Circuit Analysis for Analog Designers
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Lecture - 46
Transmission line driven by a source, power in a transmission line

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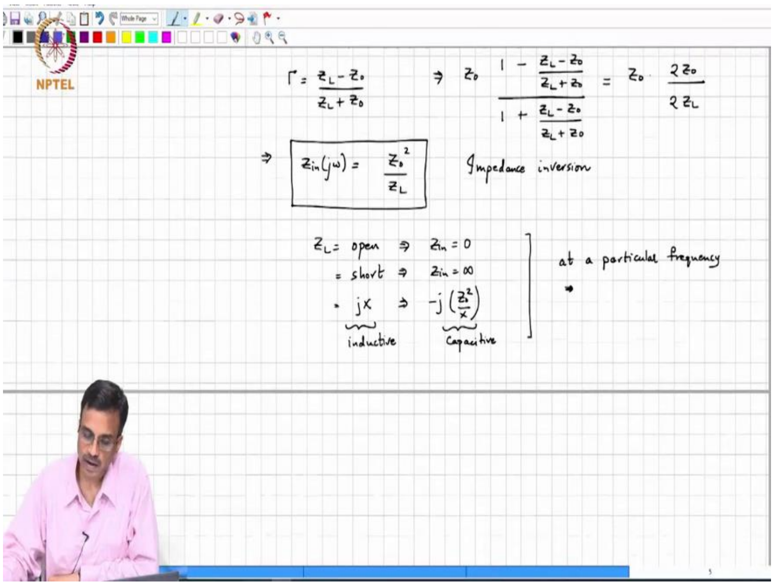

The next thing I would like to cover is what happens when again these are just examples of transmission line circuit analysis.

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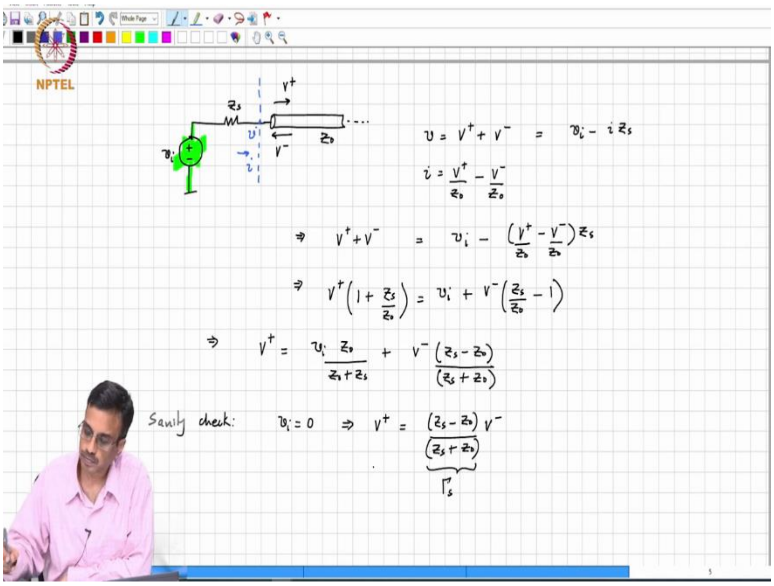

So, let us say we have a v i and a source impedance source resistance rather Z s and let us say we have a transmission line of course, we do not know where it goes. So, let us see what constraints the source impedance imposes on the incident and reflected waves ok.

So, again the trick is to realize that at the boundary uh the source and the source impedance will impose some conditions based on kcl and kvl and they must be consistent with the conditions imposed by the transmission line Namely the telegraphers' equations must be; must be valid.

So, the voltage at the, so let us call the voltage at that point v and the current as i. So, v is nothing but V plus V minus, i is nothing but V plus y Z naught minus V minus by Z naught.

$$v = V^{+} + V^{-}$$

$$i = \frac{V^{+}}{Z_0} - \frac{V^{-}}{Z_0}$$

And so V must also be equal to v i minus i times Z s that is the constraint imposed by Kirchhoff's law.

$$v = v_i - iZ_s$$

And putting these things together we get V plus V minus that is the voltage V must be equal to v i minus V plus whole Z naught minus V minus whole Z naught times Z s.

$$\Rightarrow V^{+} + V^{-} = v_i - \left(\frac{V^{+}}{Z_0} - \frac{V^{-}}{Z_0}\right) Z_s$$

Alright which basically means that V V plus therefore, times 1 plus Z s by Z naught is nothing, but v i plus V minus into Z s by Z naught minus 1.

$$\Rightarrow V^{+}\left(1 + \frac{Z_s}{Z_0}\right) = v_i + V^{-}\left(\frac{Z_s}{Z_0} - 1\right)$$

Which means that V plus is nothing but v i into Z naught by Z naught plus Z s plus V minus into Z s minus Z naught by Z s plus ok.

$$V^{+} = \frac{v_i Z_0}{Z_0 + Z_s} + V^{-}\frac{(Z_s - Z_0)}{(Z_s + Z_0)}$$

So, let us do some sanity check; one thing if say well this comes this equation comes out of the math, right. So, obvious thing that we should ask is why does it does this make intuitive sense? So, sanity check we set vi to 0 alright, if vi is 0, V plus is simply Z s minus Z naught by Z s plus Z naught times V minus.

$$v_i = 0 \Rightarrow V^+ = \frac{(Z_s - Z_0)}{(Z_s + Z_0)} V^-$$

And that makes sense simply because well if v i was 0 we should correlate this with the results we obtained when we terminated the line with a Z L there, we saw that, the reflected wave is gamma times the incident wave.

And now when you look at this picture when the source was 0, so that this is a short circuit, then the incident wave the wave that is incident on the source impedance is V minus and it gets reflected as V plus. So, it makes sense that V plus is the reflection coefficient this is what we will call gamma s corresponds to the source reflection coefficient, right. So, Z s minus Z naught by Z s plus Z naught times V minus, now that makes sense.

$$\Gamma_s = \frac{(Z_s - Z_0)}{(Z_s + Z_0)}$$

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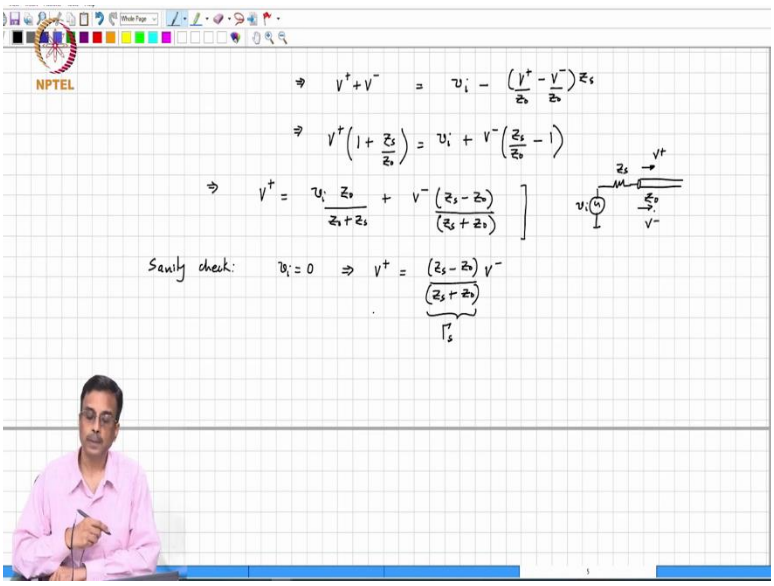

Now, let us see. So, all that this is telling us is this telling us is that the forward going wave when we have an input voltage through Z s exciting a transmission line with a characteristic impedance Z naught is following.

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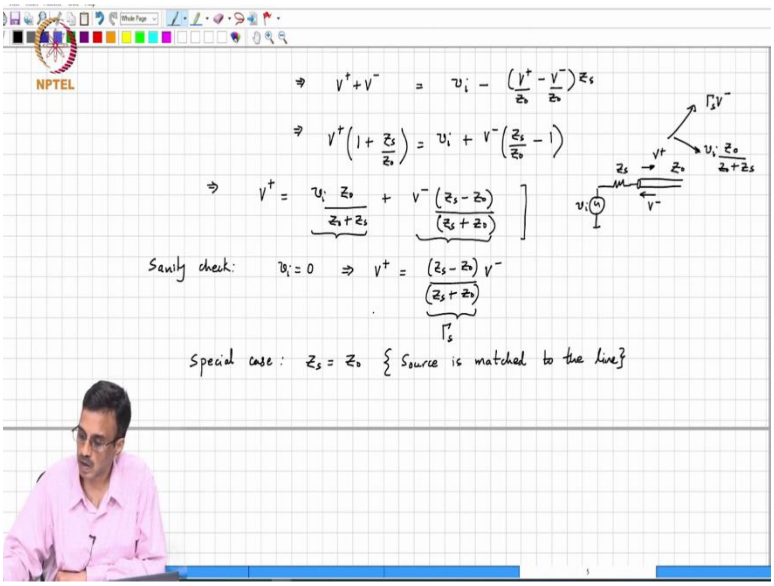

So, V plus is caused by two aspects; one is the reflected version of V minus and V minus is the wave that is incident on that junction between the transmission line and the terminating impedance.

So, V plus has got two components; one component is nothing, but gamma times V minus gamma s just to remind ourselves that that is the source reflection coefficient and given by Z s minus Z naught by Z s plus Z naught. There is another component; obviously, due to the input voltage and that is given by Z naught by Z naught plus Z s that makes intuitive sense because if you assume that the line is infinitely long right, then what happens? There is no reflected wave.

So, in other words if v i if V minus was 0, then V plus would simply be the potential divided version of the characteristic impedance Z naught and Z s. So, that is basically v i times Z naught by Z naught plus Z s ok. So, that is so that is indeed the case as we see from the math. In a special case when the source is matched to the transmission line, what do we see?

$$Z_s = Z_0$$

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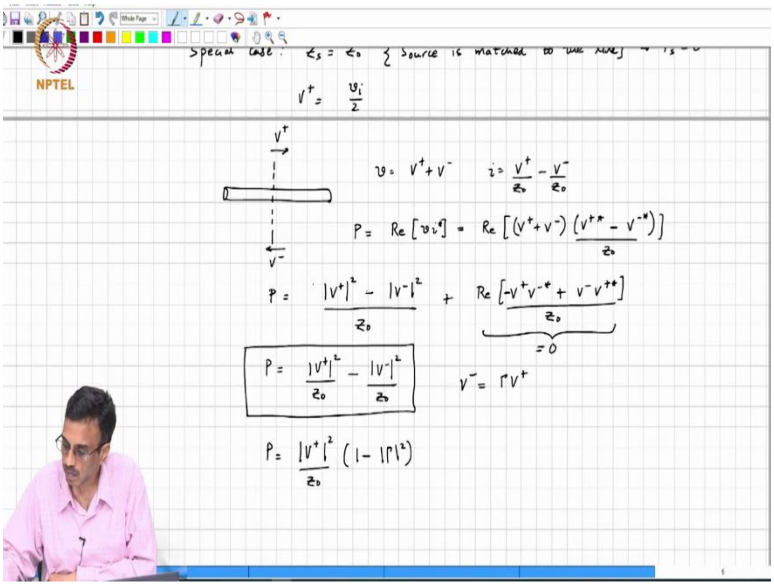

V plus is nothing but v i by 2 plus 0 because the gamma the reflection coefficient gamma s is then 0.

$$\Rightarrow V^+ = \frac{v_i}{2}$$

$$\Gamma_s = 0$$

So, what this means is that V minus is incident on the junction, but nothing gets reflected back and the only wave that is going forward is due to the input and has an amplitude of v i by 2 ok, alright.

The next thing I would like to talk about is power along a transmission line along an ideal transmission line. So, let us say you have an ideal transmission line we do not know where it comes from, where it goes, but at that black line I have drawn black dotted line I have drawn the forward going wave is V plus and the backward going wave is V minus and the power.

So, the voltage is V plus plus V minus the current is V plus by Z naught minus V minus by Z naught and the power is nothing but the real part of v times i and that is nothing but real part of V plus plus V minus times V plus v i actually sorry the the power is the real part of v i star. So, real part of V plus plus V minus times V plus star minus V minus star divided by Z naught.

$$v = V^+ + V^-$$

$$i = \frac{V^+}{Z_0} - \frac{V^-}{Z_0}$$

$$P = Re[v_i^*] = Re\left[(V^+ + V^-)\left(\frac{V^{+*} - V^{-*}}{Z_0}\right)\right]$$

And its clear that the power is nothing but mod V plus the whole square minus mod V minus the whole square by Z naught plus the real part of V plus minus V plus V minus star and plus V minus into V plus star by Z.

$$P = \frac{|V^+|^2 - |V^-|^2}{Z_0} + Re\left[\frac{-V^+V^{-*} + V^-V^{+*}}{Z_0}\right]$$

And ($Re\left[\frac{-V^+V^{-*}+V^-V^{+*}}{Z_0}\right]$) this is a purely imaginary quantity the quantity in the brackets is a purely imaginary quantity and therefore, this is 0.

So, therefore, the power flowing towards the right is nothing but mod V plus the whole square by Z naught minus mod V minus the whole square by Z naught.

$$P = \frac{|V^+|^2}{Z_0} - \frac{|V^-|^2}{Z_0}$$

So, one way you can interpret this equation is to say that you can think of power going towards the right and that is given by V plus square by Z naught and power coming back getting reflected by whatever is there on the right side at the load side.

And so we know that V minus if the line is terminated somewhere with some Z L we know that V minus is nothing but gamma times V plus.

$$V^- = \Gamma V^+$$

And therefore, the power is nothing but V plus the whole square by Z naught times 1 minus mod gamma the whole square ok.

$$P = \frac{|V^+|^2}{Z_0}\,(1 - |\Gamma|^2)$$

And so this is the reflection coefficient and this is. So, this reflects this one minus gamma square is basically telling us that of the power that is being pumped into the that is you know into the load a fraction gamma square is getting reflected back and the net power being dissipated in the load is the incident power times 1 minus gamma square mod gamma square, alright.

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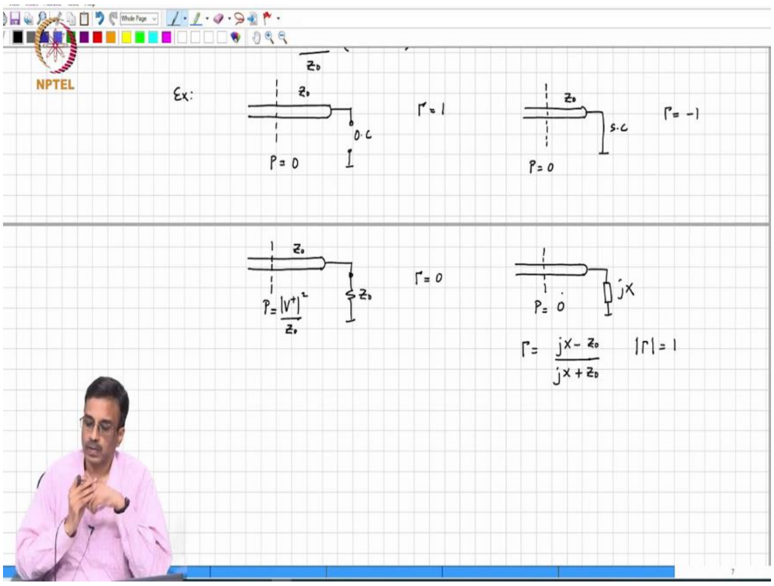

So, let us again take some examples. So, we have an open circuit let us say you have an open circuit intuitively of course, its very clear that no power can be dissipated to the I mean to the right of this dotted line and that is obvious, but that is also comes out to the math.

Because we know that gamma is 1. So, gamma square is 1 and therefore, the power therefore, going in or being dissipated to the right here is basically 0, the average power is 0 right. Similarly with a short circuit gamma is minus 1 and therefore, the power being dissipated again is 0 simply because the entire power is getting reflected ok.

Likewise let us say this is Z naught the power being dissipated is mod V plus square over Z naught alright and gamma is basically 0 that makes sense because whatever power is going towards the right none of it gets reflected everything is absorbed in the load and therefore, the entire power that is incident is dissipated towards the right of that dotted line.

Now, if you have a purely reactive load let us call this jx where x can be positive or negative depending on whether its an inductor or a capacitor. What is gamma? Gamma is jx minus 1 that is Z L minus Z naught sorry by Z L plus Z naught

$$\Gamma = \frac{jX - Z_0}{jX + Z_0}$$

And in general, its a complex number. However, the magnitude of this complex number is 1 and therefore, the power dissipated towards the right of that dotted line is simply 0 right.

Again, because recall that a pure reactance cannot dissipate any power. So, whatever power is being is incident must get reflected back, alright.

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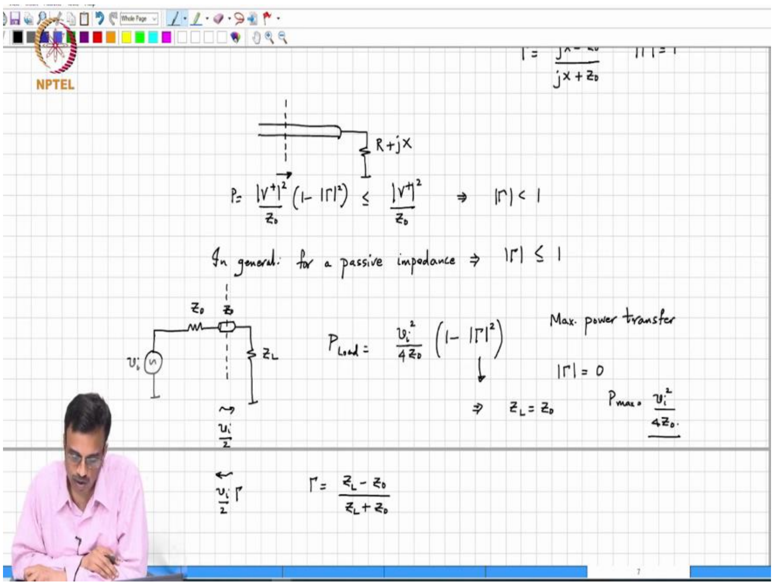

So, now, the last case. So, let us say you have a passive impedance, so you have some r plus j X. The power being dissipated is V plus of course, there will be now some power being dissipated in the resistance, so that 1 minus mod gamma whole square this must be less. So, less than the incident power right. So, which therefore, means that mod gamma must be less than ok.

$$P = \frac{|V^+|^2}{Z_0}(1 - |\Gamma|^2) \leq \frac{|V^+|^2}{Z_0} \quad \Rightarrow |\Gamma| < 1$$

So, right basically all that is saying is that the power that is being disappeared in the resistor on the resistor part of the impedance cannot be greater than the power that is being that is incident in the first place. So, this means that the reflection coefficient must be less than 1. So, in general for a passive impedance which cannot a passive impedance is one which cannot generate energy or power on its own. So, for a passive impedance the magnitude or the reflection coefficient must be less than or equal to 1.

$$|\Gamma| \leq 1$$

Equal to 1 corresponds to a purely reactive impedance right a magnitude less than 1 basically means that the terminating impedance the impedance has got some resistive part which is responsible for the dissipation of power alright. Now, the last aspect I would like to discuss is the following, this notion of incident and reflected power can also be used to kind of give us a new perspective on power transfer between two impedances.

So, let us assume for argument for simplicity we assume that these are real impedances. So, this is v i the source resistance and the load resistance. Now, let us assume let me call this Z naught and Z L and for arguments sake I can think of this connection as being an infinitely small infinitesimal transmission line with a characteristic impedance of Z right, it’s a really really tiny line with the characteristic impedance of Z, right. So, what is the power dissipated to the right of the boundary or that of the dotted line.

The we know from our discussion earlier in this lecture that the forward going wave is nothing, but v i by 2 right and the backward going wave is nothing, but v i by 2 times gamma where gamma is nothing but Z L minus Z naught by Z L plus Z naught ok. And what should be the power dissipated to the right of the boundary?

I mean the power is only dissipated in the terminating impedance, it cannot be dissipated on the transmission line and why cannot it be dissipated on the transmission line? Because we assume the transmission line is lossless, it consists only of inductors and capacitors infinitesimally small inductors and capacitors, ok. So, the power dissipated in the load is nothing but the incident power.

That is v i square by four Z naught times 1 minus mod gamma square ok and alright.

$$P_{load} = \frac{v_i^2}{4Z_0}(1 - |\Gamma|^2)$$

And if we want the maximum power to be dissipated you might also recall that this is what is called the maximum power transfer theorem from your earlier classes if you want the maximum power to be dissipated in the load resistance Z L, what would we do? We want to make this as large as possible.

So, that basically means that gamma mod gamma must be as small as possible and the smallest mod gamma can be is 0 right which therefore, means that the load must be matched to the source in other words gamma must be 0. So, that is possible only when Z L is equal to Z 0 ok and.

So, if Z L is not equal to Z naught then part of that incident power the maximum power that that can be dissipated in the load is simply v i square by 4 Z naught right and this of course, you know we knew all this from our earlier classes where you know you write the voltage you write the current and then you multiply them and you differentiate with respect to Z L and so on right. This is another way of looking at the same thing you know this on the other hand seems a lot simpler to see what is going on.

So, if you want maximum power to be dissipated in the load you want the reflection coefficient gamma to be 0 which means that the load resistance is matched to the source right and the maximum power that can be dissipated is nothing but v i square by 4 Z naught alright.

$$P_{max} = \frac{v_i^2}{4Z_0}$$

So, with this I will stop, we will continue in the next class.