Circuit Analysis for Analog Designers
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Lecture - 07
Inter- reciprocity in linear time-invariant networks

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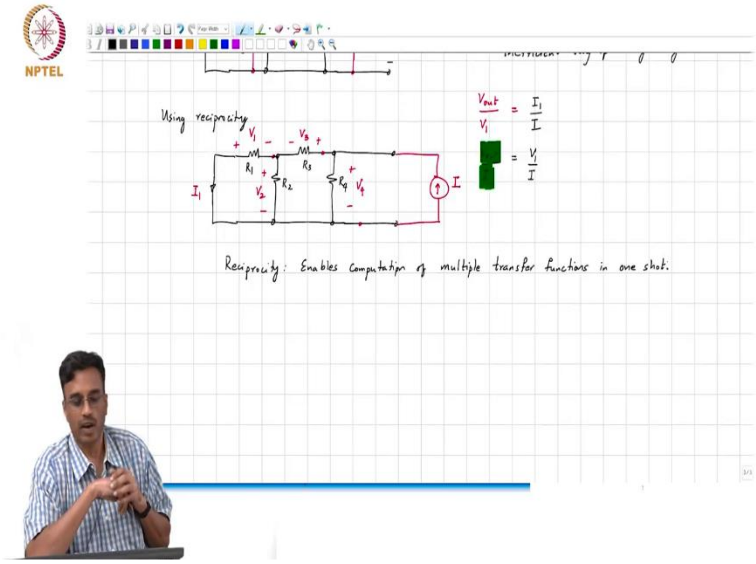

Ok so; obviously, it would be great if you could find a fix to you know find some way of being able to use this concept of reciprocity in networks that contain control sources ok.

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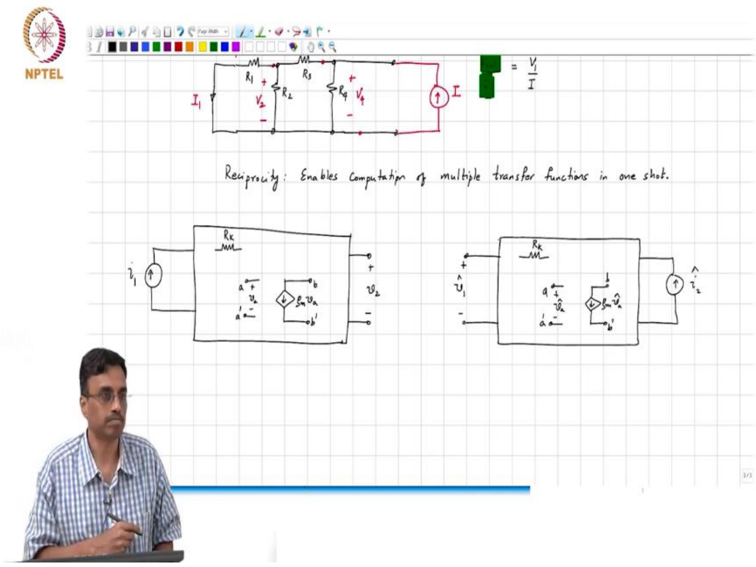

Now, let us see what we can do and again let me start with an example where we have you know as usual we have our friend this network N and as usual let us try and excite the network with a current i_1 and measure the voltage v_2.

And now we say apart from resistors there is also a control source I mean you know it makes sense to start with simple things you know one at a time. So, let us say this is one control source once you figure out what you can do with one control source you can figure out what you can do with multiple control sources.

So, let us say you have a voltage controlled current source. So, let us call this a, a prime, b, b prime and this is $g_m v_a$ alright. So, this is our network N and now we say let us try our luck. So, as you know fortune favors the brave. So, we just basically say you know it is a free country let me try and apply Tellegen’s theorem to you know what we did with the passive network right where we interchange the location of the excitation.

So, we will call this say $\hat{i}_2$, this is $\hat{v}_1$, there are resistors which I call R sub k and let us see what happens here a prime this is v_a this is $\hat{v}_a$ actually and this is $g_m v_a$, this is b and this b prime.

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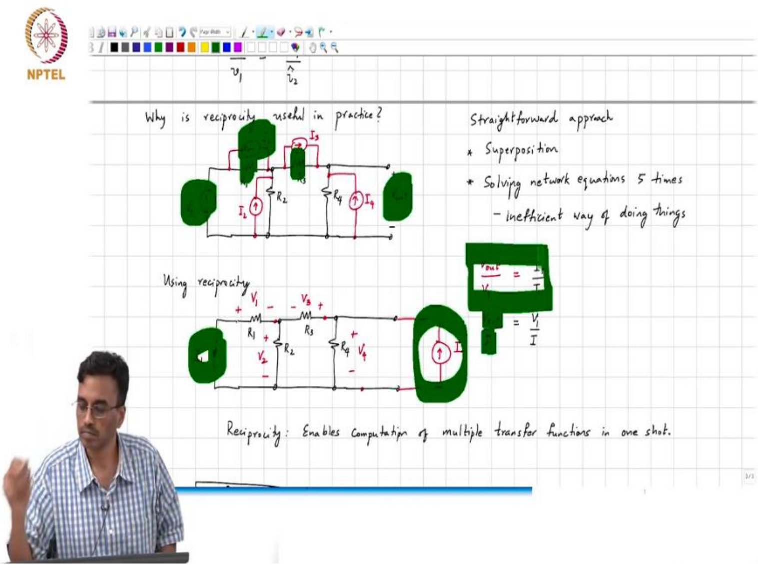

I like to find this now since he brought it up let me you know kind of digress a little bit it is always confusing to figure out whether you know I flows you know this which way or that which way right. So, what I usually like to do is the following.

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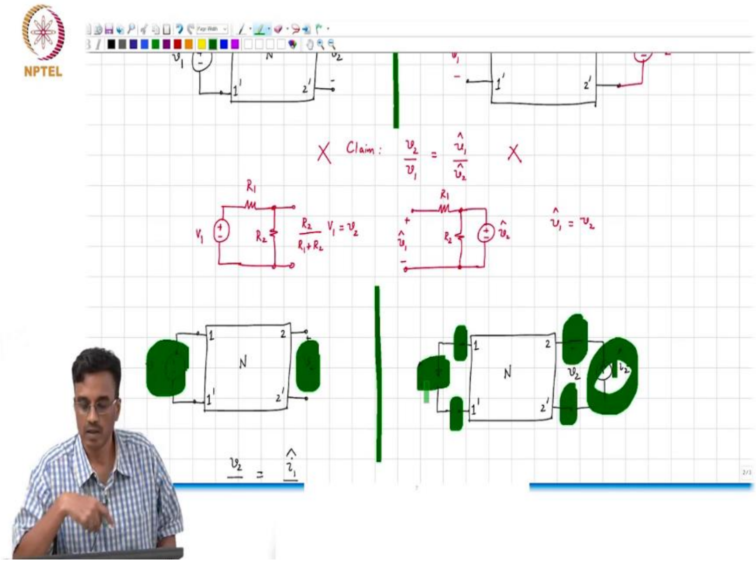

So, if you apply v_1 here it is easier with this example right if I apply v_1 here right let us assume I mean and v_2 develops here. So, if on the other hand if I apply $\hat{\imath}_2$ here there is some voltage developing you know with this terminal greater than this terminal right. So, you in the absence of a short you would expect that this voltage would be larger than this voltage correct.

So, the moment you make the short circuit current flows which way current flows downwards right I mean this is not rigorous by any means, but, but this is a you know this is a within quotes if you like to think of it these are some kind of mnemonic which will prevent you from making errors because I mean you do not want to sit and write Kirchoff's and Tellegen's theorem every time you want to find out whether the sign of this is positive or negative correct.

So, what you call what you want to think of doing is when you replace the excitation, I mean the measurement port with the excitation or there is a voltage developed in that direction right which would cause normally cause this voltage to be higher than that the lower potential and therefore, when you short it, the current would flow from the upper terminal to the lower terminal right.

So, that is just the way I like to remember it you can come up with your own. So, now, you know given that right if I apply v_1 here and if I get a positive voltage v_{out} right it turns out that you know if I apply a current like this, the current must flow in this direction for this theorem to be backed right ok.

So, coming back to the limitations of reciprocity as we have seen so far stemming from the fact that we only had you know elements which you know typically R L and C elements inside the network and while we like R L and C in electronics we are dealing with control sources all the time.

And we would like to see if we can make any headway with control and that is what we are going to see next and you know as I said fortune favors the brave. So, we just be brave and you know write the same equations the same approach that we took for the passive case and see where that leads us and if all things work out you know we are happy family and go home right.

If they do not work out, we figure out why they do not work out and fix it right and see if it can be fixed alright.

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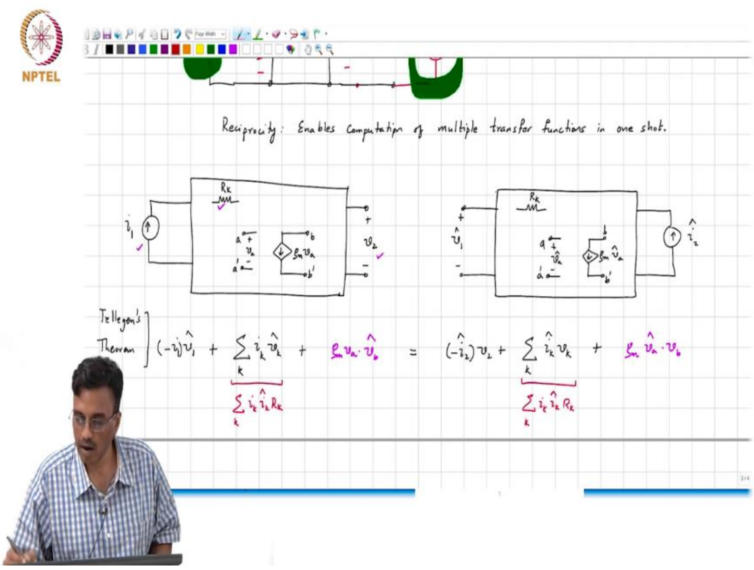

So, now help me with this now. So, i 1 times v 1 hat right plus i 2 which is; what is I 2? I mean we were just writing Tellegen’s theorem now i 1 times v 1 hat plus I 2. What is the i 2? 0. So, you know v 2 hat is irrelevant plus sum over all internal branches i k v k hat right that would be all that would we have we would have to do on the left-hand side if there was no controlled source.

Now, we have a controlled source. So, what you think we should have here plus. So, this will so this will cover we have covered this, we have covered this port, we have covered all the resistive branches. What should we cover now? The only thing left is the controlled source right. So, i k that is the current in the port here on the port a is what? It is a voltage controlled current source. So, the current in the in this port is 0. So, that is of no concern right plus g m v a times v b hat sorry that must be v b hat correct and this must be equal to. Sorry yeah sorry thank you right. So, what is the right-hand side of the equation?

i 2 so, i 1 hat what is i 1 hat? i 1 hat is 0. So, we do not worry about that. So, what we need to do is we need to worry about i 2 hat it is actually minus i 1 times v 1 hat. So, this is minus i 2 hat times v 2 plus sum over all resistors inside which is i k hat times v k plus g m; well, we need to do i k hat right here i k is 0 right as far as the first port is concerned a port is concerned. The current is 0. So, we only have to worry about the b port right and that is what now? g m times v a hat times v b ok.

$$(-i_1)\ \hat{v}_1 + \sum_k i_k\ \hat{v}_k + g_m v_a\ \hat{v}_b = (-\hat{\imath}_2) v_2 + \sum_k \hat{\imath}_k\, v_k + g_m\, \hat{v}_a\, v_b$$

And these what comment can we make about this character and this character? This is simply nothing, but $\sum_k i_k\ \hat{\imath}_k\ R_k$ and this is s$\sum_k i_k\ \hat{\imath}_k\ R_k$ ok.

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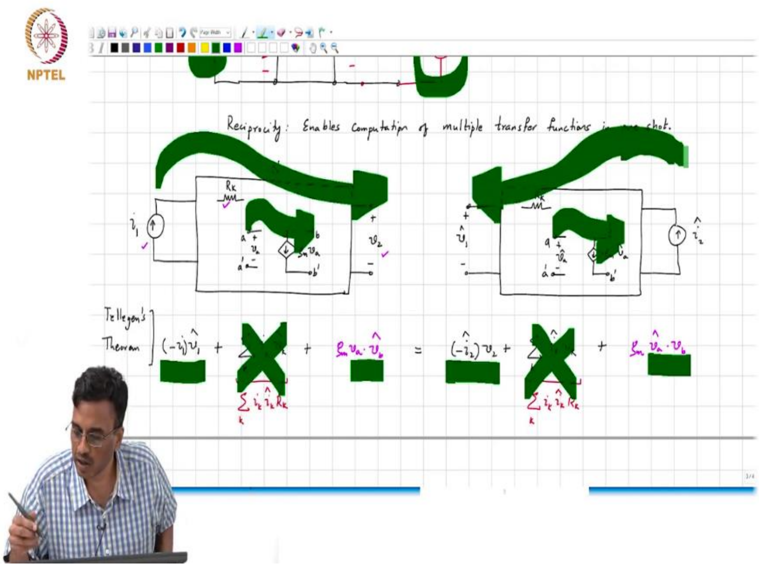

So, what comment can we make about these two quantities? They cancel like before no surprise, correct and we would be able to get our desired relationship between the port voltages and currents, except that there is this little irritant which is $v_a \ \hat{v}_b$, on this side and this is $\hat{v}_a \, v_b$.

Are these two the same? I mean can we move the hat from you know a to b and then you know be happy? I mean are these are those two equivalents I mean the quantities in magenta are they equal?

There is no reason why they should be alright ok. So, now the question is you know what do we do we scratch our head and say oh well you know it almost works except for this extra term right. Now, we say ok, well I mean we think about it a little bit and then say oh well you know it was naive for us to expect that it would work in the first place correct remember what is reciprocity saying you know in English what does it mean you have a network you apply an excitation here you get some response you apply the same excitation here you get back the same response here right.

But if you have a controlled source the job of the controlled source is to make sure that the output is controlled by the input, but it does not mean that. You go and apply the same input at the output port and you get the response at the input port, I mean the job of the controlled source is to do that. So, you know it would be terrible if actually reciprocity work with a controlled source because the controlled source job is to allow signal to flow only in one direction and reciprocity is saying that you know it flows in this direction and this direction in the same way right, ok.

So, if you have to have any hope of making this work I mean what reciprocity you are hoping to get is to find that the transfer from here to here is the same as the transfer from here to here correct ok.

If that were to happen right you can see that from a to b signal flows like this right b is influenced by a, if you want the same transfer function you know when you apply the excitation on this side you know here the signal still flow can only flow in that direction. So, that clearly you know it is apparent why that does not work right.

So, what is your hunch the first thing that you would like to do to try and see I mean if you wanted to have the same transfer function from the right to the left right what would you try.

Well, I mean the hunch is to say well you know here the signal flows from left to right ok, whereas, the excitation flows from anyway excited on the right and we expect it to flow to the left the same amount as it does on the left hand side right, but clearly there is a difference in the orientation of the controlled source. So, let me see if I flip the orientation of the controlled source in this direction.

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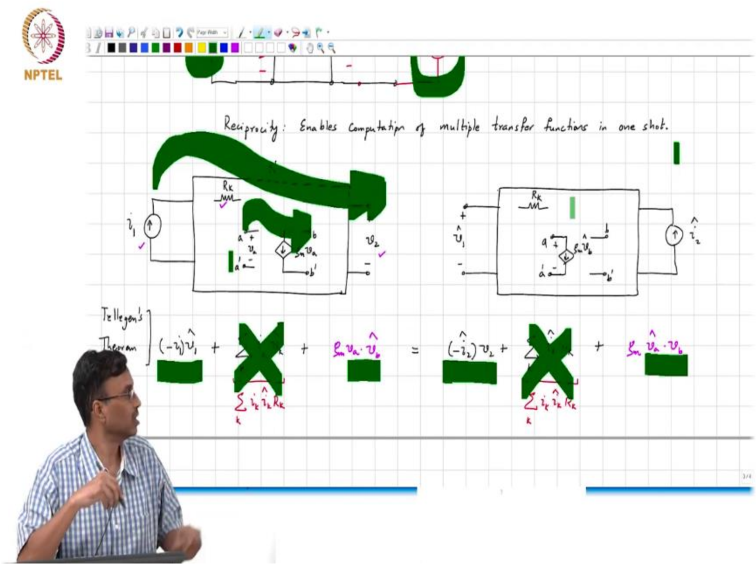

So, this is $g_m \times \hat{v}_b$, alright and see what happens at least it seems to make intuitive sense because here you have apply the excitation on the left and the controlled source has a controlling port on the left and a controlled port on the right. It seems reasonable that you know given that the controlled the control source is the source of all our problems.

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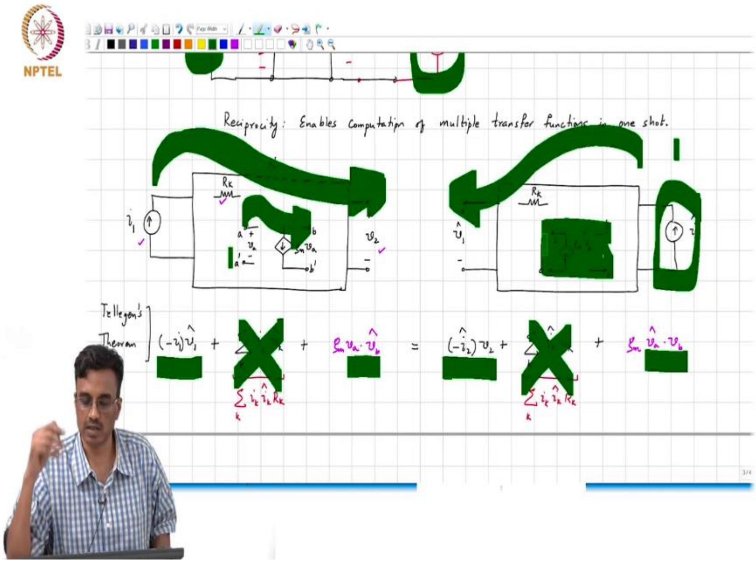

If we put in an excitation on the right and measuring the response on the left then you know the what you call it seems reasonable to flip the orientation of the controlled source and we just try at this point it is just a hunch.

Now, so, what comment can be now, so, now the Tellegen's theorem expressions which expressions will change, which are the only terms that will change? The ones in sitting in magenta right.

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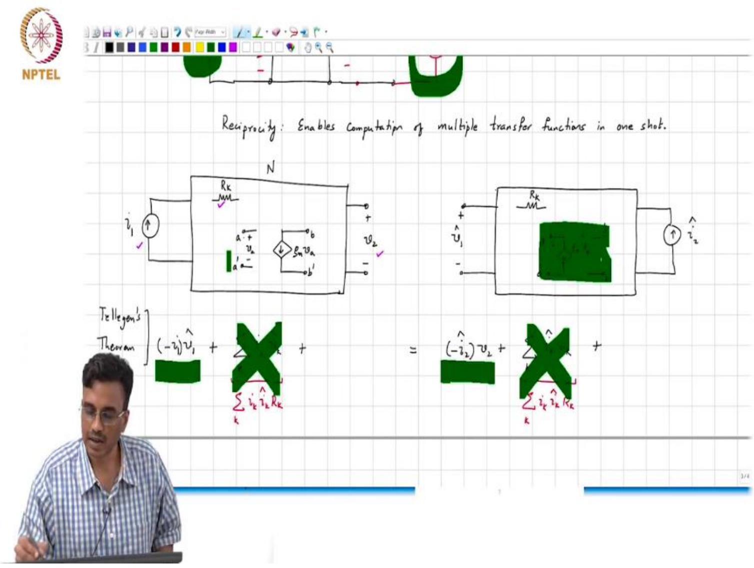

So, help me know figure that out. So, i 1 I mean i a is still. i a is still 0. So, and so, therefore, i a times v a hat is 0 i b is g m times v a Times v b hat very good right and on the right-hand side what do we see?

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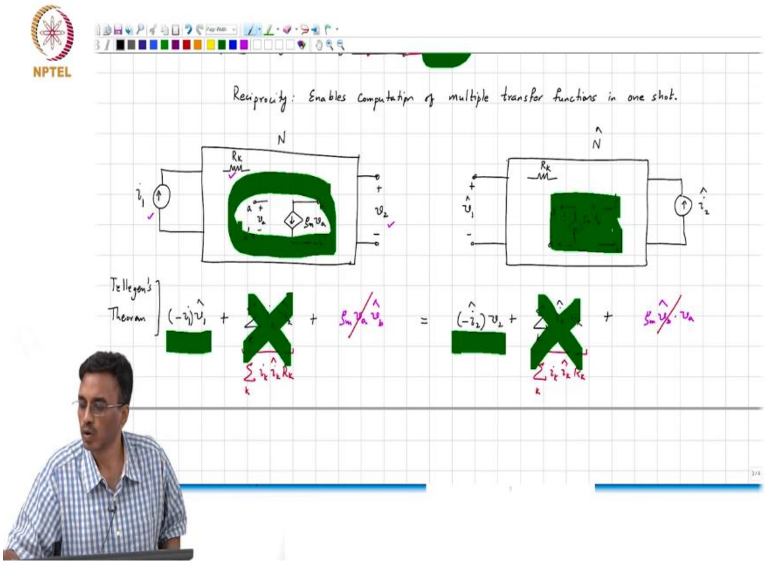

i i a is g m v b hat times v a plus i b hat is 0. So, that goes away. So, what do we see now? Well, we are not able to thank god we are now able to cancel off these two guys; does it make sense ok?

$$(-i_1)\ \hat{v}_1 + \sum_k i_k\ \hat{v}_k + g_m v_a\ \hat{v}_b = (-\hat{\imath}_2) v_2 + \sum_k \hat{\imath}_k\, v_k + g_m\, \hat{v}_b\, v_a$$

So, basically this means that well we have an original network with some controlled source inside ok; however, if you interchange the location of the excitation and the response not for the same network, but another network where you simply flip the orientation of the voltage controlled current source, then you will find that you know v 2 by i 1 is the same as v 1 hat by I 2 hat, does make sense.

$$\frac{v_2}{i_1} = \frac{\hat{v}_1}{\hat{\imath}_2}$$

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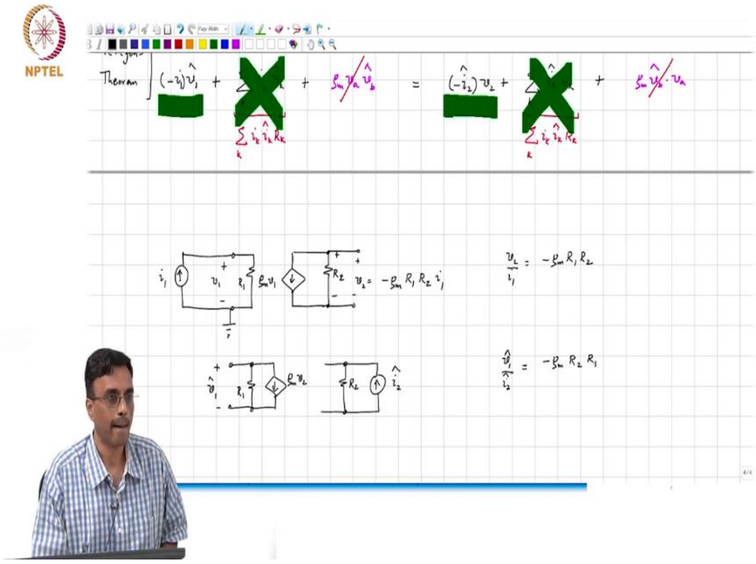

So, here let me take you know as a quick example the most trivial example is of course, just one controlled source, let us say this is our sorry. So, this is $g_m v_1$ and this is R. So, what is v_2? Sorry we put in a current right. So, let us call this R_1, i_1 ok. So, let us call this R_2. So, what is v_2? $-\, g_m\, R_1\, R_2\, i_1$ ok.

Now, what I am going to do, I mean basically all the resistors must remain the same remember what did we do for the controlled source? We just flipped its orientation. So, what should I put here? The controlled port becomes the controlling port and vice versa. So, this is R_1. This is $g_m v_2$, this is R_2 and this is $\hat{v}_1$ and this is $\hat{\imath}_2$.

So, v 2 by i 1 in the first case happened to be minus g m R 1 R 2, v 1 hat by i 2 hat turns out to be what? Minus g m times R 2 times R 1 and as you can see the transfer functions are the same alright ok.

$$\frac{v_2}{i_1} = -g_m\, R_1 R_2$$

$$\frac{\hat{v}_1}{\hat{\imath}_2} = -g_m R_2 R_1$$

So, I mean. So, if we had one voltage controlled current source, we just flipped that one voltage-controlled source the other way round and you know we have another network N hat where reciprocity like relationship is well ok.

Now, if you have 100 voltage controlled current sources what would you do? You flip the orientation of all the 100 ok.

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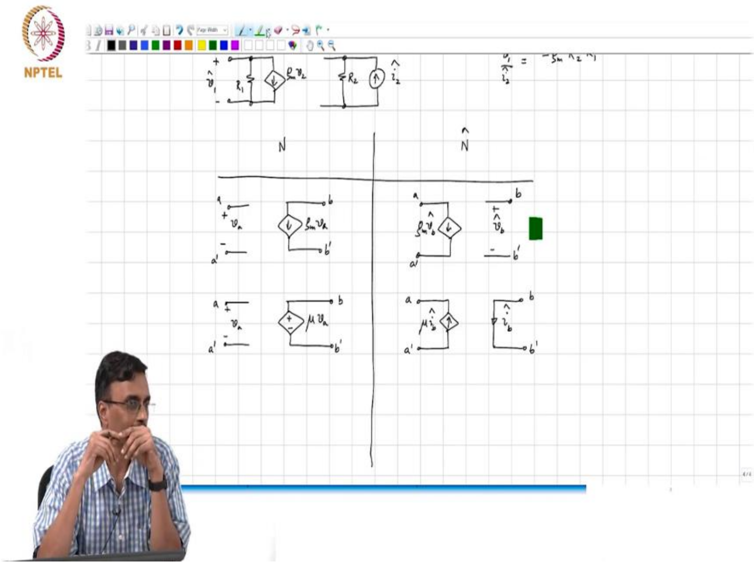

Now if you had. So, in other words the rules as the following if you have a voltage controlled. So, this is N if we had a voltage controlled current source in N to get a reciprocity like relationship going, we must flip the orientation of the current sources or control sources as shown ok.

Now, the next obvious question is what? What is the next obvious question? I mean this is not the only controlled source. We know more current controlled sources the next thing for instance what happens when you have a voltage-controlled Voltage source ok. So, a, a prime, b, b prime. So, this is v_a and let us call this gain $\mu \times v_a$ alright. What you think you will do here?

I mean the use the same intuition as before right the what do you call the controlling and the controlled ports must be reversed in N direction correct. So, what comment can you so which must be the controlling port? It must be port b ok, another aspect that I would like to draw your attention to is the following right what comment can you make about the port impedances here and here.

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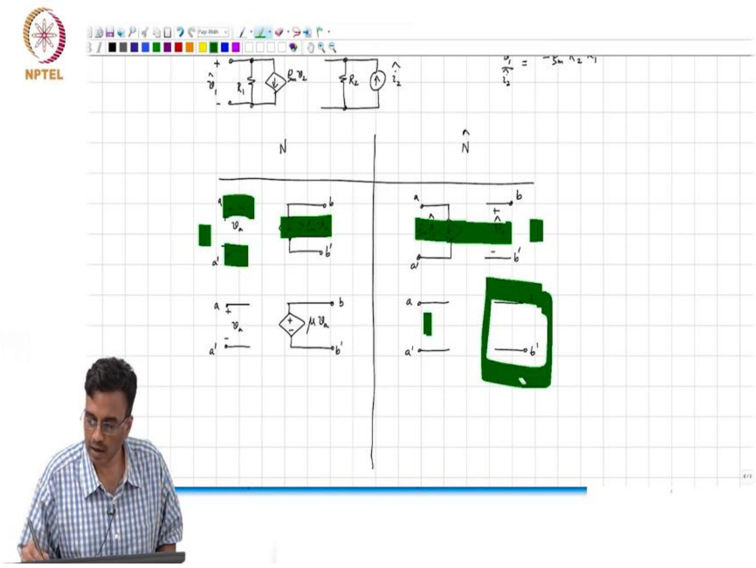

So, in other words if you set g_m to 0 right what comment can you make about the impedance here across a and a prime? Infinite it is an open circuit and likewise with b and b prime and what about here? It is the same thing correct alright now.

So, we know two things one is we know that this is now the controlling port and this is the controlled port we also would like to make sure that the when μ is set to 0.

When μ is set to 0 the port impedances are the same in both directions in both cases. So, when μ is set to 0 what happens to v a is it an open circuit or a short circuit? On the left hand side when μ is set to 0 I mean is the port impedance between a and a prime is it.

It is infinite what about between b and b prime. It is 0 right. So, b and b prime is a controlling port whose impedance is 0 correct. So, what must what kind of controlling port must it be? It must be sensing a current right.

So, this is $\hat{\imath}_b$ and what the impedance between a and a prime on the right-hand side must be infinite when μ is 0. So, what kind of controlled port must that be? Current source very good. So, this is $\mu\,\hat{\imath}_b$.

We still should work for every single value of mu; we should work for every single value of μ. Yeah they work for every single value of μ. All I am saying is that you know again to figure out especially when you have voltage controlled voltage sources and when you have current controlled current sources it is easy to forget you know which side is you know whether it you need to put a voltage controlled voltage source on the other side or a I mean a tempting thing to do would be to put a voltage controlled voltage source on the other side where the controlling and controlled ports are flipped right ok.

The way to remember that is that when you set the controlling parameter μ to 0 the impedances on both sides must be the, must be the same on the left side as well as on the right side.

Oh ok. So, I mean I think that comes back to his question we which is you know so what right, I mean you know we have done all this so what, why is this used? Correct and again it comes back to the fact that in many practical situations one is interested in finding the transfer function from multiple sources to a single output right.

In the passive case one use reciprocity to advantage. So, you took the same network you just simply interchange the location of the excitation and in one shot you are able to calculate all these transfer functions that you are looking for right. Here now what would you do?

All that we are going to do as we will show in the next class is that we have this original network with controlled sources, we want to find the transfer functions from multiple sources to one output. What we are going to do is to form this new network right that is still in our heads. It need not be done in practice this is just a technique for computation and determination, we are going to create something with where all the controlled sources are flipped as per our recipe, we have shown here we need to still you know consider two more controlled sources.

And in that network, we are going to interchange the excitation port with the response port right. And as we will see going forward it basically boils down to, I mean you know rather than use the original network when you use this new network right where the reciprocity kind of relations hold and therefore, again with one shot you can get all the transfer functions. Does it make sense people? Is that clear? Good.