**Circuit Analysis for Analog Designers**
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**Lecture - 47**
**The Smith chart**

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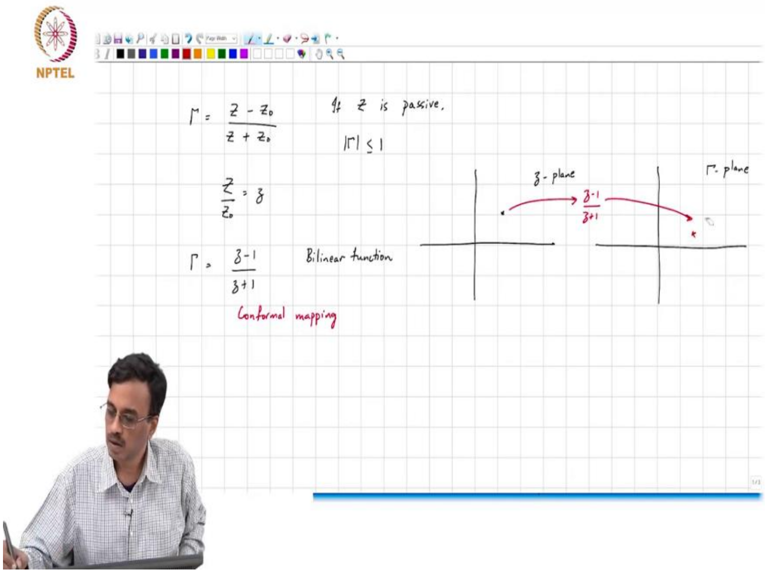


In the last class we were basically wondering about the reflection coefficient and we said that this gamma is some corresponding to some impedance Z is nothing, but minus Z naught by Z plus Z naught.

$$\Gamma = \frac{Z - Z_0}{Z + Z_0}$$

Further we said that if Z is passive then mod gamma is less than or equal to 1.

$$|\Gamma| \leq 1$$

And this simply follows from energy conservation. So, basically the incident power part of it is reflect I mean is reflected and the rest of it is dissipated in the impedance, and therefore, if you know that the impedance is passive the power dissipated in it must at least be 0 and therefore, mod gamma is less than or equal to 1.

And another thing that I would like to point out is the following. So, if you think of this the impedance normalized to the characteristic impedance is small z. Then gamma as you can see is simply nothing but z minus 1 small z minus 1 by small z plus 1.

$$\frac{Z}{Z_0} = z$$

$$\Gamma = \frac{z-1}{z+1}$$

And have you seen this before? This is a complex function right. I mean the small z is a complex number at a certain frequency correct and gamma is; obviously, also a function of a complex number right. So, this is gamma is a function of a complex variable and. So, therefore, this is an example of what you call a, have you seen this before? This function z minus 1 by z plus 1 in some other context. Where?

Yes, no I mean you either seen it or you have not seen it? (Refer Time: 02:42) Ok. Well, this is an example of what is called a bilinear function, which is simply a complicated way of saying that the numerator and denominator are both first degree polynomials in small z.

Now, if you have the z plane right which where you have the real part of z on the X-axis and the imaginary part of z on the Y-axis. For each point on in the z plane you can think of this is the gamma plane, you basically will have you will have a point in the gamma plane. Yes, because you simply plug the small z into this formula and you will get another complex is small z is a complex number right.

You compute small z minus 1 by small z plus 1 which happens to be the reflection coefficient in our case right. You will get another complex number right. So, you can plot that complex number in the you know graph on the right and that is the plane that shows that is also a complex plane, but it indicates the X-axis indicates the real part of gamma and Y-axis indicates the imaginary part of gamma correct.

So, if you trace a curve here what will happen on the in the gamma plane, you will have some other curve here right I mean ok. So, but there is a special property of a bilinear function I mean of any transformation of the form a z plus b divided by c z plus d.

And it is what is called conformal mapping I will explain what; that means, but a couple of things that I would like to first point out.

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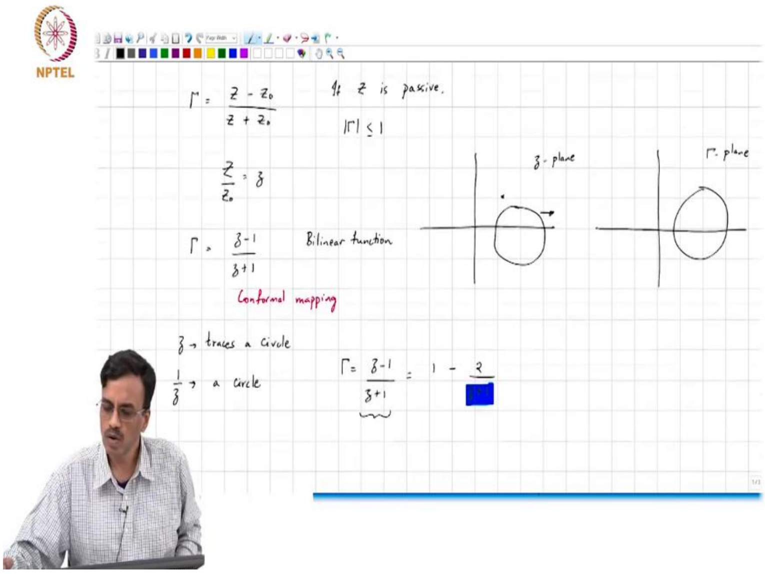

Um The mapping 1 over z right this is the only thing you need to know I mean you can prove it easily, but I am not going to spend time proving this here right.

If it turns out that if this 1 over z is I mean if you take complex variable z and you compute a new variable 1 over z. Then if z traces a circle right, then it turns out that gamma will also 1 over z will also trace a circle ok. So, if z traces a circle 1 over z will also be a circle.

And you can calculate what the radius and all this stuff will be, but the key point is to understand that if z is a complex number and if z goes around in a circle. Then if you plot 1 over z then that will also be a circle correct. And remember that a straight line is simply a special case of a circle with an infinite radius.

So, all that this means is that if z is a straight line or a circle 1 over z will also be a straight line or a circle right. I mean a straight line in z can become a circle in 1 over z and vice versa, but the key point is to basically you know remember that 1 over z basically will simply be a circle if z is a circle ok.

Now, this can be easily proven and I leave it to you as an exercise right, but you know this is one of those facts of life that you know is good to remember now yeah. So, if this reciprocal mapping is you know what do you call you know that it straight line becomes a circle and I mean whatever straight lines and circles remain straight lines and whatever when, you find the reciprocal it also becomes either a straight line or a circle.

Then it is easy to see why this gamma when you compute z minus 1 by z plus 1 which can be written as 2 minus oh sorry 1 minus 2 over z plus 1 correct and.

$$\Gamma = \frac{z-1}{z+1} = 1 - \frac{2}{z+1}$$

So, you can think of this as the following way, if let us say you had a circle in the z plane right ok. How does z plus 1 look like?

It is a circle which is shifted by 1 unit correct ok. Then what are you doing? You are finding the reciprocal of that complex number right. So, what will the how will the reciprocal look like, it will also look like a circle right then you are multiplying it by 2. So, what are you doing to that circle. Well, you are making the circle bigger right and then you are multiplying it by minus 1. What do you do what does that may happen it is getting flipped from? Left to right ok. Then you are adding one to it. So, what is happening? Shifting in the horizontal direction right.

So, bottom line is that if you had a circle to begin with then doing this will also result in, a circle ok alright. So, that is the. So, if you have a circle here you basically will it will turn out that you will be some circle like this.

We do not know the details of the circle, but we know it is going to be a circle alright.

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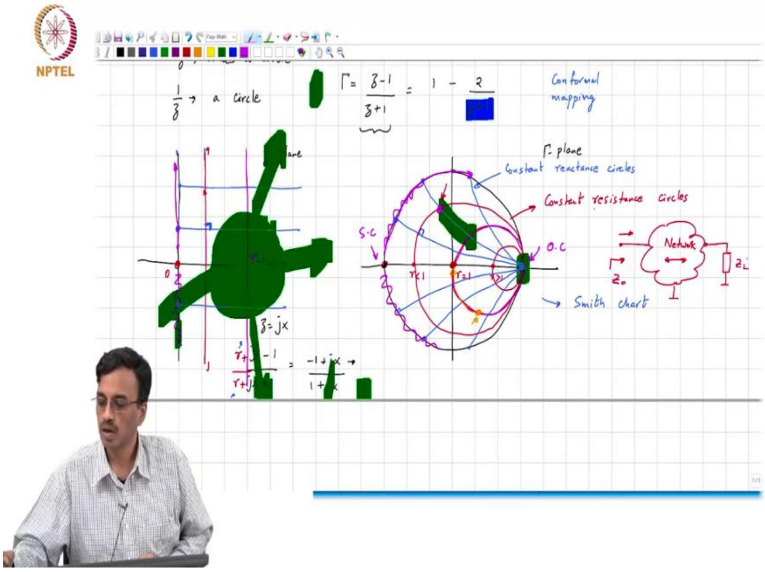


So, with this background let us see what happens when we have a passive impedance. If you have passive impedance you know, you where is the locus of all possible impedance is. Pardon? In the z plane were. Yeah. So, basically you know you are only in the. You are only in this area right in the rhb ok. The real part is positive the imaginary part can be either positive or negative, does it make sense right ok.

Now let us we know that, if we now plot what happens with some crucial points on the z plane in the gamma plane right this is al also called the reflection coefficient plane right. So, what comment can you make about the Y-axis in the z plane or the imaginary axis in the z plane right.

So, basically that corresponds to z equal to some j x and gamma therefore is nothing but j x minus 1 by j x plus 1 where x goes all the way from minus infinity to infinite.

$$z = jX$$

$$where\ \Gamma = \frac{jX - 1}{jX + 1}$$

So, how will this look like? Well, when x is 0 where are we? Oh well we are at minus 1 ok when x is infinity plus infinity where are we? So, this is minus sorry this is minus 1 plus j x by 1 plus j x.

$$\Gamma = \frac{jX - 1}{jX + 1} = \frac{-1 + jX}{1 + jX}$$

So, as x becomes infinity where are we what is the angle of this guy? Is plus 90 and the bottom 1 is also plus 90. So, basically it is at plus 1 that is here right ok. When j x is somewhere in the middle let us say it is large and positive how is it going from minus 1 to plus one.

You can see that this the radius of this for whatever value of x you have the radius is always what is the magnitude of this complex number? 1. So, this is basically a circle with the radius of 1 ok, not a particularly nice circle, but let me see, if I can look like there is no rule here which can draw circle never mind ok alright. So, this corresponds and the origin basically corresponds to this point corresponds to that point alright.

And that makes sense right because if the impedance is a perfect shot, then the reflection coefficient is reflection coefficient is minus 1 alright. Now if you have a line like this what do you think will happen? This is 0. I mean well what curve inside the unit inside the circle. It should touch ok as you go to plus infinity in this direction what happens in either direction what should happen eventually plus or minus if you go along this line where you have a positive real part and an imaginary part which goes to plus infinity or minus infinity what would you expect to see for the reflection coefficient what is. So, in other words you have r plus j x minus 1 by r plus j x plus 1. As x tends to infinity what you know what comment can you make?

It will be 1 ok. When you have x is 0 where what comment can you make? r minus 1 by r plus 1. So, where is that going to be. It will be on the real axis right ok. So, depending on where you what you choose r to be, if r is exactly equal to 1 where will it where will you be. You will be at the origin right.

So, this corresponds to r equal to 1 ok. And this is this part is for r less than 1 and this part corresponds to r greater than 1 correct. So, as you keep going depending on the value of r the starting point will change, but eventually they will all end up at 1 plus 1 right. So, and we know that all these must be, when this is a straight line and after transformation it must become either a straight line or a circle right.

So, this basically it is clear that there must be circles, which all meet at infinity ok. And all these are low psi of where the resistance is constant right it is only the reactance varies. So, these are often what are called constant resistance circles alright.

The next thing is what comment can you make about this entire space therefore, we knew this already. Pardon? Yeah, I know what will happen where will that entire space map to in the gamma plane.

Yeah, it does. So, basically it is inside the unit circle right ok. And if your kind of mentally imagine you know what you are doing is you know taking this long line and kind of you know kind of pushing it to make it a circle. I mean imagine you have you know you make a cut along here and then you kind of take that line and then make it a circle. Whatever is to towards the right of the line will become will go. Will be inside the circle right.

Now ok, what comment can we make about say this guy here now, where the resistance is varying, but the reactance is constant. So, now, r varies from 0 and you know this keeps going to and keeps going to infinity r varies from 0 all the way to positive infinity.

So, when r is 0 where are we? We are here correct. And therefore, on this picture where are we? Yeah, where when the unit circle is a big place man. It is like saying you are somewhere on iit campus.

Yeah. So, basically remember that what we have done is simply taken this vertical line and kind of squeezed it and made it a circle. So, whatever happens here will happen somewhere there on that axis exactly where it depends on where exactly you were here, but this part corresponds to this part alright. It is a non-linear mapping of course, right.

But this part corresponds to this part and this part corresponds to that part is that clear people ok. So, ok that we know what happens there and as r goes to infinity where are we? See whether x goes to infinity or r goes to infinity it basically is like an open circuit where the reflection coefficient is plus 1. So, at as r tends to infinity you must go there right and how will it how will it go there?

It will go like a circle like this ok and I will and why it goes like a circle like this. It turns out that bilinear mapping this is also what is called conformal mapping and any of you who is who is done you know complex analysis course probably has seen this before right.

And what it means is that if you have two curves and you have an angle between these two curves here. When you map these two curves using this bilinear mapping then the angle between these two curves here will be preserved here ok.

And again, I mean this is uh It is easily proven right. And I encourage you to try it out. So, as you can see therefore, I mean the angle between the red line here and the blue line here is 90 degrees and. So, at the point of intersection you will find that you know these circles are basically this the those two at the point of intersection the angle is still 90 degrees.

So, you have now a family of, for different values of reactance you will have different parts of circles like this ok. And remember these two points in the z plane they, where do they meet where do the two straight parallel straight lines meet folks? Infinity right and. So, they meet at a point the infinity in z plane you know maps on to what do you call plus 1 in the gamma plane. So, you know all these vertical lines and all the horizontal lines all meet at infinity. So, they all basically meet at plus 1 right the angle between the blue lines in the z plane right ok is always they meet, but when they meet what is the angle between them?

Angle between them is 0 correct. So, basically you know they all kind of approach the in plus 1 in that fashion. And when they meet at infinity the red line is still perpendicular to the blue line right. So, you can see that you know when all the red lines you know the red circles approach infinity plus 1 in this way, all the blue circles approach infinite in the horizontal direction the angle between them is still 90 degrees.

So, the red circles are all the so, called constant resistant circles and the blue lines the blue circles are all constant reactance circles and alright ok.

So, and I mean this is of a historical importance and also and those of you who are using r f measurement equipment will also see this, or reading r f papers will see this often this diagram the reflection coefficient diagram often plotted. And this is often called the Smith chart alright.

And you know the question is you know you know why should I bother about this and you know why is this useful. Remember that you know if the if z is you know very very large compared to 1. That I mean that the small z which is the normalized impedance which is the impedance normalized to the characteristic impedance z naught. If it is if it deviates from z equal to 1 or z equal to z naught by a lot.

So, this is let us say this is let us say this is the line that corresponds to r equal to 1 ok in here it will be this circle here. If you have an impudence and if it deviates a lot from this r equal to 1 right I mean whether it is you know here or you know yeah you know a large part of this paper is pretty much gives us almost no information as to I mean the most of the power is reflected correct ok.

So, you know you I mean what you can say I suspect I mean, I is basically that the farther you are from this r equal to 1. I mean the details of where exactly the impedance is hardly matter as far as, as far as, the amount of power being reflected or absorbed are concerned right.

So, one way of thinking about it is that, oh well if you have you know a certain space to convey information. Most of that space here is wasted, because well all the action is around r equal to 1 if you want to, I mean if you want. For instance, for to design a load which basically absorbs all the power I mean all the power that is delivered by that is incident on it by a source right.

You eventually to want to make sure that you do whatever it takes to get to this point r equal to 1 right ok. And that was you know for example; you know very important thing in and still remains an important thing right and that is called you know impedance matching where the idea is as we have seen yesterday you want to minimize the amount of power that is reflected by the load right.

You want it to absorb everything for instance you might be at the front end of an r f receiver and you know some r f power is coming in. And you want to absorb as much of it as possible that is one possibility. You might be an amplifier driving an antenna and there it is imperative that whatever you put out is all transmitted you know by the antenna and is absorbed by the antennae. So, that absorbed by the antenna means that it is transmitted out into space right.

There you want to make sure again that the amount of and the power reflected back from the load is as small as possible right and. So, you know in other words you really care about the reflection coefficient being as close to 0 as possible right. So, in other words you know you are particularly worried about you know impedances in that region there right ok.

And even say a small deviation from say r equal to 1 to say r equal to 2, already basically means that you are reflecting you know say 30 percent of your power back to the back to the source ok. So, in other words the what do you call this is where you know all the action is happening right.

And if you think I mean if you have an amount of space to show an impedance basically, I mean you know all the action is here, but most of the space is just simply getting wasted you know showing parts of the z plane which we know are not really relevant ok.

And if you want to calculate the reflection coefficient every time you know need to go and do the z minus 1 by z plus 1 alright. So, you know more convenient thing where the where this area right is basically dedicated a lot more space and the farther you move away from this space right from that area, I mean you know that you know that part is not really important right.

So, whether your impedance is here or here it hardly matters because the reflection coefficient is 1 in both cases right. So, it is so the reflection coefficient chat is a lot more useful in practice and. So, in the early days when you know you did not have you know calculators which would do the z minus 1 by z plus 1 with complex numbers quickly right.

If you had a chart which basically plotted the reflection coefficient plane rather than the impedance plane itself. Then you would be able to kind of you know this area basically here the critical area here basically translates to a large part of that picture. So, basically you dedicate a large portion of that area to within quotes the information bearing part of the z plane right.

And all the periphery is basically reserved for within quotes you know unimportant parts are the of the z plane right. And you I mean and what is the, so let us say you had some impedance. You can plot it once you know it is you know resistance and reactance you can plot it you plot the point immediately.

For example, right let us say your resistance is less than 1 and basically your reactance is some positive thing, I you can say well my oh this is my impedance z right plotted in the reflection coefficient plane. And I can immediately read off the reflection coefficient, how will I be able to read off the reflection coefficient?

This is the impedance red what is the reflection coefficient how would I be able to find the reflection coefficient? I mean this is a gamma plane right, I have marked the point here what are the magnitude of the reflection coefficient?

It is simply the radius of this the circle right the distance from the origin is simply the reflection coefficient alright ok. So, what do you call and then you know it turns out for instance if you had an impedance and here for instance ok and. So, this is Z L alright.

And let us say I need to do something to make sure that this impedance is actually Z naught. So, I will try and put a network between my source and the load in an attempt to make they looking in impedance equal to Z naught. And this load better this network better be lossless because I want whatever power is being drawn from the source to go eventually into the load. So, this is you can think of it as a transformer which transforms Z L into Z naught right, but this only works at 1 frequency unlike the transformer that you are familiar with which works at all frequencies ok alright.

So, for instance if you are here and you want to get to, I mean if the looking in impedance has to be Z naught. You eventually have to get where on the Smith chart, you have to get to the origin right. So, one way of thinking about it, is basically that you can well you can travel for instance like this right.

If you want to get from that point to that point, what do you think I could do well I you mean in principle you could add serious resistance, but you know that basically defeats the purpose of the matching network right ok.

So, what you would want to do actually it turns out that you know you add a I mean once you plot the impedance chart you can also kind of figure out what the constant admittance circles are right. I think it is it is not particularly useful in this class and therefore, and those if you are interested in go and look up Smith chart and.

You know how you can design matching networks which basically allow you to realize an input impedance of Z naught even when Z L is not what you want it to be right.

I mean a trivial example would be for example, if you were here right ok. What do you think you could do to get to clearly the reactance the resistance is one the reactance is not what you want it to be. So, what do you think you could do?

You add the appropriate reactants to go from you are on a constant resistance circle. And so basically you go like you know you add the appropriate reactants to cancel out the reactants that Z L has already and you will end up with a matched input impedance right.

Uh If you are not on a constant, I mean on the circle corresponding to r equal to 1, you can add appropriate elements in shunt and series to basically get on to this circle. And then eventually add a series element to get back to Z equal to 0 I mean sorry gamma equal to 0.

So, I what do you call there is a lot of literature on you know how you can do this right. But the Smith chat is something that you know you need to be aware of because you are going to see this a lot of times when you read you know r f papers or do r f work or use r f instruments and ok.

So, if you measure the reflection coefficient of for instance a capacitor where does you think it will lie on the Smith chart.

Pardon? Very good right. So, basically the reactance of a capacitor is negative. So, you are going to be somewhere here and that part corresponds to the lower half semicircle right. If you measure an inductor you are going to be its going to show up somewhere on the upper semicircle right.

Uh The magnitude as you can see is always going to be 1. If you have an open circuit you are going to be or a short circuit you are going to be, if you see a if you mentioned an open circuit where are you going to be?

This is the open circuit right, and if you have a short circuit where are you going to be?

Well, this alright. And if you have a fixed inductance, but sweep frequency where is where do you think what will be the locus of the points on the Smith chart. Yeah, fixed inductance put you sweep frequency at each frequency you plot the reflection coefficient where are you going to be.

Well on this picture you are going to be here right. As you keep increasing frequency the magnitude of the reactance will keep increasing. So, here you will basically the inductor will basically be will go along you will see a curve on the upper semi circle and likewise a capacitance you will see a curve on the lower semi circle ok.

So, that is all that I wanted to say about the Smith chart. The next thing I mean the Smith chart is useful and is also historically important.