Circuit Analysis for Analog Designers
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Lecture - 60
Weak nonlinearities in circuits: Calculating non-linear components using the method of current injection

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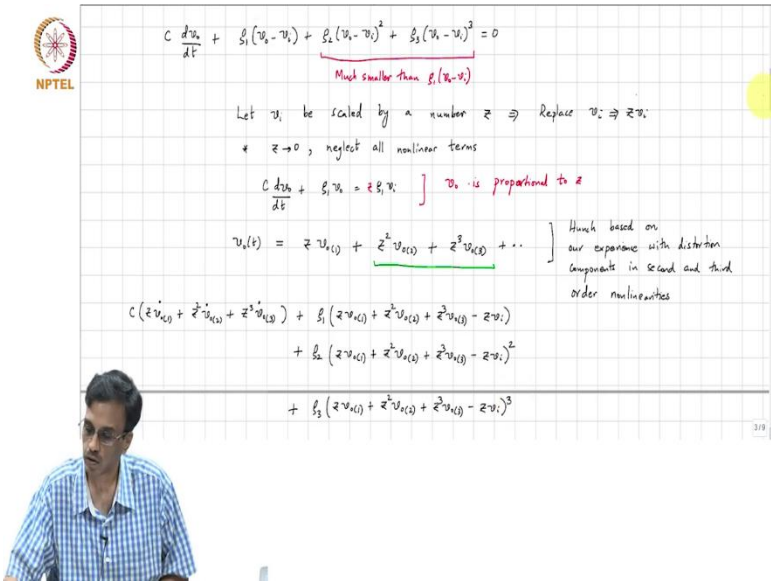

The hope is that, we do not have to solve this non-linear differential equation right which can only be done numerically. You understand. So, what do we do? Well, we know that the output is of this form. So, what you do? You go and back substitute this in the original equation and see what we get.

So, C times I am going to use the notation v dot to denote the derivative. So, this is z times v o 1 dot plus z square v o 2 dot plus z cube v o 3 dot ok etcetera plus g 1 times z times v o 1 plus z square times v o 2 plus z cube times v o 3 Minus v i right plus g 2 times Sorry yeah exactly thank you. So, this is minus z times v i remember we are applying an input which is z times v i and so, let me copy this right. So, this times

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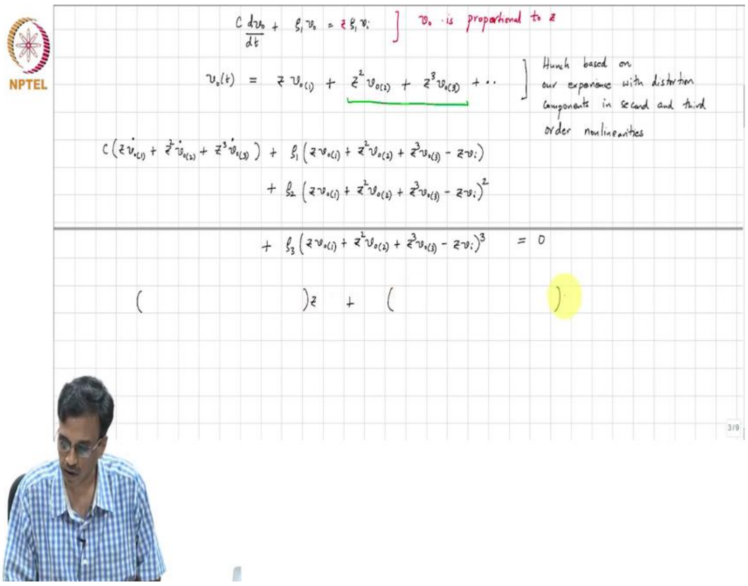

Square plus g 3 times cube must be equal to 0 alright.

$$C\left(Zv_{o_{(1)}}^{\cdot} + Z^2\, v_{o_{(2)}}^{\cdot} + Z^3\, v_{o_{(3)}}^{\cdot}\right) + g_1\left(Zv_{o_{(1)}} + Z^2 v_{o_{(2)}} + Z^3 v_{o_{(3)}} - Zv_i\right)$$
$$+ g_2\left(Zv_{o_{(1)}} + Z^2 v_{o_{(2)}} + Z^3 v_{o_{(3)}} - Zv_i\right)^2$$
$$+ g_3\left(Zv_{o_{(1)}} + Z^2 v_{o_{(2)}} + Z^3 v_{o_{(3)}} - Zv_i\right)^3 = 0$$

And what comment can we make about z? z is an arbitrary number. Correct? z can be 10 power minus 3, z can be 10 power minus 4, z can be 10 power minus 2, z can be 1 right. So, if this equation has to this differential equation non-linear differential equation has to be if you have to have a solution then it must be valid for all values of z right. So, that basically means that you have something if I can group all the terms with z right ok, and then I can group all the terms with let me do that.

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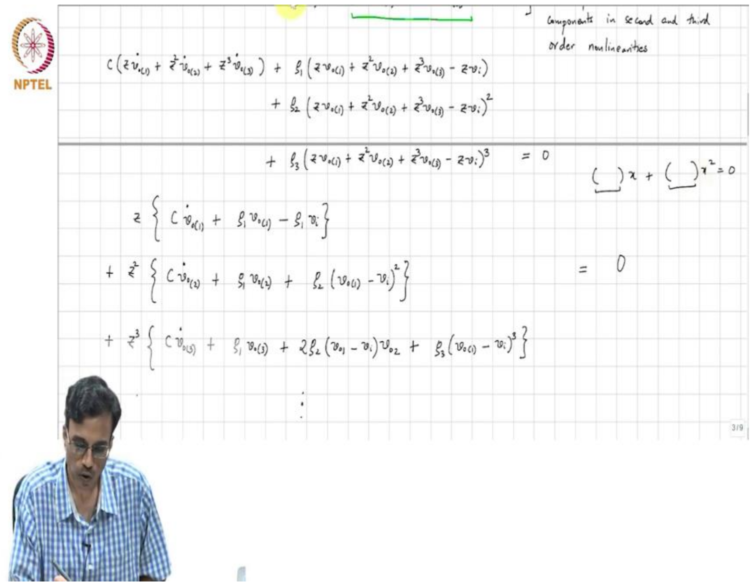

So, if I can rewrite this equation as if I group all the terms with z what do I get? I get C times v 0 1 dot. What about the z square term and z cube term will they come in here?

So, I am grouping all the terms which have only z in them. Do you understand? Right. So, what will happen to these two terms, will they are they relevant when we group only the terms with z together?

Student: It is one like z is would not be come in to the z 2 and z 3 times like square and q are there.

No, you are not understanding what I am saying. What I am asking you is, what I mean do you understand what we are trying to do?

Student: Yeah, we are looking the all the terms which have to z.

Which we have z. So, now, I am asking you what happens with these two terms, do we need to worry about them when you are grouping all the terms with z or you do not have to worry about them.

You do not have to worry about ok right ok. So, what terms do we have to worry about in the in the g 1?

g 1 times v 0 1.

No.

Will have a v i term correct. So, basically minus g 1 times v i right. Do we have to worry about?

g 2 the lowest order term in z is what would be the lowest degree of z in the if I expand out the g 2 term? It will get z square. And therefore, you know you basically have a we do not worry about any of that stuff and likewise with g 3 right. Then let us correct all the z square terms, correct how does that look like? So, there is C v 0 2 dot. Right plus g 1 times v 0 2. Is there anything else we need to worry about? We do not because everything else is either first order or third order correct ok. Then we come to the g 2 terms, what do we see there?

g 2 times v 0 1 minus v I the whole square right ok. And the other terms we will all have degree which I which is higher than 2. Do you understand?

Right ok alright plus z cube times C v 0 dot 3 dot plus g 1 times v .0.3 plus with g 2 now we should see if we should expect any third order term. Do we see any third order term needed? Because you have a z term a z square term and a z cube term.

So, basically when you multiply the z square and the z cube you will basically I mean sorry the z and the z square term you will get a z cube. So, that is basically 2 g 2 times v 0 1 minus v i times v 0 2 ok alright ok that is not all we have the g 3 term also plus g 3 times v 0 1 Minus v i the whole cube right plus is there something else? All the other terms will be, we will have order which is higher right. Plus, you know you will have you know for each order of z you can go on writing this stuff plus blah blah blah must all be equal to 0. Does make sense?

$$
\begin{aligned}
Z\left\{C\,\dot{v_{o_{(1)}}}+g_1\,v_{o_{(1)}}-g_1 v_i\right\}+Z^2\left\{C\,\dot{v_{o_{(2)}}}+g_1\,v_{o_{(2)}}+g_2\left(v_{o_{(1)}}-v_i\right)^2\right\} \\
+Z^3\left\{C\,\dot{v_{o_{(3)}}}+g_1\,v_{o_{(3)}}+2g_2\,\left(v_{o_{(1)}}-v_i\right)v_{o_{(2)}}+g_3\left(v_{o_{(1)}}-v_i\right)^3\right\}+\cdots \\
=0
\end{aligned}
$$

Yeah ok. Now this must be valid for any value of z. So, if that has to be true what conclusion can we draw from this?

If something into x plus something into x square is equal to 0 for all values of x. What comment can you make about this something and that something?

They must be 0 correct. So, if these terms are 0 for all values of z right.

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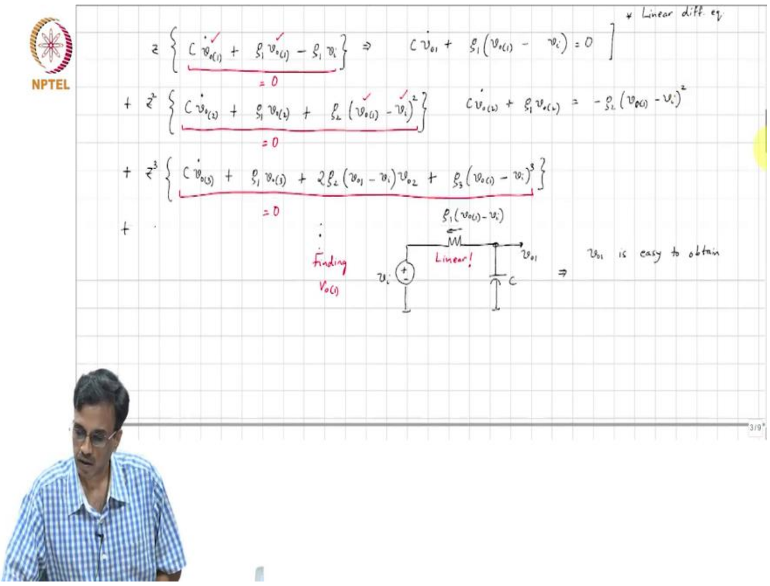

It must therefore, follow that, $(C\,\dot{v_{o_{(1)}}} + g_1\,v_{o_{(1)}} - g_1 v_i)$ this must be 0, $(\dot{v_{o_{(2)}}} + g_1\,v_{o_{(2)}} + g_2\left(v_{o_{(1)}} - v_i\right)^2)$ this must be 0, this must be 0, $(C\,\dot{v_{o_{(3)}}} + g_1\,v_{o_{(3)}} + 2g_2\left(v_{o_{(1)}} - v_i\right)v_{o_{(2)}} + g_3\left(v_{o_{(1)}} - v_i\right)^3)$ this must be 0 and so on alright. So, now let us see what that means. So, what this means is that, C times v 0 1 dot plus g 1 times v 0 1 minus g 1 v i sorry minus v i equal to 0.

$$C\,\dot{v_{o_{(1)}}} + g_1\left(v_{o_{(1)}} - v_i\right) = 0$$

So, what does this mean? I mean can you recognize? What this is a linear differential equation. So, what kind of? So you can therefore, draw a network which basically satisfies that. And what is that network do you think?

Very good. So, basically this is basically the same as our original network. Except that what is the difference then? The output will be v 0 1, but what is the difference between the original network and this network? The resistance is linear right. So, this basically is nothing, but g 1 times v 0 1 minus v i ok, $g_1 \left(v_{o_{(1)}} - v_i\right)$. So, this is remember this is a linear network linear element alright. So, is v o 1 easy to solve or difficult to solve?

Right. So, v o 1 is easy to solve because, because the network is linear we know what to do. Does that make sense? Alright. So, therefore, we know we are now in a position. So, the moment you solve this you are now in a position we know that. Now let us go to the second term right. So, this is finding v 0 of 1 alright. Remember what are we trying to find? We are trying to find.

Yeah, we are trying to find v 0 1, v 0 2, v 0 3 etcetera right. So, basically, we found v 0 1 right. Now how do we find v 0 2? Well, we look at this here. What do we get? The since this is 0, what comment can you make about this differential equation? This is also a differential equation.

Is it linear or non-linear? Which is the known and which is the unknown? If the unknown is v 0 2 right. What about v 0 1? From the previous step we know v 0 1. So, v 0 1 is known. What about v i is known or not known? v i? What is vi? It is the input. So, vi is known.

Do you understand? Right. So, what is the unknown? v 0 2 ok. So, now, is this a linear differential equation or a non-linear differential equation? It is a linear differential equation right and so, therefore, C v 0 2 dot plus g 1 v 0 2 equals? Minus g 2 v 0 1 Minus v i the whole square.

$$C\, \dot{v_{o_{(2)}}} + g_1\, v_{o_{(2)}} = -g_2 \left(v_{o_{(1)}} - v_i\right)^2$$

So, how does this look like? So, the input is that a current or a voltage? What is this equation man, is this a Kirchhoff’s current law or a Kirchhoff’s voltage law? Yeah. So, which. So, this term here it is going to be?

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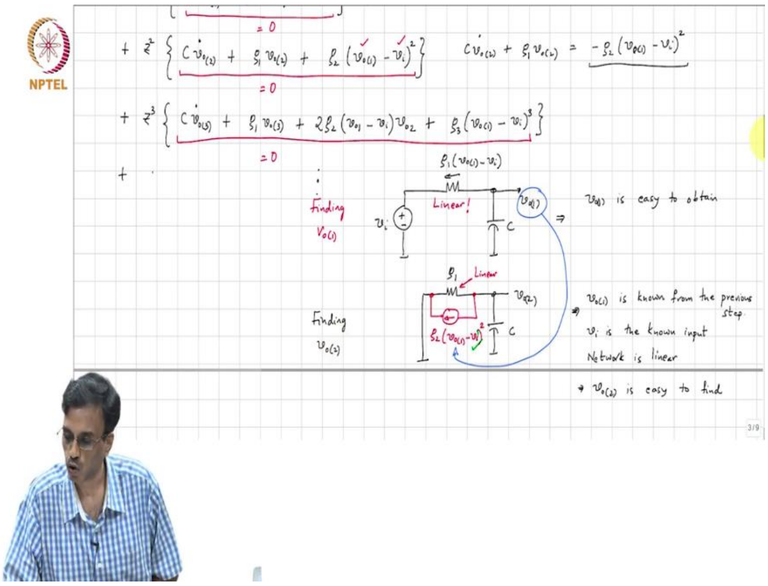

Current ok. So, what is the network? What does this represent? This represents the linear network.

Correct? So, this is the capacitance right. Ok and I mean. If this if this excitation was 0, if the right-hand side was not there, what would the network look like? g 1 is the resistance very good. And where would that go? It will go to 0. Does that make sense?

And now what are we adding in addition to. So, basically that saying now i mean we have this network and we are exciting the capacitor node v o 2 with a current which is. It is a known current. Which is basically, how much is that current? It is going between ground and this and this is connected across like this and this is g 2 times v 0 1 minus v i the whole square, $\left(g_2\left(v_{o_{(1)}} - v_i\right)^2\right)$. Does make sense? Right. And is this resistance linear or non-linear?

This is also a linear resistor right. And this is of course, the input and therefore, this is known and v o 1 is known from v o sorry this is we must be written as v o 1, that is known from the previous step alright. So, therefore, v o 1 is known from the previous step, v i is the known input and this is a network is linear.

Right. So, v o 2 is easy to find right. So, this is step 2 right. So, finding v 0 2 alright ok. Now we know v 0 1 and v 0 2. So, what comment can you make about v 0 3 now?

So, this differential equation must be 0. Is this a linear differential equation or is it a non-linear differential equation? Its again a linear differential equation. What is the unknown? v 0 3 is the unknown and all v 0 1 is known from step 1. v i of course, is known, v 0 2 is known from? Step 2 v 0 1 is known from step 1, v i is known right. So, basically, we have everything that we need correct. So, what is the network now?

Student: Similar to second linear network we have to have (Refer Time: 19:55) process.

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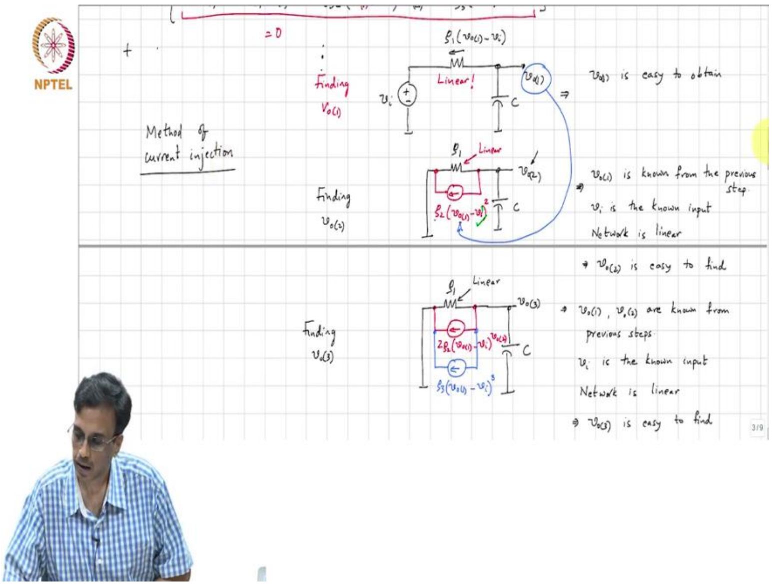

Now very good. So, we have a linear network again right and we have ok this is finding v 0 3 and. So, this is v 0 3 this is C this is g 1. And what are we supposed to add? We are supposed to add a current which is, we are supposed to add 2 currents 1 which is Minus 2 g 2 times v 0 1 minus v i times v 0 2, $\left(-2g_2\left(v_{o_{(1)}}-v_i\right)v_{o_{(2)}}\right)$. You understand?

Student: Sir that to have we are writing the KCL and the output current leaving is positive. C dv dt this goes to the other side know?

Sorry, this must be minus here sorry, $\left(-2g_2\left(v_{o_{(1)}}-v_i\right)v_{o_{(2)}}\right)$. Correct? Alright. And we also need to add another current source which is? g 3 times v 0.1 minus v i the whole square right, $\left(g_3\left(v_{o_{(1)}}-v_i\right)^3\right)$. And if you solve this linear network. So, v 0 1 and v 0 2 are known from previous steps right. v i is the input is the known input and the network is linear.

So, therefore, v o 3 is easy to find right. So, like this you can go on I mean, if we assume that you know the output can be written in this form then you can plug this form back into the non-linear differential equation and go on getting v 0 1, v 0 2, v 0 3 and so on and once you find all this the output v o of t is simply nothing but v 0 1 of t plus v 0 2 of t plus v 0 3 of t and so on correct ok.

$$v_o(t) = v_{o_{(1)}}(t) + v_{o_{(2)}}(t) + v_{o_{(3)}}(t) + \cdots$$

So, in essence we have transformed the problem nonlinear differential equation to solution of a linear differential equation with multiple iterations right.

So, one obvious question that comes up is, you know why cannot I use this for, when the nonlinearity becomes strong? Right. And what happens when I mean you know this sounds like a nice approach where I am you know avoiding having to solve a non-linear differential equation all that I am doing is solving a linear differential equation multiple times which is much easier to do.

So, the question is you know obviously, this must break down somewhere and when the equation when the nonlinearity is no longer weak right and that it turns out that what; that means, in mathematical terms is that this series will not converge.

Right. So, it will only converge when the weekly non-linear condition is met. Fortunately, we in circuits a lot of the times you are you are basically trying to I mean distortion terms are very small compared to or are needed to be very small compared to the linear term anyway right. So, this is a very good way of estimating.

You know a quick way of estimating distortion and understanding where distortion is coming from and so on. And as you can see basically the first order term is that v 0 1 is simply the linear output right. In this to get v 0 2, what are we doing? We are the input source is made 0.

Correct? So, we are deenergizing all the input sources right. And in parallel with every non-linear element or at the output or across every non-linear element, you are injecting a non-linear current alright, which you know depends on the details of the nonlinearity of that element as well as you know the voltage in this case the voltage across that element in the linear network alright.

So, that therefore, will cause the output here therefore, will be a function of I mean its; obviously, a function of both v 0 1 right ok and v i because the current through g 1 basically in the original network in the linear network depends on v 0 1 minus v i ok.

So, the third order response, to find the third order response you basically again you have the same network linear network right, but in parallel with the non-linear element you now inject a different set of currents I mean basically you see all the third order currents that are being injected here correct and so on and so on right.

Most of the time you know, if you do I mean in a within codes circuit that works properly right, the second and third order distortion terms are often the most dominant and therefore, 2 iterations are good enough right. And since this method basically you know within quotes is adding or injecting current right, in parallel with every non-linear element right, this is called the method of current injection alright. And so, I mean we will do a couple of examples perhaps in the next class.

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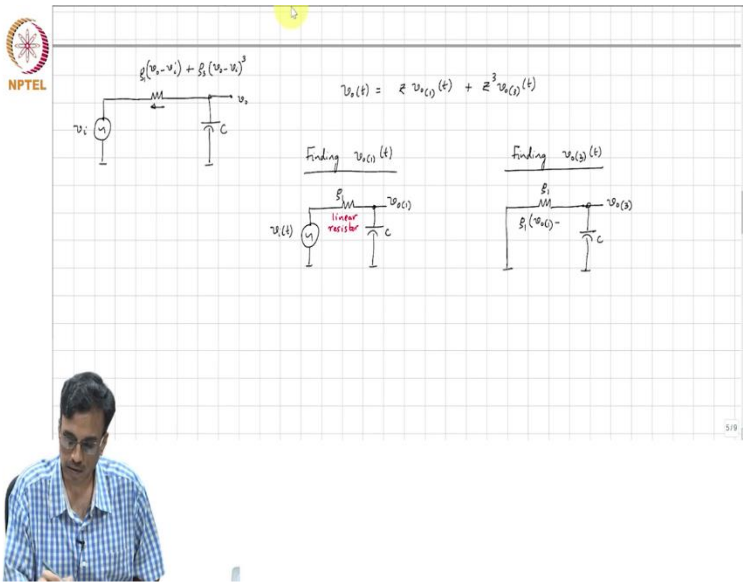

But one thing I would like to point out is that, let us say the resistor only had. So, let us do a simple example where the resistance had only odd order nonlinearity. So, this is basically v i this is C this is v o. So, this is v o minus v i plus g 3 or g 1 times v o minus v i g 3 times v o minus v i the whole cube alright, $(g_1(v_o - v_i) + g_3(v_o - v_i)^3)$. So, now, there is no second order nonlinearity to talk about. So, what do you think I mean. So, there is one of course, we can rewrite all the equations again, but based on what we have seen right what would we expect the output to look like? So, v o of t. Must be of the form z times v 0 1 of t Plus z cube v 0 3 of t alright.

$$v_o(t) = Z\, v_{o_{(1)}}(t) + Z^3 v_{o_{(3)}}(t)$$

Now how do you find? So, finding v 0 1 of t, what do we do? We will take only the linear network. So, that is easy. So, we have v i of t. So, this is g 1 and this is v of 1 right. So, this is remembered this is linear ok. The next step is to find v o. 2 of t, we have v 2 of t is 0. So, v of 3 of t. What do we do?

We will set v i to 0 this is the capacitor this is v 0 of 3 this is g 1, and this current is g 1 times v 0 of 1. Sorry v 0 2 is 0. Correct. So, what do we need to put in? We only need to add the third order term that is basically.

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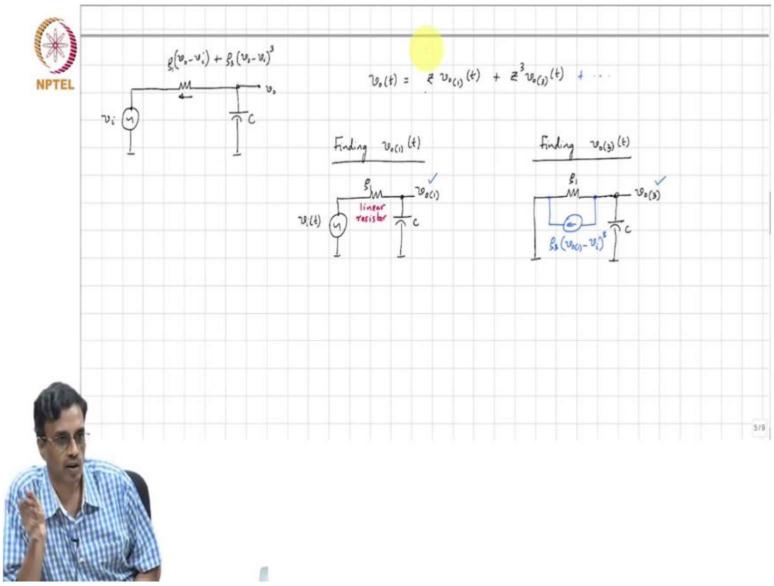

g 3 times v 0 1 minus v i the whole cube ok and v out. So, once you know this and once you know this well the output can be written as Put z equal to 1. So, v o of t is nothing, but v o 1 of t Plus v o 3 of t plus higher order terms, but the hope is that you know if the resistance is only weakly non-linear the higher order terms can be neglected.