Circuit Analysis for Analog Designers
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Lecture - 05
Tellegen's Theorem and reciprocity in linear resistive networks

People, good evening and welcome to advance electrical networks, this is lecture 3.

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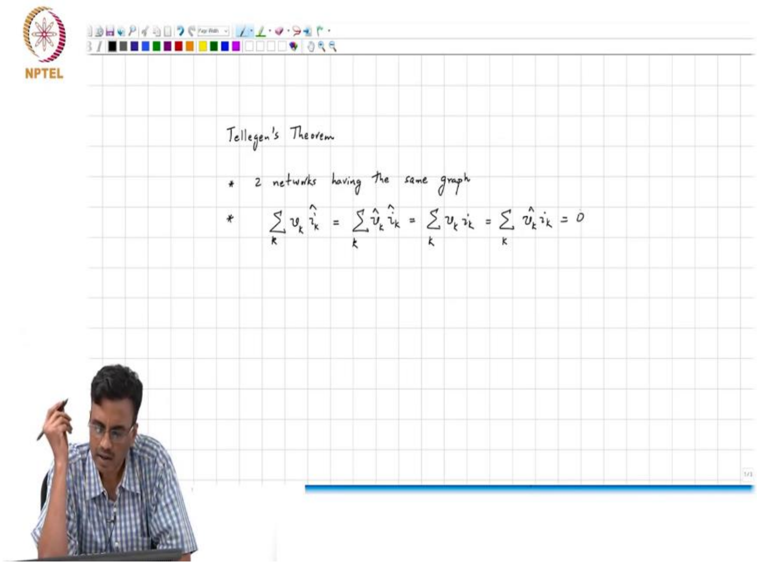

In the last class we looked at Tellegen's theorem and it pertains to two networks having the same graph and is it necessary that the two networks be linear? There is no nothing that says that the networks have to be linear or time invariant. So, the only requirement is that the two networks have the same graph. And what does it say? It says that you know sigma over all the branches v k times i k hat right is the same as sigma over k, v k hat times i k ok must be equal to sigma v k times i k as which is the same as v k hat times i k is all equal to 0 and we saw how this comes about right.

$$\sum_k v_k \, \hat{i}_k = \sum_k \hat{v}_k \, \hat{i}_k = \sum_k v_k i_k = \sum_k \hat{v}_k \, i_k = 0$$

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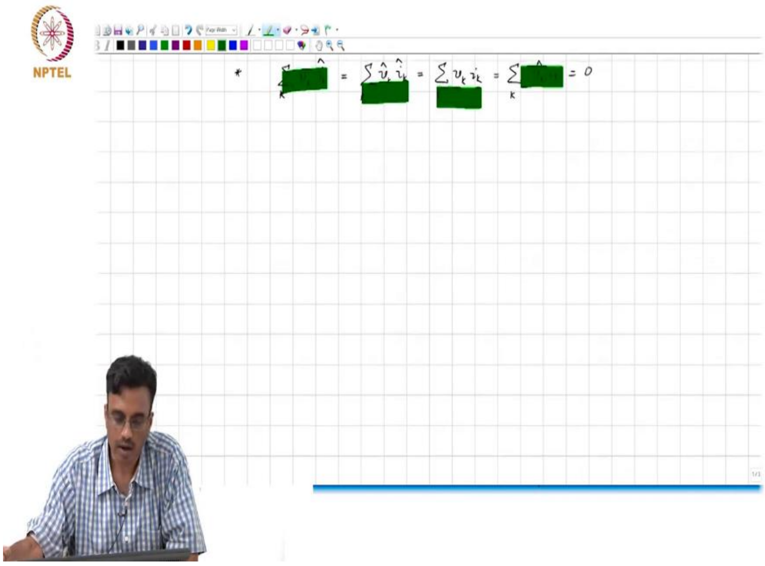

These two are of course, you know immediately understandable and apparent. You may say well you know at any instant of time it seems reasonable that power is conserved right. These on the other hand at first sight seem little puzzling, but we saw a simple network construct that helped us get intuition into why the physical reason why this makes sense right.

All that we did was to put in parallel with every branch of the network, a current which flows in the opposite direction with the value i k hat and if you now apply power conservation to this network its apparent that you basically get the other two identities. Now, it turns out that Tellegen's theorem is also very useful to prove a whole lot of interesting theorems.

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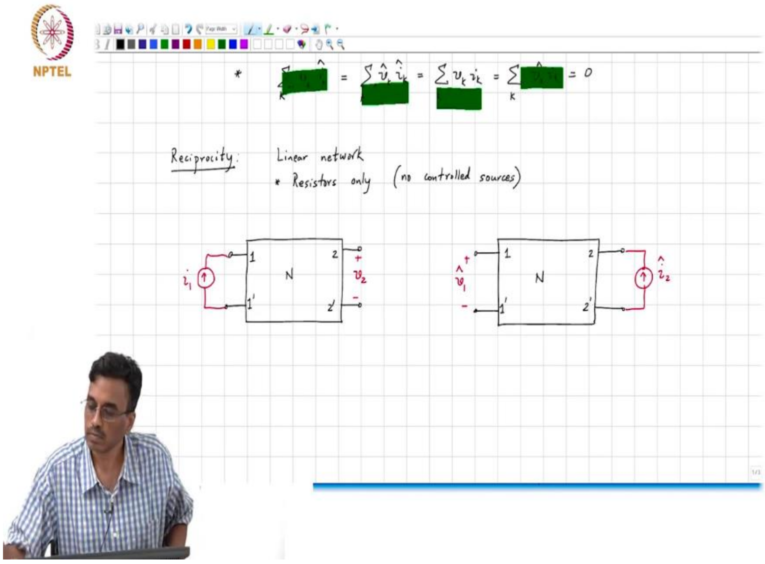

The first one that I like to draw your attention to is reciprocity and the idea is the following. So, let us say you have a linear network now ok and we have the same another copy of the linear network, its let us assume its it has two ports 1, 1 prime and 2, 2 prime. This is 1, 1 prime and 2, 2 prime and this is i 1, this is v 2 and here I inject i 2 hat, and measure the voltage at port 1 is v 1 hat.

Further let us assume that this a linear network which consists of resistors only, in other words there are no controlled sources inside the network. And we apply Tellegen's theorem to both these networks and let us see what happens.

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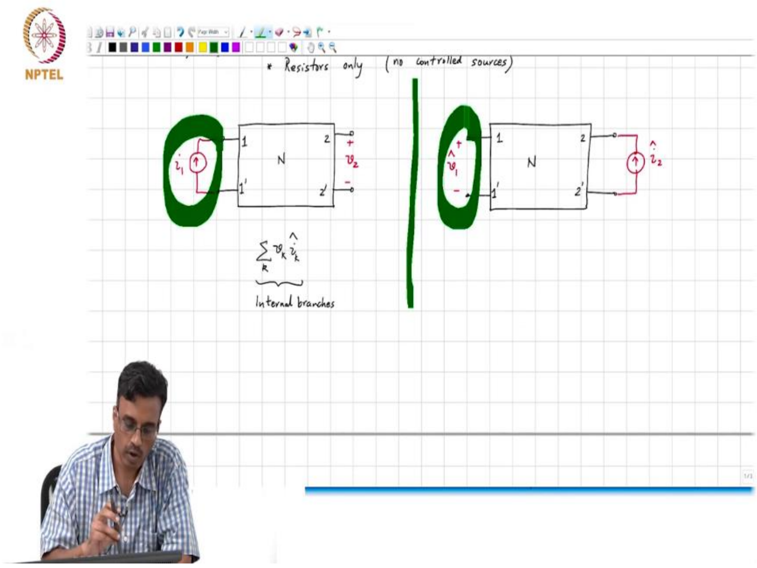

Remember sigma over all branches ok, inside this network N of $\sum_k v_k \, \hat{\imath}_k$ which correspond to the branch currents in the; remember the unhatted quantities correspond to the network on the left, the hatted quantities correspond to the network on the right. So, and the k runs over all the branches which are sitting inside the box.

So, if I want to form the product $v_k \, \hat{\imath}_k$ for all the; for all the branches in this entire network you have this which corresponds to the internal branches right and what comment can you make about $v_k \, \hat{\imath}_k$ as far as the port branches and currents are concerned, what is; we do not know the voltage reforming the product $v_k \, \hat{\imath}_k$.

We do not know the voltage across this branch. Right, but we need to find the product of this voltage and this current and what does that happen to be? Well, the current in port 1 of the network on the right is 0. So, therefore, it must follow that the product of v 1 times i 1 hat is 0.

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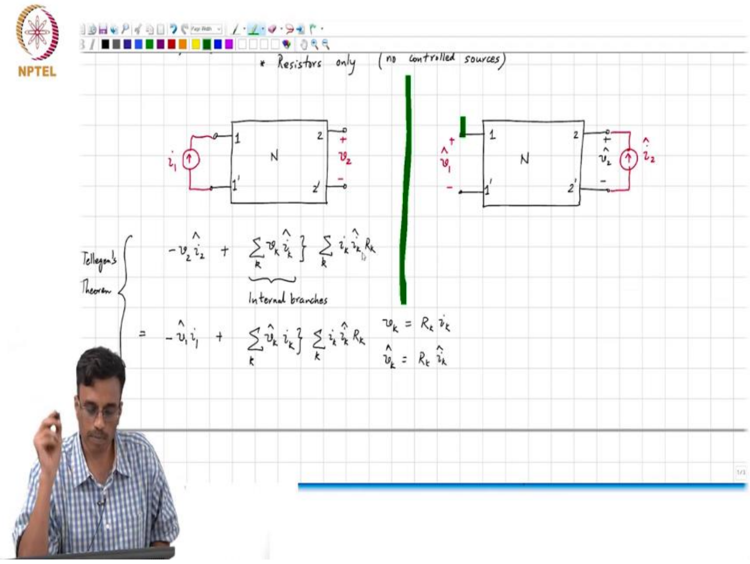

And the only other quantity to consider is $v_2\ \hat{\imath}_2$, ok. Actually, I mean this is our voltage v_2 hat, it must actually be $-\hat{\imath}_2$ simply because if we assume v_2 to be in that direction, branch current should flow in the opposite direction. So, it is $-v_2\ \hat{\imath}_2$. And this must be equal by Tellegen's theorem, sigma over all internal branches v k hat times i k correct.

$$= \sum_k \hat{v}_k\ i_k$$

That is the all-internal branches then, we need to deal with the external the port quantities and, so what is $\hat{v}_k$? For the first one it must be $\hat{v}_1 \times -i_1$ ok and what comment can we make about the second port? v 2 hat times the current in port 2 which happens to be? 0 correct. So, this is what Tellegen's theorem throws up.

$$-v_2\ \hat{\imath}_2 + \sum_k v_k\ \hat{\imath}_k = -\hat{v}_1 i_1 + \sum_k \hat{v}_k\ i_k$$

We still have not exploited the fact that the network is linear and that every branch is a resistor right. So, v k therefore, how is it related to the how is the branch current related to the branch voltage? Right, simply nothing but some R k times i k and likewise v k hat is R k times i k hat.

$$v_k = R_k i_k$$

$$\hat{v}_k = R_k\, \hat{\imath}_k$$

So, this therefore, means that this part will become sigma sum over all branches inside $i_k\, \hat{\imath}_k\, R_k$ and this will also become sigma over all $i_k\, \hat{\imath}_k\, R_k$. And therefore, what comment can we make about what does this throw up? If you; so, these two quantities evidently are the same. So, these two are therefore, the same and therefore, what do we get?

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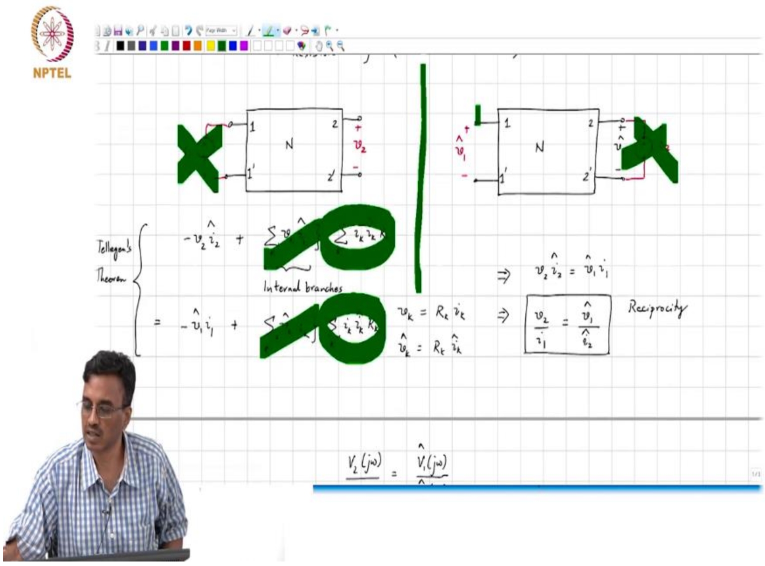

v 2 times i 2 hat equals v 1 hat times i 1 which is equivalent to saying that v 2 by i 1 is the same as v 1 hat by i 2 hat.

$$v_2\, \hat{\imath}_2 \;=\; \hat{v}_1\, i_1$$

$$\frac{v_2}{i_1} = \frac{\hat{v}_1}{\hat{\imath}_2}$$

So, this is the reciprocity right because we now only did this for network with resistors. It follows I mean it is you go through the same proof and show that it is easy to see that.

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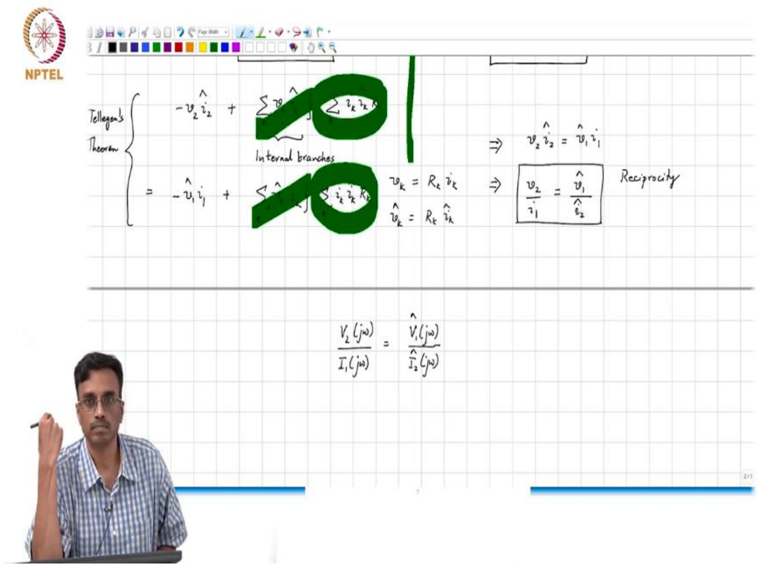

If you are if you have impedances ok, V 2 of j omega by I 1 of j omega is V 1 hat of j omega divided by I 2 hat of j omega. The proof is exactly the same. Make sense? Yes.

$$\frac{V_2(j\omega)}{I_1(j\omega)} = \frac{\hat{V}_1(j\omega)}{\hat{I}_2(j\omega)}$$

So, in other words the transfer function I mean the what do you seen in the past is basically you know the one way to say it is that well if you have a linear network with only resistors or impedances then you can interchange the location of the excitation and the response and you get the same transfer function.

One thing that I would like to point out is that if you de energize all the sources ok, so, in other words this is not around and neither is this what comment can you make about the network about both the networks? So, if I remove i 1 on the left and i 2 hat on the right, if I look at the two pictures what are they? It is exactly the same.

So, please bear that in mind when you apply a reciprocity right. If the graphs of the network are not the same then you know you will not get, I mean this condition will not be satisfied. So, we have seen current input and a voltage output, it turns out that you know you can work this out at home.

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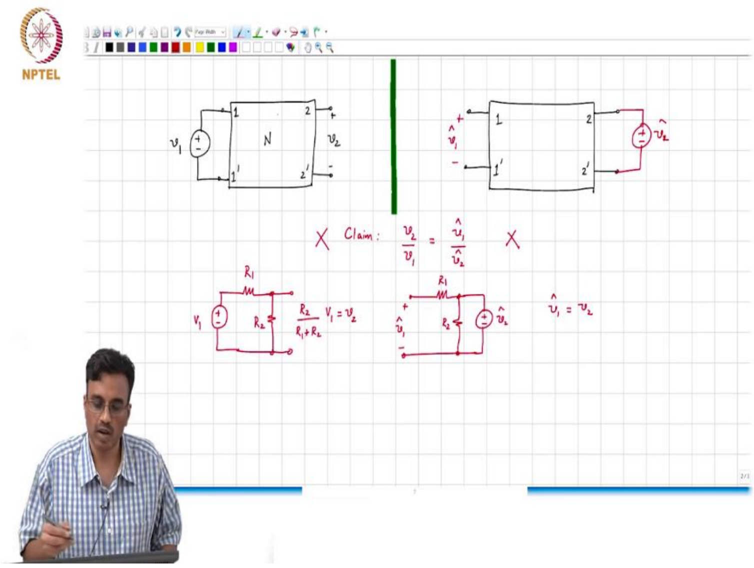

But the same thing happens when you have or rather, I would say a similar thing happens when you have voltage input. So, this is v 1 and this is v 2 and what should I do? Does somebody know what I should do? Ok, so well I have the answer I get is ok how about exiting this and the sorry this is v 2 hat and this is v 1, the claim is that v 2 by v 1 is v 2 hat v 2 by v 1 is the same as what? v 1 hat by v 2 hat. This is the claim right and I see all of you nodding your heads you know, it is correct?

Exactly right. So, this is tempting to kind of say, but this is actually not correct alright, remember that the graphs of the network when your de energize the sources must be the same. So, when you de energize the sources what do you get on the left side which port is shorted?

Port 1 is shorted in the on the network on the right port 2 is shorted. So, you do not have the same graph on both the sides and therefore, you know the this is definitely not correct. And if you want a simple analogy a simple counter example here it is this is R 1. I still see people kind of incredulously looking at the result and saying how can that be. So, this is v 1 ok.

So, what is the output voltage? R 2 by R 1 plus R 2 times v 1 ok.

$$\frac{R_2}{R_1 + R_2} V_1$$

Now, I take the same network as you guys suggested. I will apply v 2 hat here and what is v 1 hat? v 1 hat is? v 2 ok. So, clearly you can see that the transfer functions are not the same right and the reason is that you know the graph of the network is ok. So, now that this is wrong what is right is what we are do.

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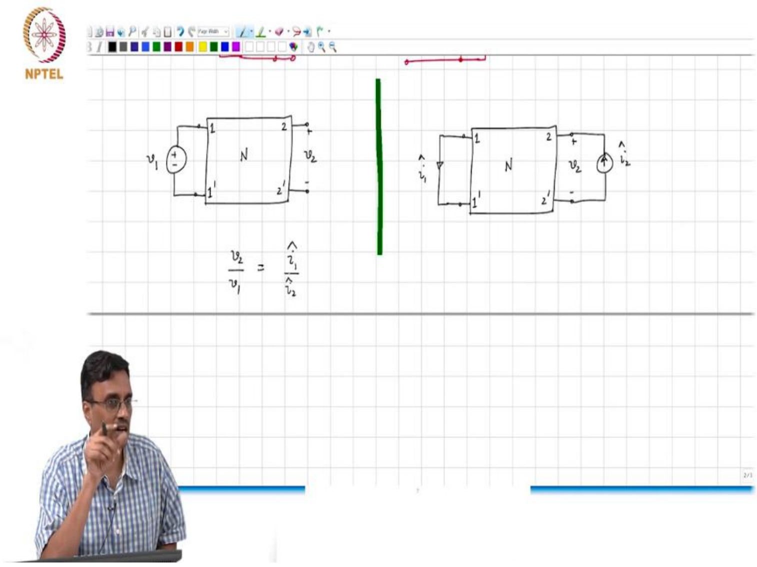

So, what do you think the correct answer is? Short ports. So, the correct expression of reciprocity is basically to look at i 1 hat and you apply what should I apply on this side? Remember v 1 and v 2 the input is a voltage the output is a voltage. Correct. So, what you have to apply is a current i 2 hat and measure the current i 1 hat ok. And it so turns out I mean you can simply go through the same arguments that we went through to prove the other result. I leave that for you as home work and convince yourselves that v 2 by v 1 equals? i 1 hat by i 2 hat ok.

$$\frac{v_2}{v_1} = \frac{\hat{\imath}_1}{\hat{\imath}_2}$$

The voltage transfer function in the network on the left is the same as the current transfer function on the network on the right and whenever you do this you always will be in good stand if you run the sanity check namely when you de energize the sources the graphs must be the must be the same. No, the I we said already that N only consists of resistors right.

An impedance right, but impedance is not a source right. It reciprocity applies to a linear network where all the branches currents can be expressed linearly as a function of with the branch voltages right. So, if you look at see the comment was does it hold for any linear network, it holds for all network. The key point in the argument is, remember, Tellegen's theorem is valid for any network, it does not you know need linearity right.

What is allowing us to basically cancel out all the powers you know the pseudo powers kind of thing dissipated in the branch internal branches. We are able to do that because we are able to express the branch voltage as some R k times i k right or in general if there are impedances the branch you know v k of j omega is some z k of j omega times i k of j omega.

As long as that is satisfied you will say you know reciprocity holds. Now one you know practical case where this is automatically satisfied is when you have a linear network which is also passive right when you have only R, l and c in them right which are the commonly known you know the passive elements that we use and it is automatically satisfied.

And you know there is a general misconception that it is only applicable to passive networks right, but that is not the case because for instance if you know I somehow manage to create an impedance which was negative right all that these relationships saying is that v k is R k times i k, the sign of the R could be negative right.

If the resistance is negative, it means that the network is actually active because the resistor is actually pushing out power rather than dissipating it, but that does not affect reciprocity at all, right because these products will cancel out and you still end up with you still end up with the transfer functions being identical.

Control sources is a you know we have still not as far as we are concerned at this point there can be no controlled sources inside the network. Only two terminal linear two terminal elements whose branch currents and branch voltages are linearly related to each other, is that clear.