# Circuit Analysis for Analog Designers
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## Lecture - 51
## Scattering matrices properties

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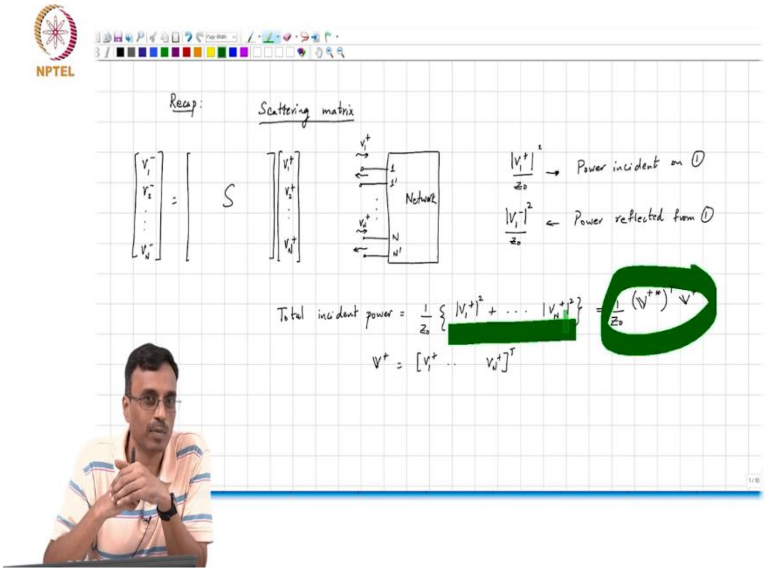


A quick recap of what we were doing in the last class. We were talking about the Scattering matrix and if we have an N port and if the characteristic impedance corresponding to the transmission lines attaching the N port to the outside world is all the same and equal to Z naught. Then we can simply basically relate the amplitude of the complex magnitude of the incident wave, at every port to the reflected wave at that port through a matrix.

So, that is V 1 minus V 2 minus blah blah blah, V N minus is the scattering matrix times the reflected waves are basically the V N plus alright.

$$\begin{bmatrix} V_1^- \\ V_2^- \\ \vdots \\ V_N^- \end{bmatrix} = \begin{bmatrix} & & \\ & S & \\ & & \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \\ \vdots \\ V_N^+ \end{bmatrix}$$

And we saw that there is a need for these scattering parameters right and the need is motivated by practical realities, when you want to measure admittance parameters or in impedance parameters of an N port and most commonly a two port.

One has to terminate the port with either an open circuit or a short circuit and of course, in reality it means that you will not be able to place the short exactly at the boundary of the port. You will have to connect it through a cable the cable is modeled as a transmission line and if you terminate a transmission line with a short circuit, at the other end of the line you can see an impedance which can go anywhere from being a short circuit to an open circuit to anywhere in between.

And consequently, you basically end up with you end up in a situation where, if you are trying to measure an active device right. For example, a transistor and stuff like that, you are you will be in a situation where the whole setup namely the transmission line at the transistor and the entire measurement setup starts to oscillate.

That basically means that you know you will not be able to measure the parameters properly and why because remember a trans a transistor is basically a non-linear device and whenever we talk about impedance or admittance parameters what exactly do we mean. That basically you linearize that transistor about the operating point and get a small I mean get a two port which is linear.

And then we assume that the signals that we impress on these two ports are sufficiently small to satisfy the small signal approximation and the and then you know all the other theory makes sense.

But if the whole setup is oscillating and you know once something starts to oscillate it only limits when something somewhere in the setup saturates, then its no longer small signal and this notion of small signal parameters basically does not make any sense right, that is the best case.

The worst case of course, is that the amplitude of oscillation becomes so large that the power dissipated in the transistor becomes, you know either the power dissipated becomes larger the voltage across the transistor the current through the transistor becomes so large so as to breach its maximum permissible voltage in current values: and that will cause the destruction of the transistor.

And sometimes if the transistor is a large high-power device, then its entirely possible to have an accident. So, the way around it is to avoid I mean the root cause as you can see is that, even though we terminate the cable or the transmission line with an open or a short circuit at the other end it can look like something completely different.

And the only way to get around this problem is in other words we need to terminate the transmission line you know in some magic impedance and we want to be able to see when you look at the impedance on the other side of the transmission line you want to make sure that, I mean what do you want is that the impedance should be independent of the length of the transmission line right.

And the only way to do it is if you have a transmission line with characteristic impedance Z naught then you basically terminate it also with Z naught. And this way you can have a transmission line which is which can be in principle arbitrarily long and I mean then.

So, therefore, the real-world difficulty of having to run a cable from the port to the termination basically or to the measurement equipment for that matter right that problem goes away. And you are now able to measure terminating the cable with 50 ohms is equivalent to terminating the ports with 50 ohms at the location of the port even though the actual port maybe a mile away from your terminal.

Does it make sense right, very good. And we also you know that you know V 1 square by Z naught or mod V 1 square by Z naught, $\left(\frac{|V_1^+|^2}{Z_0}\right)$, quantifies therefore, is the power going in the forward direction or the incident power. And likewise on port 1 and likewise V 1 square by Z naught, $\left(\frac{|V_1^-|^2}{Z_0}\right)$, is the power reflected from port 1 right yes.

Student: Sir for this over line impedance will not match to the impedance that at in terminating impedance both are not scattered of converting.

Of course, you know now he brings up an interesting point he says. Well, this is all great in theory right, where you know the characteristic impedance of the line is exactly the same as the terminating impedance right. So, what happens you know if of course, in reality you will never be able to make the characteristic impedance of the line exactly the same as the terminating impedance. Now what do you think will happen well we can do the math right, but what do what would you expect.

It will become dependent, but what comment can we make about that dependence. I mean you should expect that as you get closer as the terminating impedance gets closer and closer to the characteristic impedance of the line, the reflection coefficient becomes closer and closer to 0.

And therefore, you basically end up with something which is, I mean yeah it does indeed change over you know over the length of the cable right and it will the impedance will no longer at the other end of the line will no longer be equal to Z naught right. There will be some frequency dependence, but that frequency dependence the that frequency dependence will be small compared to what you would have seen if you terminated with a short circuit right. And you know, but this is indeed a practical problem as far as measurement accuracy is concerned right. And we will see what is done in practice to fix this problem right.

So, the long and short of it is that if the terminating impedance is not exactly the same as that of the line right. Yes, that is it is correct that you know the impedance will now be a function of the length of the line as well as the frequency right. However, it will almost be equal to Z naught right.

And the variation in impedance with respect to frequency or with the length of the line will be small. And compare that with you know when you terminate with a short circuit you know you thought you had a short circuit, but you can actually see something which is all alright ok.

So, now, the. So, let us consider this N port again I would like to kind of draw your attention to you know a property of the scattering matrix for a general N port. What comment can we make about the sum total of the incident power, on this network I mean is this is this like. So, simple that you know you think its below your dignity to answer it or is it. So, difficult that you know you have no clue what is happening.

Very good ok alright. So, I will think of it I think your answer to be the former and move on right. So, V 1 square plus blah blah blah right alright.

$$Total\ incident\ power = \frac{1}{Z_0}\{|V_1^+|^2 + \cdots |V_N^+|^2\}$$

Which can be written if you write the incident vector if you put all these V 1 through V N as a vector its simply nothing but let me call that I do not know I call that V plus alright.

So, this is nothing but V 1 plus V N plus is the column vector of all in the amplitude to the incident waves.

$$\mathbb{V}^{+} = [V_1^{+} \quad \dots \quad V_N^{+}]^T$$

And therefore, the incident power in terms of this column vector can be written as 1 by Z naught times V plus conjugate times sorry V plus I am getting.

Now, there are is plus the star there is transpose right at this times V plus.

$$Total\ incident\ power = \frac{1}{Z_0}\{|V_1^{+}|^2 + \cdots |V_N^{+}|^2\} = \frac{1}{Z_0}(\mathbb{V}^{+*})^T\mathbb{V}^{+}$$

Does it make sense I mean just I mean you know yeah; I mean, I am sure all of you agree that you know this looks a lot more intimidating than simply writing this right ok. And that is the whole purpose of going to university right, where you make something simple and make it look all the more complicated.

Because if you go tell your friends if the power in one port is V 1 square and N port is V 1 square plus V 2 square plus V 3 square all the way up to V N square. I guys says why do I need you to tell me that is not that obvious right.

Whereas, now if you write this complicated looking notation with star and transpose and you know plus all of a sudden starts to look, very impressive though there is nothing much to it right ok. Now so now let us you know pretend to be sophisticated and write an expression for the reflected power is what now.

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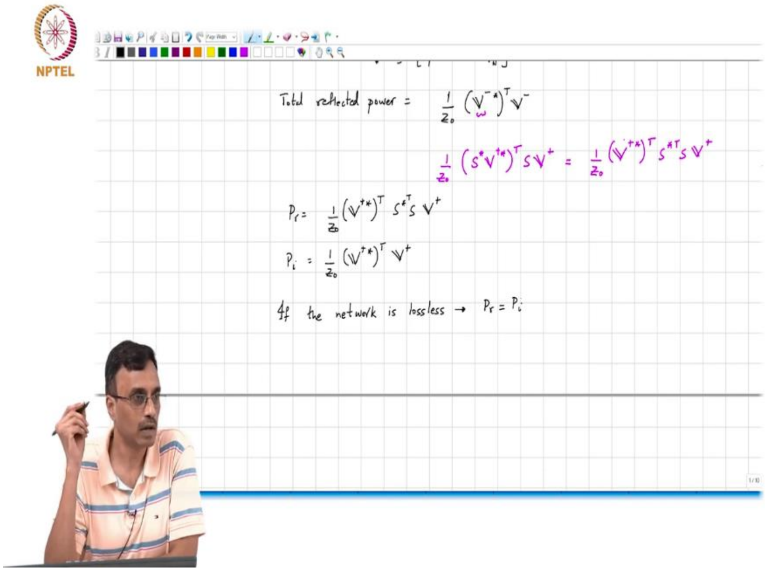


Well, if the input power is 1 over Z naught times V plus star transpose times yeah, the vector V plus what will be the reflected power. Well does not require a genius to figure that out that is basically V minus star transpose times P minus alright.

$$Total\ reflected\ power = \frac{1}{Z_0}(\mathbb{V}^{-*})^T\mathbb{V}^-$$

And what do we know about V minus in terms of V plus?

This is well this is nothing but S times V plus,

$$\mathbb{V}^- = S\mathbb{V}^+$$

And what is this guy what is this guy actually? That is S times V plus again that times star. So, that should give you that is conjugate its conjugate which is basically come on folks, I mean a b whole conjugate is. Now, people are thinking as b conjugate a conjugate or a conjugate b conjugate come on folks.

Ok alright. So, this is S conjugate as S star times V minus star. Sorry, V plus star correct and this transpose. So, our reflected power is this times S times V plus alright ok. And this must be equal to you can write this as, well this will be V I mean sorry I forgot the 1 over Z naught. So, this is 1 over Z naught times V plus star transpose times, transpose of the product is nothing but you have to flip the order and you would expand it out. So, that is basically S transpose S star transpose times S times V plus does make sense people.

$$Total\ reflected\ power = \frac{1}{Z_0}(\mathbb{V}^{-*})^T\mathbb{V}^-$$

$$= \frac{1}{Z_0}(S^*\mathbb{V}^{+*})^T S\mathbb{V}^+ = \frac{1}{Z_0}(\mathbb{V}^{+*})^T S^{*T} S\mathbb{V}^+$$

So, therefore the reflected power is 1 over Z naught V plus star transpose S star transpose, times S times V plus and the incident this is the P r the reflected power

$$P_r = \frac{1}{Z_0}(\mathbb{V}^{+*})^T S^{*T} S\mathbb{V}^+$$

P i again for your reference is simply 1 over Z naught times V plus star transpose times V plus alright.

$$P_i = \frac{1}{Z_0}(\mathbb{V}^{+*})^T\mathbb{V}^+$$

So, if the network is loss less and what comment can you make, the reflected power must be the same as the incident power right ok.

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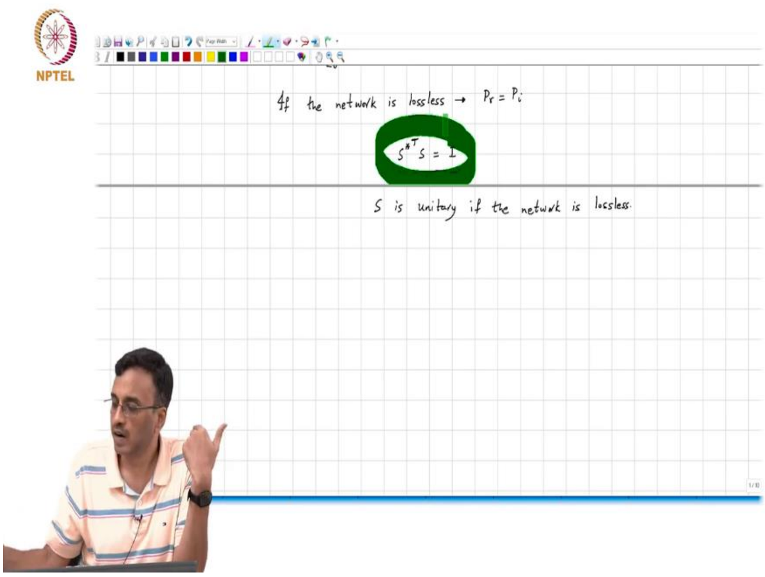

And therefore, this will happen if S star transpose times S equals the identity matrix right, identity matrix is of size N.

$$S^{*T}S = I$$

And it turns out I mean have you I mean do you know I mean if you have taken a course on linear algebra you probably know this already right. So, if you have a matrix and S star transpose times S is equal to I what do you call that such a matrix unitary matrix right. So, S is ok unitary ok, if the network is lossless right. In English what does this mean yeah which is equivalent to saying I mean you know English now basically is now become right. He says all the rows are orthogonal to each other which is sounds just as you know, Latin and Greeky to the uninitiated.

So, which yeah that is correct, but what does it mean? If you take the ith row or the ith column complex conjugated multiply it with itself and add up all the terms that you get you will get one if you do that with the column which is not your own right, you will get 0 ok, that is all that there is do it ok.

So, this is a you know something to bear in mind and again as you can see, I mean you know it sounds like something very deep, but as you can see all that it means is that what is you know, how do you get this what you push in is the same as what you get out. And in a less if you wanted to if you want to say this in a less pretentious way you just basically say sum of V 1 square through V N square plus is sum is the same as V 1 square through V N square minus alright. That is that another quick point.

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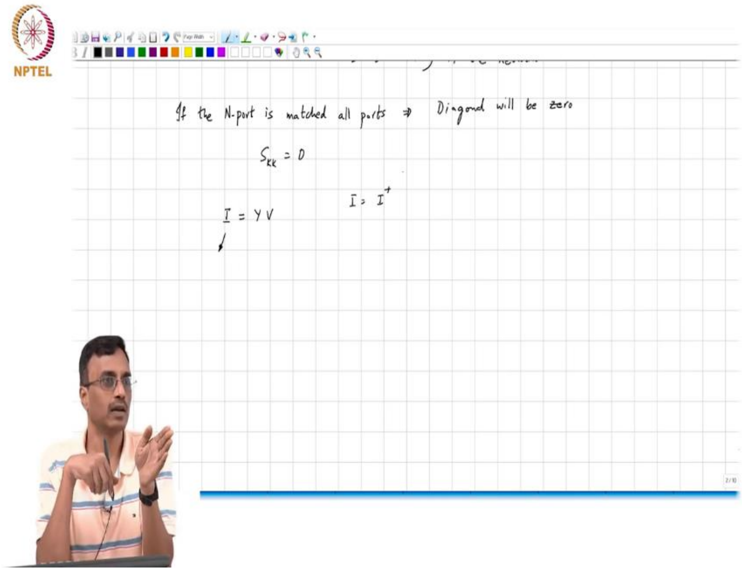


If the N port is matched at all ports, what comment can you make about S matrix. What comment can we make when we say N port is matched at all ports, what does it mean? That basically means that whatever is incident on that port nothing gets reflected and.

So, what you know what term in the S matrix quantifies what is reflected when you put something into the Nth port.

Yeah, you know S KK where K is the Kth port. So, S KK equal to D.

$$S_{kk} = D$$

So, basically if the N port is matched to all ports, then diagonal will be 0 alright. And remember we started off this entire discussion of this S parameters by saying we wanted to measure why, but then you know we; you know we found that to be a practical difficulty.

So, we came up with this scheme of you know measuring in coming up we came with the new parameter set that is the S parameter set. Now the question is how is the S parameter set related to yeah either the Y or the Z or you know any of these you know bazillion other things right.

So, let us just do say for instance how the S parameters are related to the Y parameters. So, the Y matrix basically relates of an N port basically relates currents to voltages right where I Y and V are all I and V I mean I and V are column vectors with N terms and Y is an N-by-N square matrix.

$$I = YV$$

Now the current is simply the sum of the current at the ports can be written as we are trying to relate it to the S parameter set correct and the S parameter set only recognizes you know forward and backward going waves. So, we need to express the current at the port in terms of the forward and backward going voltage ways right.

So, what comment can you make about I remember I is nothing but I plus, plus or minus, the total current at a point ok how will you express this in terms of; in terms of V plus and V minus ok. Ok.

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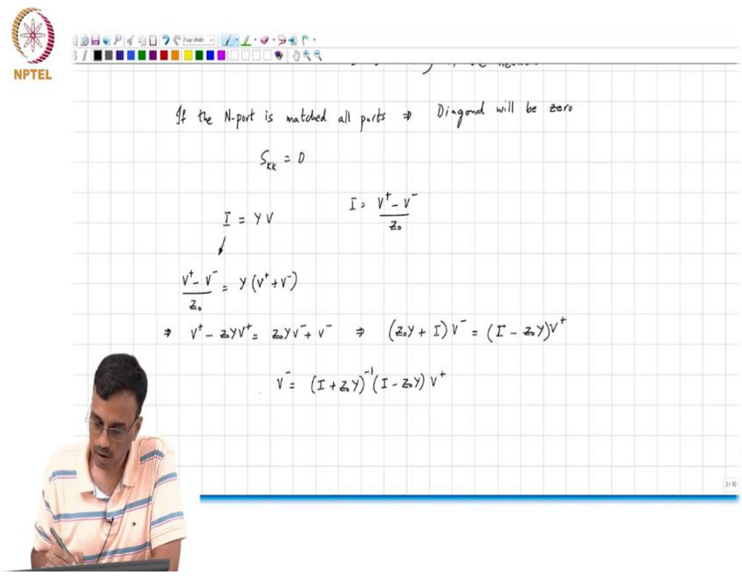


Yeah ok. So, V plus minus V minus over by Z naught.

$$I = \frac{V^+ - V^-}{Z_0}$$

So, this is V plus minus V minus by Z naught and that is Y times. What is the voltage? V plus, plus V minus correct.

$$\frac{V^+ - V^-}{Z_0} = Y(V^+ + V^-)$$

And therefore, we are interested in finding we take you know push all the V minus to one side and then V plus to the other side and then I just play with this.

So, V plus minus Z naught, times Y times V plus equals Z naught times Y times V minus plus V minus alright. So, Z naught Y plus I times V minus equals I minus Z naught Y times V plus that makes sense I hope I am not messed up.

$$\Rightarrow V^{+} - Z_0 Y V^{+} = Z_0 Y V^{-} + V^{-}$$

$$\Rightarrow (Z_0 Y + I)V^{-} = (I - Z_0 Y)V^{+}$$

So, V minus therefore, equals I plus Z naught times Y inverse times I minus Z naught Y times V plus.

$$V^{-} = (I + Z_0 Y)^{-1}(I - Z_0 Y)V^{+}$$

And therefore, what comment can we make. So, how was the given the Y matrix how will you get the S matrix?

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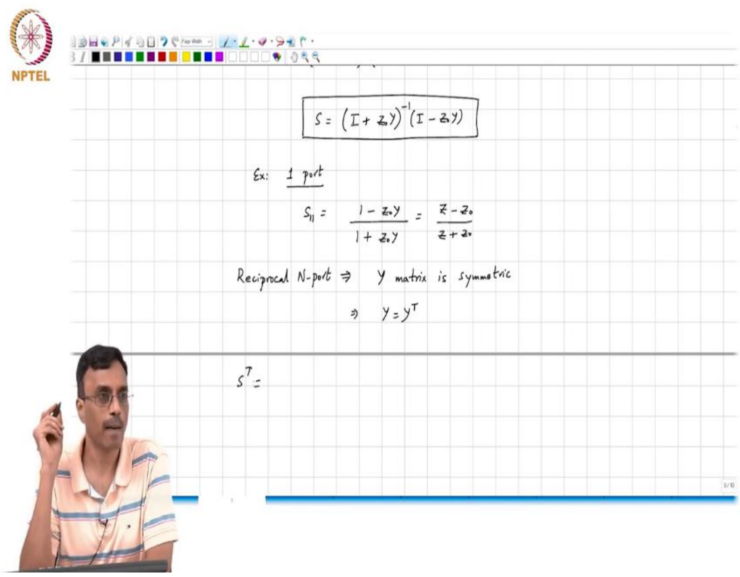


Well the S matrix is nothing but I plus Z naught times Y, remember that Z naught is the scalar this inverse times I minus Z naught right.

$$S = (I + Z_0 Y)^{-1}(I - Z_0 Y)$$

Sanity check, what is the easiest number of ports to work with for the sanity check. 1 port right ok.

So, for a 1 port S 1 1 equals 1 plus Z naught times Y inverse which is 1 plus Z naught times Y times 1 minus Z naught times Y which is nothing but Z minus Z naught by Z plus Z naught. Which is which makes sense, because this is Z minus Z naught by Z plus Z naught is nothing but the reflection coefficient which also happens to be the you know the lower I mean S 1 1 of that 1 port alright.

$$S_{11} = \frac{1 - Z_0 Y}{1 + Z_0 Y} = \frac{Z - Z_0}{Z + Z_0}$$

Next thing if the 2 port is reciprocal let us say I mean for example, ok. So, if the N port. So, if you have a reciprocal N port what comment can you make about the Y matrix, yes people. It is, the Y matrix is symmetric which basically means that Y equals Y transpose right.

$$Y = Y^T$$

So, what does that mean as far as the S matrix is concerned? Remember S is given by this guy, so what is the S transpose? S transpose is nothing but rather yeah. What is S transpose? Or rather if you replace Y with Y transpose here I mean, what do you; what do you think you will get? Or rather ok.

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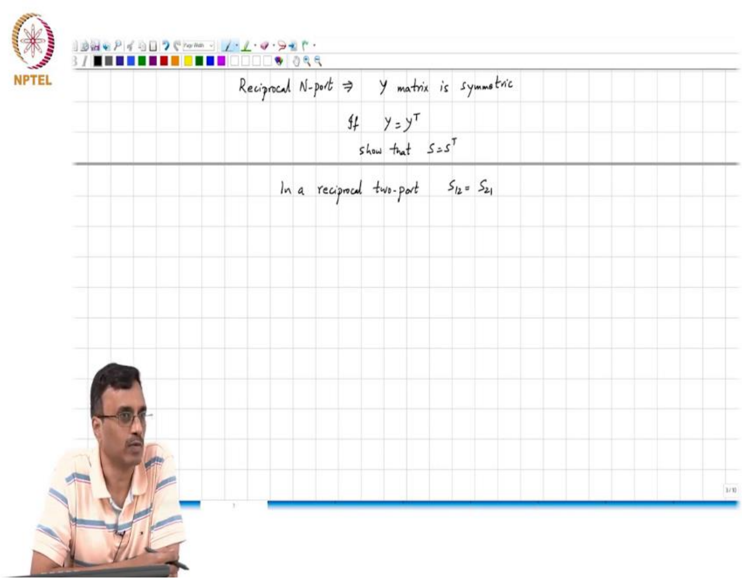

So, basically I leave it as an exercise for you to prove that s. If Y is Y transpose show that S equal to S transpose hm.

$$If\ Y = Y^T$$

$$show\ that\ S = S^T$$

So, that is. So, it turns out that if the a the. So, in a 2 port in a reciprocal 2 port S 1 2 must be equal to S 2 1 ok

$$S_{12} = S_{21}$$

And you can come up with you know a similar way of basically converting from Z to Z to S and Z to I mean I as S to any arbitrary set of parameters that you would like ok. It does not make sense for me to go through all those parameter conversions.