

Control Systems
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Lecture – 48
Frequency Response
Part – 2

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The image shows a series of handwritten mathematical derivations on a grid background. The derivations are as follows:

$$P(j\omega) = |P(j\omega)| e^{j\phi}, \quad \phi = \angle P(j\omega) \Rightarrow P(-j\omega) = |P(j\omega)| e^{-j\phi}$$

$$\Rightarrow a_1 = -\frac{U_0}{2j} |P(j\omega)| e^{j\phi}$$

$$a_2: * (s-j\omega) \Rightarrow \frac{P(s) U_0 \omega}{(s+j\omega)} = \frac{a_1 (s-j\omega)}{(s+j\omega)} + a_2 + \frac{n_1(s)}{d(s)}$$

$$\xrightarrow{s=j\omega} \frac{P(j\omega) U_0 \omega}{2j\omega} = a_2 \Rightarrow a_2 = \frac{U_0}{2j} P(j\omega) = \frac{U_0}{2j} |P(j\omega)| e^{j\phi}$$

$$Y(s) = \frac{a_1}{s+j\omega} + \frac{a_2}{s-j\omega} + \frac{n_1(s)}{d(s)} \Rightarrow y(t) = a_1 e^{-j\omega t} + a_2 e^{j\omega t} + \sum_{l=1}^k \sum_{m=0}^{(k_l-1)} c_{lm} t^m e^{p_l t}$$

$$\lim_{t \rightarrow \infty} y(t) = a_1 e^{-j\omega t} + a_2 e^{j\omega t} = -\frac{U_0}{2j} |P(j\omega)| e^{-j\phi} e^{-j\omega t} + \frac{U_0}{2j} |P(j\omega)| e^{j\phi} e^{j\omega t}$$

As $t \rightarrow \infty$, these terms tend to 0.

What can I say about this term as t tends to infinity. So, what can I say about all these terms as t tends to infinity? The system is stable right, system stable system means all poles are on the right of complex plane. So, what can you say about the exponent exponential term, what will happen to that? Does not matter, whether the poles are real or complex conjugates, does not matter what is going to happen to all the terms as t tends to infinity. We are going to tend to 0 right. So, since my system is stable, the plant transfer function has all poles in the left half complex plane ok all poles P_1 through P_n have negative real parts ok.

So, consequently, what is going to happen as t tends to infinity, all these terms will tend to 0 right, so that is what is going to happen to this term ok, so that is something which is very important for us. Is it clear, how we got this result ok? So, once again I am just using the fact that we are dealing with a stable system. So, what is going to happen, so this means that limit t tending to infinity, y of t ok, which is the steady state output right.

P of minus j ω , P of minus j ω is the magnitude of P of j ω times e power minus j ϕ .

Student: (Refer Time: 04:42).

Magnitude, see the magnitude is the same right yeah. So, see because P of minus j ω will be the magnitude of P of minus j ω e power minus j ϕ , but the magnitudes remain the same, it does not matter whether it is P of j ω or P of minus j ω that is how we got a magnitude of P of j ω to be common, yes, yeah.

Student: as t tends to infinity remain t tends to infinity $\sin \omega t$ plus while simply term (Refer Time: 05:09) it oscillates.

Oscillates yeah, so that is what we are finding the limit ok. So, you can converge to a single value, because it really depends right so what I can say is that this is the function, to which it oscillates ok. So, as opposed to a unit step response of a stable system in a unit step response, you will go to a constant value right, for a sinusoidal input, if you have a stable LTI system, it would be oscillatory ok. It will keep on oscillating that is what we can that is the frequency response ok, yeah.

Student: So, for a sinusoidal network initially the exponential terms were (Refer Time: 05:46).

Dynamics exactly.

Student: and then

And then, it will ultimately settle down to a sinusoid ok, but there is an important feature here ok. It will settle down to a sinusoid of the same frequency as that of input ok, we are just going to look at that here right so let me discuss those points right. So, this is the expression for the steady state output right so this is the steady state output. Please note that the steady state output is not a constant value here, but rather it depends on time, but you it is a what is a sinusoidal pattern right. So, this is the expression for the steady state output of the system ok.

Now, what is important here? Right so, you look at the frequency, the steady state output is a sinusoidal what to say a function. But, the frequency of the steady state output is also

the same as that of the input $u(t)$, so that is an important feature right. So, we immediately see that the frequency of the steady state output or let me or write it in this way. The steady state output is a sinusoidal signal of the same frequency as that of the input $u(t)$, but scaled in magnitude by the magnitude of $P(j\omega)$, and shifted in phase by ϕ , which is the phase of $P(j\omega)$ that is the observation we can make right so from this derivation right.

Once again, please go through this derivation pretty straightforward. Only thing is that like instead of numbers, we just worked with general variables, and functions and so on right, so that is what we did. So, you after the from the derivation, what we can see is that if you look at the steady state output of this class of systems, the steady state output is also a sinusoidal signal having the same frequency as the input frequency ω , that is extremely important $u(t)$. However, the steady state output is scaled in magnitude by $|P(j\omega)|$ why, because the magnitude of the steady state output is going to be $|U| |P(j\omega)|$, magnitude of $P(j\omega)$.

Student: (Refer Time: 09:06)

So, $|U|$ was the magnitude of the input signal right so, $|P(j\omega)|$ is the scaling factor as far as a steady state output magnitude is concerned. And that the steady state output is shifted in phase by ϕ right, because ϕ is the phase of $P(j\omega)$. So, the steady state output is shifted from the input sinusoid by a phase of ϕ $u(t)$, so that is what we have.

And this is a property of stable linear time invariant systems $u(t)$, so that is something which is very important $u(t)$, so that is for stable linear time linear systems. If you give a sinusoidal input and you wait and measure the steady state output, the steady state output also will be a sinusoid of the same frequency that is extremely important. And we will see that we will exploit this fact to discuss methods for the experimental determination of transfer function $u(t)$. And towards the end of this course, we will do a case study right. So, where I will give you data, and then like you can use these techniques to of experimentally determine the transfer function $u(t)$.

Till now, what the way we have been doing the determination of transfer function is through analytic means right. So, what did we do, we essentially wrote down equations of motion governing equations in the form of linear ODEs with constant coefficients.

Then we took Laplace transform applied 0 initial conditions, then we calculate at the transfer function right. So, we will also figure out how to empirically determine that ok there you will see that the step response and the frequency response are going to play a vital role ok, when we do that ok, is it clear, what is called as frequency response. So, this so what is frequency response? Frequency response is nothing but the response of the system, where to a sinusoidal input that is why it is called frequency response.

And please note that the scaling factor magnitude of P of j omega and the phase shift phi, which is the phase of P of j omega depend on omega right. Depending on the input frequencies, these numbers are going to be different ok. So, please remember that ok, so that is something, which you will address shortly ok yeah. So, this P of j omega is what is called as a sinusoidal transfer function ok. So, when you when you have P of s, it is a transfer function right. When you substitute s equals j omega, it essentially becomes a sinusoidal transfer function ok.

So, how do we get P of j omega, let us say as an example, let us say P of s let us take it as 1 by s plus 1 right, so that is a stable system right. So, then what happens P of j omega becomes 1 by j omega plus 1, this can be written as 1 minus j omega divided by 1 plus omega square right by multiplying and dividing by the conjugate right. So, this I can rewrite as 1 divided by 1 plus omega squared minus j times omega divided by 1 plus omega square right.

(Refer Slide Time: 12:52)

$|P(j\omega)| = \frac{1}{\sqrt{1+\omega^2}}$, $\angle P(j\omega) = -\tan^{-1}(\omega)$.

$P(j\omega) = \frac{n_1(j\omega) \dots n_m(j\omega)}{d_1(j\omega) \dots d_n(j\omega)}$, $|P(j\omega)| = \frac{|n_1(j\omega)| \dots |n_m(j\omega)|}{|d_1(j\omega)| \dots |d_n(j\omega)|}$, $\angle P(j\omega) = [\angle n_1(j\omega) + \dots + \angle n_m(j\omega)] - [\angle d_1(j\omega) + \dots + \angle d_n(j\omega)]$

Q: How can we visualize $P(j\omega)$ as ω is varied?

\rightarrow Bode Plot $\begin{cases} \text{MAGNITUDE PLOT} \rightarrow |P(j\omega)| \text{ vs } \omega. \\ \text{PHASE PLOT} \rightarrow \angle P(j\omega) \text{ vs } \omega. \end{cases}$

\rightarrow Nyquist Plot $\rightarrow \text{Re}(P(j\omega)) \text{ vs } \text{Im}(P(j\omega)) \text{ in the complex plane (P(s) plane)}$

So, consequently what can I have? So, how can I evaluate the magnitude of P of $j\omega$, I can do square root of real squared plus imaginary squared right. You take the real component square plus imaginary component square take the add and take the square root. Alternatively, I can also look at the function ok, if you have a ratio of complex functions, what is going to be the magnitude? It is going to be the same ratio of the individual magnitudes right. So, from here, I can immediately rewrite the magnitude of P of $j\omega$ divided to be equal to 1 divided by square root of 1 plus ω^2 right.

Of course, even if you do real part squared plus imaginary part squared take the square root, you will get the same answer you can double check ok, so you will get the same answer. And what is the phase of P of $j\omega$? Once again, you can do tan inverse of imaginary part by real part or you can just take the algebraic sum of the individual phases right. So, if the ratio is 1 by $j\omega + 1$, what will be the phase, the phase of 1 is 0.

Student: (Refer Time: 14:03).

And then since, what is the phase of $j\omega + 1$, it is going to be tan inverse of ω , but that is in the denominator right. So, you have a minus ok so this is going to be the phase of P of $j\omega$ for this example ok, so that is how we calculate the magnitude and the phase ok. So, once again you know like just as a general what to say generalization right. So, let us say if P of $j\omega$ is going to be $\frac{n + j\omega}{d + j\omega}$ (Refer Time: 14:49) all the way till $n + j\omega$ divided by $d + j\omega$ all the way till $d + j\omega$.

So, what we just did was it like magnitude of P of $j\omega$ is essentially going to be just the same structure, but the individual take the individual magnitudes right, so that is what we are going to have so it is going to be this. And what about the phase, the phase of P of $j\omega$ is just going to be the algebraic sum, so that is going to be the phase of $n + j\omega$ plus all the way till phase of $n + j\omega$ minus the phase of $d + j\omega$ all the way till the phase of $d + j\omega$ ok, so that is what we get right ok, so that is what we have oops sorry ok, so that is how we calculate the sinusoidal transfer function right. Now, are there any questions?

Student: (Refer Time: 16:12)

Yeah, tan inverse of minus omega, we just write it as minus tan inverse omega right, so ok, so that is it. Yeah, if you do tan inverse of imaginary by real, you will get tan inverse of minus omega, you just take it out right. Of course, it is a periodic function right, depending on how you want rewrite it ok. So, you will see that that is that is exactly my next point ok. Now, the question is you know like all these equations are fine, so from an engineering perspective, how are we going to visualize it ok. So that your question leads to my next topic of discussion, which we are going to continue from tomorrow but I am going to leave you with one question ok.

So, the questions which I am going to leave you with is the following right. So, how can one visualize a P of $j\omega$ as ω has changed, as ω is varied right. So, because let us say you know like from a design perspective right, so from an analysis perspective you know like this fine right. So, we have some expressions, we have some formulae and so on right. So, I have the plan sinusoidal transfer function as a function of ω and so on right.

So, now but you see that all these are like as a system order increases, you know like you are seeing that the sinusoidal transfer function becomes more or more complex in structure right. So, then you know like how does one interpret right, and use it for design ok. So, for that prospective from the prospective, you know like it becomes quite important to figure out, how one can visualize right the sinusoidal transfer functions variation as ω is vary.

Question is first of all we need to ask ourselves, can we visualize graphically, when this P of $j\omega$, how P of $j\omega$ changes as ω is vary, if so how? And the answer is yes, we can write and broadly there are three visualizations that typically are used. So, the first one is what is called as a Bode plot ok, so a Bode plot is one, where there is something called as a magnitude plot ok. A magnitude plot where the magnitude of P of $j\omega$ is plotted, this is ω ok.

But, of course on a different scale, we will discuss it tomorrow ok. We will use a logarithmic scale, and a logarithm measure. And we have what is called as a phase plot, where the phase of P of $j\omega$ is plotted against ω ok, so that is the first visualization ok. Of course, we are going to as I told you we are going to use logarithmic

measures and logarithmic scales right. So, we will discuss more about that tomorrow ok. So, a Bode plot is very commonly used ok.

So, a second visualization is what is called as a Nyquist plot. So, a Nyquist plot is essentially a plot of the real part of P of j omega, this is the imaginary part of P of j omega in the complex plane ok. What is called as a P of s plane so right? So, we will see what this is right, as we go further.

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Q: How can one visualize $P(j\omega)$ as ω is varied?

→ Bode Plot

↔ MAGNITUDE PLOT → $|P(j\omega)|$ vs ω .

↔ PHASE PLOT → $\angle P(j\omega)$ vs ω .

→ Nyquist Plot → $\text{Re}(P(j\omega))$ vs $\text{Im}(P(j\omega))$ in the complex plane ($P(s)$ plane).

→ Nichols Plot → ?

The 3rd plot is what is called as a Nichols plot ok. So, what is this Nichols plot? I leave it you as a question ok, just search, and find out and tell me the answer tomorrow ok. So, by enlarge, we are going to focus on the Bode plot ok. The Bode plot is something, which you will see that it is very frequently used in engineering applications as far as studying frequency response is concerned ok, so that is something, which we are going to concentrate to a great degree.

And we are also going to learn how to use the Bode plot for design purpose ok. So it does not matter, you know like what is your domain automotive biomedical, you know like or even other domains right. So, you will see that you know like Bode plot is something, which is frequently used ok, in the design process.

So, we are going to study Bode plot in detail. A Nyquist plot, and Nichols plot, I will just explain what it what they are ok, so that will be the extent of essentially discussion that

we will do. Because, we want to get into design as using frequency response for which Bode plot becomes pretty important right. So, we will start with the Bode plot in the next class and then, we will see how given a transfer function, we can get the Bode plot of that transfer function ok.

Then you know we will also ask the inverse question, if say through some means I give you the Bode plot of a system right, which is which can be obtained through experiments can you figure out the transfer function all right, so that is the inverse question to ask ok. So given P of s what is a Bode plot that is something, which you will learn and then given the Bode plot, can you figure out P of s that is a question we will ask ourselves right, because as an engineer that is pretty important right. So, both problems are both questions are equally important to us fine right. So, I will stop here, and then like we will continue from this place tomorrow.