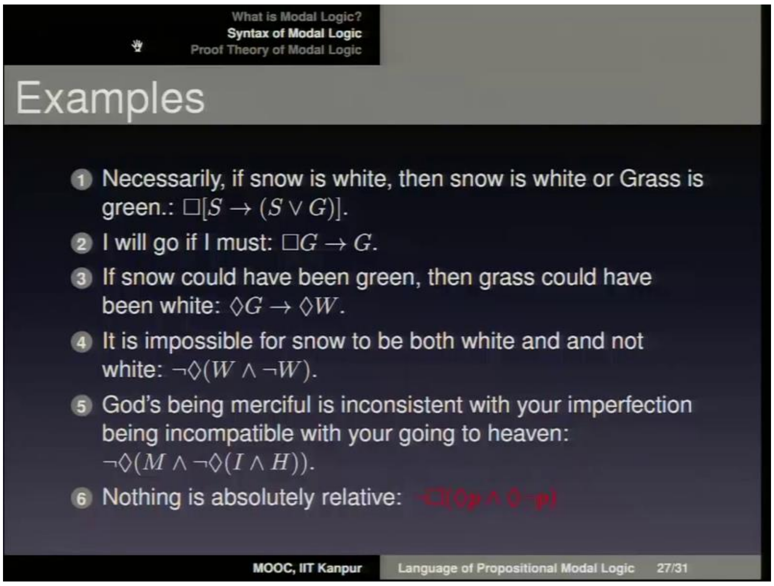


This goes like this necessarily, if snow is white the snow is white, and then grass: is grass is green. So, now, here the necessity operator operates over the whole condition the whole condition is S in plus S r is G now the second is I will go if I must. So, this is translated as it is necessary that G implies G like this we translate like this and coming back to the fifth example gods being merciful is inconsistent with your imperfection of being incompatible with your going to heaven it seems to be little bit complicated kind of sentence, but we need to little bit will have to break it suppose you need to read it like this what is incompatible with going to the heaven that is I and h these are incompatible I and H is incompatible means this is the formula it is not possible that I and H no now this one together with gods being merciful has to be consistent inconsistent. So, this together with this one it has to be inconsistent.

So, this consistent you put like this, but if it is inconsistency you put negation here this shows that this part and this one are not consistent to each other the sixth example is the one which we have already seen nothing is absolutely relative things are relative is possible that it is possible that P and it is also possible that not P and this is not related to nothing is absolutely related means whatever is the case possibility of P and possibility of not P need not have to be necessary.

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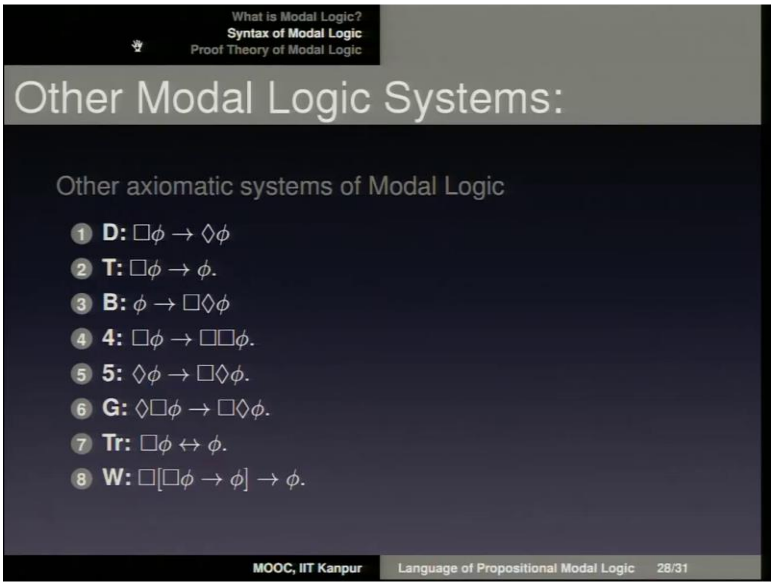


So, while translating etcetera we need to take care of the scope of the model operator. So, now, based on we just spoke about Minimal Modal Logic and till 1960s the attempt was this that modal logicians are trying to come up with various kinds of logical systems and they are named after one logician or the another for example, we have D, D states that it is necessary that phi implies that it is possible that phi the same kind of formula which was seeing it here it is necessary that phi implies that it is true that may not apply in D for example, if you say that you have to follow the traffic rules implies that you follow actually follow the traffic rules it may not be the case you can come out with the example you have to follow the traffic rules, but you sometimes in some cases you might break that particular kind of thing in the case of emergency you are in an ambulance or something like that you will be breaking the rules.

So, modal logicians were trying to come up with various logical system which suites some of the interesting things till the development of Kripke semantics modal logicians are constantly trying to prove theorems within this logical systems and they have come up with characteristics and these characteristics exempt are like this in the next class we will be dealing in a better you will be dealing in a these things in a systematic.

Thank you.